A new approach to equilibrium problems

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Abstract

The goal of this paper is to prove some general vector-valued perturbed equilibrium principles and some existence results of vector equilibrium points for bifunctions satisfying a new natural notion of lower semi-continuity. We obtain these results by going through a new concept of approximatively equilibrium point.

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1 Introduction

A scalar equilibrium problem is defined as follows :

$$\text{(EP)} \quad \text{Find } \overline{x} \in X \text{ such that } \forall y \in X, \;\; f(\overline{x}, y) \geq 0,$$

where X is a given set and $f : X \times X \to \mathbb{R}$ is a given bifunction. A point $\overline{x}$ satisfying (EP) is called an equilibrium point. There are many examples of such equilibrium problems (see [7] for a first survey). Let us just mention a few of them : minimization problems (where $f(x, y) := h(y) - h(x)$ and $h : X \to \mathbb{R}$), variational inequalities (where $f(x, y) :=< Tx, y - x >$, $T : X \to X^*$, and X is a normed space) and fixed point problems (where $f(x, y) :=< x - Tx, y - x >$, $T : H \to H$, and H is an Hilbert space). Let us also mention Nash equilibria in non-cooperative games and complementary problems. It is natural to extend the previous scalar equilibrium problem to a vector equilibrium problem. That is :

$$\text{(VEP)} \quad \text{Find } \overline{x} \in X \text{ such that } \forall y \in X, \;\; f(\overline{x}, y) \notin -K \setminus \{0\},$$

or, in a weaker way :

$$\text{(WVEP)} \quad \text{Find } \overline{x} \in X \text{ such that } \forall y \in X, \;\; f(\overline{x}, y) \notin -\operatorname{int} K,$$

where $f : X \times X \to Y$ and Y is a real Banach space, partially ordered by a closed convex pointed cone K. There are a lot of applications to vector optimization, game theory and mathematical economics. It is why many papers are devoted to vector equilibrium problems, see [13] and [15] (and the references therein), and for example [1, 2, 3, 14, 16]. In [5], Bianchi, Kassay and Pini provide a vector version of the Ekeland variational principle connected to equilibrium problems with the purpose to find approximate vector equilibrium points. They are then able to prove the non-emptiness of the solution set of (WVEP) without any convexity requirements on the set X and on the bifunction f. Of course, they need some usual assumptions as: the semi-continuity of the functions $f(x, \cdot)$ and $f(\cdot, y)$ for all $x, y \in X$ and, either the compactness of the domain or a coercivity condition on the bifunction.
If we consider a bifunction f such that $f(x, \cdot)$ is bounded below and lower semi-continuous for every $x \in X$, by lack of compactness, there is no reason why a vector equilibrium point should exist. In this article, we study *perturbed equilibrium principles*. That is : *results which assert the existence of a perturbation g, as small as possible, such that $f + g$ admits a vector equilibrium point.*
In the scalar case, the Deville-Godefroy-Zizler variational principle [10] solves the question for minimization problems. Let us recall this result :

Theorem (Deville-Godefroy-Zizler Variational Principle). *Let X be a Banach space and $(Z, \|\cdot\|_Z)$ be a Banach space of real-valued bounded continuous functions on X such that :*

(i) for all $g \in Z$, $\|g\|_Z \geq \|g\|_\infty := \sup_{x\in X} |g(x)|$;

(ii) Z is translation invariant, i.e. if $g \in Z$ and $x \in X$ then $\tau_x g : X \to Y$ given by $\tau_x g(t) := g(t - x)$ is in Z and $\|\tau_x g\|_Z = \|g\|_Z$;

(iii) Z is dilation invariant, i.e. if $g \in Z$ and $\alpha \in \mathbb{R}$ then $g^\alpha : X \to Y$ given by $g^\alpha(t) := g(\alpha t)$ is in Z;

(iv) there exists a bump function $b : X \to \mathbb{R}$ in Z.

If $f : X \to \mathbb{R} \cup \{+\infty\}$ is a bounded below, lower semi-continuous and proper function, then the set of all functions $g \in Z$ such that $f + g$ admits a strong minimum is a G_δ dense subset of Z.

Let us recall that a *bump function* on X is a real-valued function on X with non-empty bounded support. Corollaries of this principle are, for example, the (classical) Ekeland variational principle [11] and the Borwein-Preiss smooth perturbed minimization principle [9]. A vector-valued version of the Deville-Godefroy-Zizler variational principle has been obtained in [12] for bounded below, order lower semi-continuous functions $f : X \to Y$. And, as in the scalar case, the authors got a vector-valued version of the Ekeland and Borwein-Preiss minimization principles.

Here we study a new vector-valued version of the Deville-Godefroy-Zizler variational principle for bifunctions $f : X \times X \to Y$ which satisfy a new natural continuity property and such that $f(x, \cdot)$ is bounded below for all $x \in X$. We also get in this context the Ekeland and Borwein-Preiss perturbed equilibrium principles. On the other hand, our techniques allow us to prove the same existence results than in [5] but under weaker assumptions.

In Section 2, we recall some basic definitions and some relationships between some different vector-valued notions of lower semi-continuity.

Section 3 is devoted to a new notion of lower semi-continuity for bifunctions, called *coordinate free lower semi-continuity* (Definition 1). This notion looks quite natural since when f is defined by $f(x, y) = h(y) - h(x)$, where h is a function from X to Y, the coordinate free lower semi-continuity of f is equivalent to the order lower semi-continuity of h (Proposition 3). We prove that the coordinate free lower semi-continuity of f is weaker than the lower (resp. upper) semi-continuity of $f(x, \cdot)$ (resp. $f(\cdot, y)$) for all $x \in X$ (resp. $y \in X$) (Proposition 4 (resp. Proposition 8)). The rest of Section 3 is devoted to the study of some connections between this notion of coordinate free lower semi-continuity and the classical notions of semi-continuity.

In Section 4, we introduce the notion of *approximatively equilibrium point in the direction of an element of* K (Definition 12). The key result for applications is Proposition 13 which asserts the existence and localization of such a point if we work with a diagonal null and lower transitive bifunction f such that $f(x, \cdot)$ is bounded below for all $x \in X$.

Some applications are given in Section 5. We first establish a Deville-Godefroy-Zizler perturbed equilibrium principle (Theorem 14), and we get, as corollaries, the Ekeland and Borwein-Preiss perturbed equilibrium principles (Corollaries 16 and 18). We also prove some existence results (Theorems 19, 20 and 21) for equilibrium problems under weaker assumptions than the usual ones.

2 Preliminaries and notation

Throughout this paper, X and Y are two real Banach spaces and Y is partially ordered by a closed convex pointed cone K. No assumption is required on the interior of K (except of course when we deal with the problem (WVEP)).

For any elements $u, v \in Y$, we will write $u \leq v$ whenever $v - u \in K$. The set $[u, v] := \{w \in Y : u \leq w \leq v\}$ is called the *order interval* between u and v. We say that a sequence $(u_n) \subset Y$ is *non-increasing* whenever, for all n, $u_{n+1} \leq u_n$. The ball of center

x_0 and radius r in X will be denoted by $B_X(x_0, r)$. Let S be a non-empty subset of Y, we denote respectively by $\operatorname{int} S$ and $\operatorname{Aff} S$, the interior and the affine hull of S.
Let f be a function from X to Y, it is said to be *bounded below* (resp. *above*) if there exists some b in Y such that $b \leq f(x)$ (resp. $f(x) \leq b$) for all $x \in X$, and *order-bounded* if it is both bounded below and above. The following two notions of lower semi-continuity were introduced in [8] and [19] :

- f is said to be *lower semi-continuous* (lsc) at $x_0 \in X$ iff, for each neighborhood V of $f(x_0)$ in Y, there exists a neighborhood U of x_0 in X such that $f(U) \subset V + K$.
- f is said to be *quasi lower semi-continous* (q-lsc) at $x_0 \in X$ iff, for each $b \in Y$ such that $b \not\geq f(x_0)$, there exists a neighborhood U of x_0 such that $b \not\geq f(x)$ for each x in U.

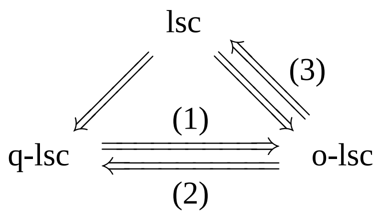
A function f is (resp. *quasi-*) *upper semi-continuous*, usc for short (resp. q-usc), if $-f$ is lsc (resp. q-lsc). A function f is lsc (resp. q-lsc) if f is lsc (resp. q-lsc) at each point of X. Let us give some well-known facts concerning these notions (see, for example, [8], [12] and [19]).

- A function f is lsc at x_0 iff, for each sequence $(x_n) \subset X$ converging to x_0, there exists a sequence $(g_n) \subset Y$ converging to 0 such that $f(x_0) \leq f(x_n) + g_n$ for all n.
- A function f is q-lsc iff for each b in Y, the set $\{x \in X : f(x) \leq b\}$ is closed in X.
- A lsc function at x_0 is q-lsc at x_0.

A new notion of lower semi-continuity, weaker than the two others, was intoduced in [12]. It is called *order lower semi-continuity* because it links, in a good way, the norm topology and the partial order of Y :

- f is said to be *order lower semi-continuous* (o-lsc) at $x_0 \in X$ iff, for each sequence $(x_n) \subset X$ converging to x_0 for which there exists a sequence $(\varepsilon_n) \subset Y$ converging to 0 such that the sequence $(f(x_n) + \varepsilon_n)$ is non-increasing, there exists a sequence $(g_n) \subset Y$ converging to 0 such that $f(x_0) \leq f(x_n) + g_n$ for all n.

These three notions of lower semi-continuity coincide for scalar-valued functions but it is not the case in the vector-valued case. Here is a summary of the different relationships for a vector-valued function $f : X \to Y$, with $\dim Y > 1$ (cfr. [12]) :



(1) (Y, K) has the Monotone Bounds Property or $\dim Y < \infty$ and f is bounded below.

(2) f is bounded below and order intervals are compact.

(3) f is order bounded and order intervals are compact.

A lot of examples are given in [12] in order to show that these conditions are also necessary.

3 Lower semi-continuity for bifunctions

In this section, we introduce a new notion of lower semi-continuity for bifunctions and we compare it with the classical ones.

Definition 1. *A bifunction $f : X \times X \to Y$ is said to be* ***coordinate free lower semi-continuous*** *(cf-lsc) at $x_0 \in X$ iff, for each sequence $(x_n)_{n\geq 1} \subset X$ converging to x_0 for which there exists a sequence $(\rho_n)_{n\geq 1} \subset \mathbb{R}_0^+$ converging to 0 such that :*

$$\forall n \geq 1,\ \forall l \geq 1, \quad f(x_n, x_{n+l}) \in -K + B_Y(0, \rho_n),$$

there exists a sequence $(\omega_n)_{n\geq 1} \subset \mathbb{R}_0^+$ converging to 0 such that :

$$\forall n \geq 1, \quad f(x_n, x_0) \in -K + B_Y(0, \omega_n).$$

The bifunction f is said to be ***weakly cf-lsc*** *if this condition is satisfied for each sequence $(x_n)_{n\geq 1} \subset X$ weakly converging to x_0.*

The following useful lemma relaxes this definition by allowing to work up to a subsequence :

Lemma 2. *Let f be a bifunction from $X \times X$ to Y. Then, f is cf-lsc at x_0 iff from any sequence $(x_n)_{n\geq 1}$ converging to $x_0 \in X$ and for which there exists a sequence $(\rho_n)_{n\geq 1} \subset \mathbb{R}_0^+$ converging to 0 such that : $\forall n \geq 1,\ \forall l \geq 1,\ f(x_n, x_{n+l}) \in -K + B_Y(0, \rho_n)$, one can extract a subsequence $(x_{n_k})_{k\geq 1}$ such that : $\forall k \geq 1,\ f(x_{n_k}, x_0) \in -K + B_Y(0, \omega_k)$, where $(\omega_k)_{k\geq 1} \subset \mathbb{R}_0^+$ is a sequence converging to 0.*

PROOF. The "only if" part is obvious. For the "if" part, let us suppose by contradiction that f is not cf-lsc at x_0. Therefore, there exist some $\varepsilon \in \mathbb{R}_0^+$, a sequence $(x_n)_{n\geq 1}$ converging to x_0 and a sequence $(\rho_n)_{n\geq 1} \subset \mathbb{R}_0^+$ converging to 0 such that : $\forall n \geq 1,\ \forall l \geq 1,\ f(x_n, x_{n+l}) \in -K + B_Y(0, \rho_n)$, and $\forall n \geq 1,\ f(x_n, x_0) \notin -K + B_Y(0, \varepsilon)$. By hypothesis, one can extract a subsequence $(x_{n_k})_{k\geq 1} \subset (x_n)_{n\geq 1}$ for which there exists a sequence $(\omega_k)_{k\geq 1} \subset \mathbb{R}_0^+$ converging to 0 such that : $\forall k \geq 1,\ f(x_{n_k}, x_0) \in -K + B_Y(0, \omega_k)$. Thus, $f(x_{n_k}, x_0) \in -K + B_Y(0, \varepsilon)$ whenever k is large enough. This contradiction ends the proof. □

Let us note that a similar result is true for the notion of order lower semi-continuity of a function $f : X \to Y$ (cfr. Lemma 4 of [12]).

In some sense, the next results justify the fact that the previous notion is a good extension of the notion of lower semi-continuity for bifunctions. The first one characterizes the coordinate free lower semi-continuity of a bifunction by means of the order lower semi-continuity of the function of one variable which defines the bifunction.

Proposition 3. *If the bifunction $f : X \times X \to Y$ is defined by $f(x, y) := h(y) - h(x)$, where h is a function from X to Y, then f is cf-lsc at x_0 iff h is o-lsc at x_0.*

PROOF. Let us suppose that the bifunction $f : X \times X \to Y$, defined by $f(x, y) := h(y) - h(x)$, is cf-lsc at x_0. Let us consider a sequence $(x_n)_{n\geq 1}$ in X converging to x_0 and a sequence $(\varepsilon_n)_{n\geq 1}$ in Y converging to 0 such that $(h(x_n) + \varepsilon_n)_{n\geq 1}$ is a non-increasing sequence. Let us consider a subsequence $(\varepsilon_{n_k})_{k\geq 1} \subset (\varepsilon_n)_{n\geq 1}$ such that the sequence $(\|\varepsilon_{n_k}\|_Y)_{k\geq 1}$ is non-increasing. So, we have :

$$\forall k \geq 1,\ \forall l \geq 1,\ \ f(x_{n_k}, x_{n_{k+l}}) = h(x_{n_{k+l}}) - h(x_{n_k}) \leq \varepsilon_{n_k} - \varepsilon_{n_{k+l}}$$

and then

$$f(x_{n_k}, x_{n_{k+l}}) \in -K + B_Y(0, \rho_{n_k})$$

where $\rho_{n_k} := 2\|\varepsilon_{n_k}\|_Y \xrightarrow[k\to+\infty]{} 0$. Since f is cf-lsc at x_0, there exists a sequence $(\omega_k)_{k\geq 1} \subset \mathbb{R}_0^+$ converging to 0 such that : $\forall k \geq 1,\ f(x_{n_k}, x_0) \in -K + B_Y(0, \omega_k)$. That means that $h(x_0) \leq h(x_{n_k}) + g_k$ with $g_k \to 0$ in Y and, by Lemma 4 of [12], h is o-lsc at x_0.
Let us consider now a function h from X to Y, o-lsc at $x_0 \in X$, and let us suppose that $(x_n)_{n\geq 1}$ is a sequence in X converging to x_0 and $(\rho_n)_{n\geq 1}$ is a sequence in $\mathbb{R}_0^+$ converging to 0 such that : $\forall n \geq 1,\ \forall l \geq 1,\ f(x_n, x_{n+l}) = h(x_{n+l}) - h(x_n) \in -K + B_Y(0, \rho_n)$. Let us consider a subsequence $(\rho_{n_k})_{k\geq 1} \subset (\rho_n)_{n\geq 1}$ such that $\sum_{k=1}^{+\infty} \rho_{n_k}$ converges. We have : $\forall k \geq 1,\ \forall l \geq 1, \exists \varepsilon_{k,l} \in B_Y(0, \rho_{n_k}) :\ h(x_{n_{k+l}}) \leq h(x_{n_k}) + \varepsilon_{k,l}$. In particular, $\forall k \geq 1$:

$$h(x_{n_{k+1}}) + \sum_{i=k+1}^{+\infty} \varepsilon_{i,1} \leq h(x_{n_k}) + \sum_{i=k}^{+\infty} \varepsilon_{i,1}$$

and $\left(h(x_{n_k}) + \sum_{i=k}^{+\infty} \varepsilon_{i,1}\right)_{k\geq 1}$ is non-increasing, with $\| \sum_{i=k}^{+\infty} \varepsilon_{i,1}\|_Y \to 0$ whenever $k \to +\infty$. Since h is o-lsc at x_0, there exists a sequence $(g_k)_{k\geq 1}$ converging to 0 in Y such that : $\forall k \geq 1,\ h(x_0) \leq h(x_{n_k}) + g_k$, that is $f(x_{n_k}, x_0) \leq g_k$. This proves, by Lemma 2, that f is cf-lsc at x_0. □

In the framework of applications to equilibrium problems, the classical hypothesis on the bifunction f are some semi-continuity conditions on the functions $f(x, \cdot)$ and $f(\cdot, y)$ for all x and y in X. We prove now that these conditions are stronger than the coordinate free lower semi-continuity.

Proposition 4. *Let f be a bifunction from $X \times X$ to Y. If $f(x, \cdot)$ is lsc at x_0 for all $x \in X$ then f is cf-lsc at x_0.*

PROOF. Let $(x_n)_{n\geq 1}$ be a sequence converging to $x_0 \in X$ and $(\rho_n)_{n\geq 1}$ be a sequence in $\mathbb{R}_0^+$ converging to 0 such that : $\forall n \geq 1,\ \forall l \geq 1,\ \exists \varepsilon_{n,l} \in B_Y(0, \rho_n) :\ f(x_n, x_{n+l}) \leq \varepsilon_{n,l}$. Let $n_0 \geq 1$ be fixed. Since $f(x_{n_0}, \cdot)$ is lsc, there exists a sequence $(g_n)_{n\geq 1}$ converging to 0 in Y such that $f(x_{n_0}, x_0) \leq f(x_{n_0}, x_{n_0+l}) + g_l$ for all $l \geq 1$. Since K is closed, we thus have $f(x_{n_0}, x_0) \in -K + B_Y(0, \rho_{n_0})$. This proves that f is cf-lsc at x_0 and ends the proof. □

When the interior of the ordering cone is non-empty we can improve the previous result by asking $f(x, .)$ q-lsc, instead of lsc.

Proposition 5. *Let f be a bifunction from $X \times X$ to Y where the partial order on Y is given by a cone K with non empty interior. If $f(x, \cdot)$ is q-lsc at x_0 for all $x \in X$ then f is cf-lsc at x_0.*

PROOF. If $\operatorname{int} K \neq \varnothing$, then there exists $e \in K \setminus \{0\}$ such that, for all $y \in Y$, $\pm y \leq \|y\|_Y e$ (see [12]). Let $(x_n)_{n\geq 1}$ be a sequence converging to $x_0 \in X$ and $(\rho_n)_{n\geq 1}$ be a sequence in $\mathbb{R}_0^+$ converging to 0 such that : $\forall n \geq 1,\ \forall l \geq 1,\ \exists \varepsilon_{n,l} \in B_Y(0, \rho_n) :\ f(x_n, x_{n+l}) \leq \varepsilon_{n,l}$. Let $n_0 \geq 1$ be fixed. For all $l \geq 1$, we have : $f(x_{n_0}, x_{n_0+l}) \leq \varepsilon_{n_0,l} \leq \|\varepsilon_{n_0,l}\|_Y e \leq \rho_{n_0} e$. Since $f(x_{n_0,.})$ is q-lsc at x_0, we have $f(x_{n_0}, x_0) \leq \rho_{n_0} e$. Since $\|\rho_n e\|_Y \to 0$, this proves that f is cf-lsc at x_0. □

In the finite dimensional setting, we can remove the assumption on the interior of the ordering cone but we have to work with a bifunction which is bounded below.

Proposition 6. *Let f be a bifunction from $X \times X$ to Y where Y is finite dimensional. If f is bounded below and $f(x, \cdot)$ is q-lsc at x_0 for all $x \in X$ then f is cf-lsc at x_0.*

PROOF. We have only to consider the case where $\operatorname{int} K = \varnothing$. Without loss of generality, we can assume that, for all $x, y \in X : 0 \leq f(x, y)$, that is $f(X \times X) \subset K \subset \operatorname{Aff} K$. Since, in the finite dimensional case, the interior of K for the topology relative to $\operatorname{Aff} K$ is non empty [18], we have: for all $(x_n)_{n\geq 1} \subset X$ converging to x_0 and all $(\rho_n)_{n\geq 1} \subset \mathbb{R}_0^+$ converging to 0 such that $\forall n \geq 1,\ \forall l \geq 1,\ \exists \varepsilon_{n,l} \in \operatorname{Aff} K \cap B_Y(0, \rho_n) :\ f(x_n, x_{n+l}) \leq \varepsilon_{n,l}$, there exists $(\omega_n)_{n\geq 1} \subset \mathbb{R}_0^+$ converging to 0 such that : $\forall n \geq 1,\ f(x_n, x_0) \in -K + B_Y(0, \omega_n)$. We have to remove the restriction $\varepsilon_{n,l} \in \operatorname{Aff} K$. Suppose that : $\forall n \geq 1,\ \forall l \geq 1,\ \exists \varepsilon_{n,l} \in B_Y(0, \rho_n) :\ f(x_n, x_{n+l}) \leq \varepsilon_{n,l}$. Let us prove that $\varepsilon_{n,l} \in \operatorname{Aff} K$. For all $n \geq 1$, let us write $\varepsilon_{n,l} := a_{n,l} + t_{n,l}$ with $a_{n,l} \in \operatorname{Aff} K$ and $t_{n,l} \in T$, where T is a topological complement of $\operatorname{Aff} K$ in Y. We thus have :

$$t_{n,l} = f(x_n, x_{n+l}) - a_{n,l} + k_{n,l}$$

for some $k_{n,l} \in K$. The fact that the right-hand side is an element of $\operatorname{Aff} K$ and $\operatorname{Aff} K \cap T = \{0\}$ imply $t_{n,l} = 0$, which ends the proof. □

Another classical hypothesis on bifunctions is the following :

Definition 7. *A bifunction $f : X \times X \to Y$ is said to be* ***diagonal null*** *iff for every $x \in X$, $f(x,x) = 0$.*

Proposition 8. *Let f be a bifunction from $X \times X$ to Y. If f is diagonal null and $f(\cdot\ , x_0)$ is usc at $x_0 \in X$ then f is cf-lsc at x_0.*

PROOF. Let $(x_n)_{n\geq 1}$ be a sequence converging to x_0. Since $f(\cdot\ , x_0)$ is usc and f is diagonal null, there exists a sequence $(g_n)_{n\geq 1}$ converging to 0 in Y such that $f(x_n, x_0) \leq f(x_0, x_0) + g_n = g_n$. This implies that f is cf-lsc at x_0. □

Remark 9. *Let us remark that the coordinate free lower semi-continuity of the bifunction f from $X \times X$ to Y is a strictly weaker assumption than the lower (resp. upper) semi-continuity of the functions $f(x, \cdot)$ (resp. $f(\cdot, y)$) for all $x \in X$ (resp. for all $y \in X$). This is very easy to show in the vector-valued case. Indeed, by Proposition 3, it suffices to consider f defined by $f(x,y) = h(y) - h(x)$ where h is o-lsc but not q-lsc at some point.*

It will be useful for our purpose to know that coordinate free lower semi-continuity is preserved under continuous perturbations.

Proposition 10. *Let f and g be two bifunctions from $X \times X$ to Y. If f is cf-lsc and g is continuous and diagonal null then $f + g$ is cf-lsc.*

PROOF. Let $x_0 \in X$, $(x_n)_{n\geq 1}$ be a sequence in X converging to x_0 and $(\rho_n)_{n\geq 1}$ be a sequence in $\mathbb{R}_0^+$ converging to 0 such that $\forall n \geq 1,\ \forall l \geq 1,\ \exists \varepsilon_{n,l} \in B_Y(0, \rho_n)$: $(f+g)(x_n, x_{n+l}) \leq \varepsilon_{n,l}$. Since g is continuous and diagonal null, there exists $(\delta_n)_{n\geq 1}$ a sequence in $\mathbb{R}_0^+$ converging to 0 such that $\forall n \geq 1,\ \forall l \geq 1,\ \exists g_{n,l} \in B_Y(0, \delta_n)$: $g(x_n, x_{n+l}) = g(x_0, x_0) + g_{n,l} = g_{n,l}$. So, $f(x_n, x_{n+l}) \leq \varepsilon_{n,l} - g_{n,l}$ and, since f is cf-lsc, there exists a sequence (v_n) converging to 0 in Y such that $f(x_n, x_0) \leq v_n$. Now, since $g(x_n, x_0) = u_n$ with $\|u_n\|_Y \to 0$, we have $(f+g)(x_n, x_0) \leq v_n + u_n$ which proves that $f + g$ is cf-lsc at x_0. □

4 Approximatively equilibrium points

The following notion of ε-*equilibrium point* has been recently introduced in [4] and [5] : the point $x_0 \in X$ is said to be an ε-vector equilibrium point of f in the direction of $e \in K \setminus \{0\}$ if

$$\forall y \in X,\ y \neq x_0,\ \ f(x_0, y) + \varepsilon \|x_0 - y\| e \notin -K.$$

Bianchi, Kassay and Pini [4, 5] prove the existence of such points under the following assumptions on f : diagonal nullness, lower boundedness of $e^*(f(x, \cdot))$ for all $x \in X$ and *lower transitivity*.

Definition 11. *A bifunction $f : X \times X \to Y$ is said to be* ***lower transitive*** *iff for every $x, y, z \in X$, $f(z,x) \le f(z,y) + f(y,x)$.*

Using this, they get existence results for the problem (WVEP) (see Section 5.2).

Let us introduce here the notion of *approximatively equilibrium point in the direction of an element of $K \setminus \{0\}$*.

Definition 12. *Let f be a bifunction from $X \times X$ to Y. A point $x_0 \in X$ is called an* ***ε-approximatively equilibrium point of f in the direction of*** *$e \in K \setminus \{0\}$ iff*

$$\exists \rho \in \mathbb{R}_0^+,\ \forall x \in X,\ \forall \xi \in B_Y(0,\rho):\ \ f(x_0,x) + \varepsilon e + \xi \notin -K. \tag{1}$$

The following Proposition will be the key tool for applications (main Theorems 14 and 19). It gives *existence* and *localization* of approximatively equilibrium points for a diagonal null and lower transitive bifunction f such that $f(x,\cdot)$ is bounded below for all $x \in X$. The proof of this Proposition follows the ideas of [12].

Proposition 13 (Existence and localization of approximatively equilibrium points). *Let f be a bifunction from $X \times X$ to Y. If f is diagonal null, lower transitive and $f(x,\cdot)$ is bounded below for all $x \in X$, then for every $\varepsilon \in \mathbb{R}_0^+$ and every $e \in K \setminus \{0\}$, there exists $x_0 \in X$ an ε-approximatively equilibrium point of f in the direction of e. Moreover, given $\tilde{x} \in X$ and $\delta \in \mathbb{R}_0^+$, one can suppose that : $f(\tilde{x}, x_0) \in B_Y(0,\delta) - K$.*

PROOF. Let $\varepsilon \in \mathbb{R}_0^+$, $e \in K \setminus \{0\}$, $x_0 := \tilde{x}$ and $\delta \in \mathbb{R}_0^+$ be fixed. If x_0 satisfies (1) then the proof is finished since $f(x_0,x_0) = 0$. If x_0 does not satisfy (1), let $\rho_1 := \frac{\delta}{2}$ and get the existence of $x_1 \in X$ and $\xi_1 \in B_Y(0,\rho_1)$ such that $f(x_0,x_1) + \varepsilon e + \xi_1 \in -K$. If x_1 satisfies (1), the proof is finished. If not, we repeat the same construction. Let us define $\rho_n := \frac{\delta}{2^n}$ for all $n \in \mathbb{N}_0$. Using the hypothesis of lower transitivity of the bifunction f, at the step $n \ge 1$, we have $x_n \in X$ such that

$$f(x_0,x_n) + n\varepsilon e + \sum_{i=1}^{n} \xi_i \in -K, \quad \text{with } \ \xi_i \in B_Y(0,\rho_i) \ \text{ for } i \in \{1,\cdots,n\}.$$

Let us suppose that, for all $n \in \mathbb{N}_0$, x_n does not satisfy (1). By hypothesis, there exists $b_0 \in Y$ such that $b_0 \le f(x_0,x)$ for all $x \in X$, and then:

$$\forall n \in \mathbb{N}_0,\ \ b_0 + n\varepsilon e + \varphi_n \in -K \ \text{ with } \ \varphi_n := \sum_{i=1}^{n} \xi_i \in B_Y(0,\delta).$$

Thus, $e \in \bigcap_{n \in \mathbb{N}_0} \big(B_Y(0,\frac{r}{n}) - K\big)$, where $r := \frac{\delta + \|b_0\|}{\varepsilon}$. Since K is closed, this intersection is equal to $-K$ and, since K is pointed, this implies that $e = 0$, a contradiction. □

5 Applications

5.1 Perturbed equilibrium principles

For our first application, let us come back to the vector equilibrium problem (VEP). Using Proposition 13 and the ideas of [12], we prove the following main result :

Theorem 14 (Deville-Godefroy-Zizler perturbed equilibrium principle). *Let $(Z, \|\cdot\|_Z)$ be a Banach space of norm bounded, bounded below, continuous bifunctions from $X \times X$ to Y such that:*

(i) for all $g \in Z$, $\|g\|_Z \geq \|g\|_\infty := \sup_{x,y \in X} \|g(x,y)\|_Y$,

(ii) Z is translation invariant, i.e. if $g \in Z$ and $x, y \in X$ then $\tau_t g : X \times X \to Y$ given by $\tau_t g(t) := g(x-t, y-t)$ is in Z and $\|\tau_t g\|_Z = \|g\|_Z$,

(iii) Z is dilation invariant, i.e. if $g \in Z$ and $\alpha \in \mathbb{R}$ then $g^\alpha : X \times X \to Y$ given by $g^\alpha(x,y) := g(\alpha x, \alpha y)$ is in Z,

(iv) there exists a continuous and norm bounded bump function $b : X \to \mathbb{R}$ and an element $e \in K \setminus \{0\}$ such that $b(0) > 0$ and $\hat{b} : X \times X \to Y$ given by $\hat{b}(x,y) := (b(y) - b(x))e$ belongs to Z.

Let $f : X \times X \to Y$ be a cf-lsc , diagonal null and lower transitive bifunction such that $f(x, \cdot)$ is bounded below for all $x \in X$. Then the set of all $g \in Z$ such that $f + g$ admits an equilibrium point is dense in Z.

PROOF. Let $\varepsilon \in \mathbb{R}_0^+$ be fixed. We want to prove the following :

$$\exists g \in Z,\ \exists \overline{x} \in X : \ \ \|g\|_Z \leq \varepsilon, \quad \forall y \in X \setminus \{\overline{x}\} : \ (f+g)(\overline{x}, y) \notin -K.$$

Without loss of generality, we can suppose that $b(0) = 1$ and $\|e\|_Y = 1$. Moreover, there exists some $r > 0$ such that $b(x) = 0$ whenever $\|x\| \geq r$.

(i) Let us define $b_1 : X \to \mathbb{R}$ by $b_1(x) := b(2rx)$, whose support is now in $B_X(0, \frac{1}{2})$. By Proposition 13, there exist $x_1 \in X$ an $\varepsilon_1 := \big(\frac{\varepsilon}{4\|b_1 e\|_Z}\big)$-approximatively equilibrium point of f in the direction of e, that is:

(2) $$\exists \rho_1 \in \mathbb{R}_0^+,\ \forall y \in X,\ \forall \xi \in B_Y(0, \rho_1) : \ \ f(x_1, y) + \varepsilon_1 e + \xi \ \notin -K.$$

Let us define $g_1 : X \times X \to Y$ by $g_1(x,y) := -\big(b_1(y - x_1) - b_1(x - x_1)\big)\varepsilon_1 e$. We have $g_1 \in Z$, $\|g_1\|_Z \leq \frac{\varepsilon}{2}$ and $g_1(x_1, y) = \big(1 - b_1(y - x_1)\big)\varepsilon_1 e$ for all $y \in X$. If we set

$$A_1 := \{y \in X : \ \exists \xi \in B_Y(0, \rho_1), \ \ (f + g_1)(x_1, y) + \xi \in -K\} \ ,$$

then, since $f + g_1$ is diagonal null, $x_1 \in A_1$ and, by (2), since $g_1(x_1, y) = \varepsilon_1 e$ if $y \notin B_X(x_1, \frac{1}{2})$, $A_1 \subset B_X(x_1, \frac{1}{2})$. If $A_1 = \{x_1\}$ then x_1 is an equilibrium point of $f + g_1$. Let us suppose that $A_1 \neq \{x_1\}$.

(ii) Let us define $b_2 : X \to \mathbb{R}$ by $b_2(x) := b_1(2x)$, whose support lies in $B_X(0, \frac{1}{2^2})$. Since $f + g_1$ satisfies the hypothesis of Proposition 13, we get (for $\tilde{x} := x_1$ and $\delta := \frac{\rho_1}{4}$) the existence of $x_2 \in X$ an $\varepsilon_2 := \big(\min\{\frac{\varepsilon}{2^2}; \frac{\rho_1}{4}\}/2\|b_1 e\|_Z\big)$-approximatively equilibrium point of $f + g_1$ in the direction of e, that is:

(3) $\exists \rho_2 \in \mathbb{R}_0^+,\ \forall y \in X,\ \forall \xi \in B_Y(0, \rho_2) : \ (f + g_1)(x_2, y) + \varepsilon_2 e + \xi \ \notin -K,$

located as follows:

(4) $$\exists v_2 \in B_Y(0, \frac{\rho_1}{4}) : \ (f + g_1)(x_1, x_2) + v_2 \in -K.$$

Without loss of generality, we can assume that $\rho_2 \leq \frac{\rho_1}{4}$. Let us define $g_2 : X \times X \to Y$ by $g_2(x, y) := -\big(b_2(y - x_2) - b_2(x - x_2)\big)\varepsilon_2 e$. We have $g_2 \in Z$, $\|g_2\|_Z \leq \min\{\frac{\varepsilon}{2^2}; \frac{\rho_1}{4}\}$ and $g_2(x_2, y) = \big(1 - b_2(y - x_2)\big)\varepsilon_2 e$ for all $y \in X$. If we set

$$A_2 := \{y \in X : \ \exists \xi \in B_Y(0, \rho_2), \ (f + g_1 + g_2)(x_2, y) + \xi \in -K\},$$

then, since $f + g_1 + g_2$ is diagonal null, $x_2 \in A_2$ and, by (3), since $g_2(x_2, y) = \varepsilon_2 e$ if $y \notin B_X(x_2, \frac{1}{2^2})$, $A_2 \subset B_X(x_2, \frac{1}{2^2})$. If $A_2 = \{x_2\}$ then x_2 is an equilibrium point of $f + g_1 + g_2$. Let us suppose that $A_2 \neq \{x_2\}$.

(iii) Let us suppose we have carried out the construction until step $n - 1$ and let us perform the step n. Let us write $\overline{g}_{n-1} := \sum_{k=1}^{n-1} g_k$ and $b_n : X \to \mathbb{R}$ the bump function defined by $b_n(x) := b_1(2^n x)$, so that $\operatorname{supp} b_n \subset B_X(0, \frac{1}{2^n})$. Since $f + \overline{g}_{n-1}$ satisfies the hypothesis of Proposition 13, we get (for $\tilde{x} := x_{n-1}$ and $\delta := \frac{\rho_{n-1}}{4}$) the existence of $x_n \in X$ an $\varepsilon_n := \big(\min\{\frac{\varepsilon}{2^n}; \frac{\rho_{n-1}}{4}\}/2\|b_n e\|_Z\big)$-approximatively equilibrium point of $f + \overline{g}_{n-1}$ in the direction of e, that is:

(5) $\exists \rho_n \in \mathbb{R}_0^+,\ \forall y \in X,\ \forall \xi \in B_Y(0, \rho_n) : \ (f + \overline{g}_{n-1})(x_n, y) + \varepsilon_n e + \xi \ \notin -K,$

located as follows:

(6) $$\exists v_n \in B_Y(0, \frac{\rho_{n-1}}{4}) : \ (f + \overline{g}_{n-1})(x_{n-1}, x_n) + v_n \in -K.$$

Without loss of generality, we can assume that $\rho_n \leq \frac{\rho_{n-1}}{4}$. Let us define $g_n : X \times X \to Y$ by $g_n(x, y) := -\big(b_n(y - x_n) - b_n(x - x_n)\big)\varepsilon_n e$. We have $g_n \in Z$, $\|g_n\|_Z \leq \min\{\frac{\varepsilon}{2^n}; \frac{\rho_{n-1}}{4}\}$ and $g_n(x_n, y) = \big(1 - b_n(y - x_n)\big)\varepsilon_n e$ for all $y \in X$. If we set

$$A_n := \{y \in X : \ \exists \xi \in B_Y(0, \rho_n), \ (f + \overline{g}_n)(x_n, y) + \xi \in -K\},$$

then, since $f + \overline{g}_n$ is diagonal null, $x_n \in A_n$ and, by (5), since $g_n(x_n, y) = \varepsilon_n e$ if $y \notin B_X(x_n, \frac{1}{2^n})$, $A_n \subset B_X(x_n, \frac{1}{2^n})$. If $A_n = \{x_n\}$ then x_n is an equilibrium point of $f + \overline{g}_n$.

(iv) Let us suppose that, for all n, $A_n \neq \{x_n\}$. Since $\|v_n\|_Y < \rho_n$, we have, by (6), that $x_{n+1} \in A_n \subset B_X(x_n, \frac{1}{2^n})$ and then (x_n) is a Cauchy sequence in X. So, there exists some $\overline{x} \in X$ such that $x_n \xrightarrow[\|\cdot\|_X]{} \overline{x}$. Also, $\|g_n\|_Z \leq \frac{\varepsilon}{2^n}$ for all n implies that there exists $g \in Z$ such that $\overline{g}_n \xrightarrow[\|\cdot\|_Z]{} g$. So, $g = \overline{g}_n + h_n$ with $\|g\|_Z \leq \varepsilon$ and $h_n := \sum_{i>n} g_i$. Let us also remark that:

$$(7) \qquad \forall k,\ \forall x, y, z \in X,\ \ g_k(x,z) = g_k(x,y) + g_k(y,z),$$

and then

$$(8) \qquad \forall n,\ \ h_n(x,y) + h_{n+1}(y,z) = g_{n+1}(x,y) + h_{n+1}(x,z).$$

(v) We want to show that $\overline{x}$ is an equilibrium point of $f+g$. In view of (6), we have:

$$(9) \qquad \forall n,\ \ (f+g)(x_n, x_{n+1}) \leq -v_{n+1} + h_n(x_n, x_{n+1}),$$

with $\|-v_{n+1} + h_n(x_n, x_{n+1})\|_Y \xrightarrow[n\to+\infty]{} 0$. Moreover, using the fact that $f+g$ is lower transitive and by (9) and (8), we have for $l > 1$:

$$\begin{aligned}
(f+g)(x_n, x_{n+l}) &\leq (f+g)(x_n, x_{n+1}) + (f+g)(x_{n+1}, x_{n+l}) \\
&\leq -v_{n+1} + h_n(x_n, x_{n+1}) + (f+g)(x_{n+1}, x_{n+2}) + (f+g)(x_{n+2}, x_{n+l}) \\
&\leq -\sum_{i=1}^{2} v_{n+i} + h_n(x_n, x_{n+1}) + h_{n+1}(x_{n+1}, x_{n+2}) + (f+g)(x_{n+2}, x_{n+l}) \\
&\leq -\sum_{i=1}^{2} v_{n+i} + g_{n+1}(x_n, x_{n+1}) + h_{n+1}(x_n, x_{n+2}) + (f+g)(x_{n+2}, x_{n+l}) \\
&\leq -\sum_{i=1}^{3} v_{n+i} + g_{n+1}(x_n, x_{n+1}) + h_{n+1}(x_n, x_{n+2}) + h_{n+2}(x_{n+2}, x_{n+3}) \\
&\quad + (f+g)(x_{n+3}, x_{n+l}) \\
&\leq -\sum_{i=1}^{3} v_{n+i} + \sum_{i=1}^{2} g_{n+i}(x_n, x_{n+i}) + h_{n+2}(x_n, x_{n+3}) + (f+g)(x_{n+3}, x_{n+l}) \\
&\leq \cdots
\end{aligned}$$

Since $f+g$ is diagonal null, $(f+g)(x_{n+l}, x_{n+l}) = 0$ and then :

$$(f+g)(x_n, x_{n+l}) \leq \underbrace{-\sum_{i=n+1}^{n+l} v_i + \sum_{i=n+1}^{n+l-1} g_i(x_n, x_i) + h_{n+l-1}(x_n, x_{n+l})}_{:=\varepsilon_{n,l}}.$$

We have $\|\varepsilon_{n,l}\|_Y \leq \rho_n$. This follows from the following estimates (already obtained in [12]) :

- $\left\|\sum_{i=n+1}^{n+l} v_i\right\|_Y \leq \frac{\rho_n}{3}$;
- $\| \sum_{i=n+1}^{n+l-1} g_i(x_n, x_i)\|_Y \leq \frac{\rho_n}{3}$;
- $\|h_{n+l-1}(x_n, x_{n+l})\|_Y \leq \|h_{n+l-1}\|_\infty \leq \|h_{n+l-1}\|_Z \leq \sum_{i=n+l}^{+\infty} \|g_i\|_Z < \frac{\rho_n}{3}$.

Since f is cf-lsc and g is continuous and diagonal null, by Proposition 10, $f+g$ is cf-lsc and then:

$$\forall n, \ \ (f+g)(x_n, \overline{x}) \leq w_n \quad \text{with} \ \ \|w_n\|_Y \to 0. \tag{10}$$

Let now $\overline{y} \in X$ be such that $(f+g)(\overline{x}, \overline{y}) \in -K$. We have to prove that $\overline{y} = \overline{x}$. Since $f+g$ is lower transitive, we deduce from (10) :

$$\forall n, \ \ (f+g)(x_n, \overline{y}) \leq w_n. \tag{11}$$

Let $n_0 \in \mathbb{N}_0$ be fixed. Proceeding as before, we have for all $n \geq n_0$:

$$(f+g)(x_{n_0}, \overline{y}) \leq -\sum_{i=n_0+1}^{n+1} v_i + \sum_{i=n_0+1}^{n} g_i(x_{n_0}, x_i) + h_n(x_{n_0}, x_{n+1}) + (f+g)(x_{n+1}, \overline{y}),$$

and using (11) :

$$(f+g)(x_{n_0}, \overline{y}) \leq -\sum_{i=n_0+1}^{n+1} v_i + \sum_{i=n_0+1}^{n} g_i(x_{n_0}, x_i) + h_n(x_{n_0}, x_{n+1}) + w_{n+1}.$$

Then, for all $n \geq n_0$:

$$\begin{aligned}(f+\overline{g}_{n_0})(x_{n_0}, \overline{y}) \leq & -\sum_{i=n_0+1}^{n+1} v_i + \sum_{i=n_0+1}^{n} g_i(x_{n_0}, x_i) + h_n(x_{n_0}, x_{n+1}) + w_{n+1} \\ & - h_{n_0}(x_{n_0}, \overline{y}).\end{aligned}$$

Since we have (using (7)) :

$$\begin{aligned}\sum_{i=n_0+1}^{n} g_i(x_{n_0}, x_i) - h_{n_0}(x_{n_0}, \overline{y}) &= \sum_{i=n_0+1}^{n} g_i(x_{n_0}, x_i) - \sum_{i=n_0+1}^{+\infty} g_i(x_{n_0}, \overline{y}) \\ &= \sum_{i=n_0+1}^{n} g_i(\overline{y}, x_i) - \sum_{i=n+1}^{+\infty} g_i(x_{n_0}, \overline{y}) \\ &= \sum_{i=n_0+1}^{n} g_i(\overline{y}, x_i) - h_n(x_{n_0}, \overline{y}),\end{aligned}$$

previous inequality implies that for all $n \geq n_0$:

$$\begin{aligned}(f+\overline{g}_{n_0})(x_{n_0},\overline{y}) &\leq -\sum_{i=n_0+1}^{n+1} v_i + \sum_{i=n_0+1}^{n} g_i(\overline{y},x_i) - h_n(x_{n_0},\overline{y}) + h_n(x_{n_0},x_{n_0+1}) + w_{n+1} \\ &\leq -\sum_{i=n_0+1}^{n+1} v_i + \sum_{i=n_0+1}^{n} g_i(\overline{y},x_i) - h_n(\overline{y},x_{n+1}) + w_{n+1}.\end{aligned}$$

Since $\|h_n(\overline{y},x_{n+1}) + w_{n+1}\|_Y \leq \|h_n\|_Z + \|w_{n+1}\|_Y < \frac{\rho_{n_0}}{3}$ for n large enough, we have

$$\left\| -\sum_{i=n_0+1}^{n+1} v_i + \sum_{i=n_0+1}^{n} g_i(\overline{y},x_i) - h_n(\overline{y},x_{n+1}) + w_{n+1} \right\|_Y < \rho_{n_0}. \tag{12}$$

for n is large enough. Therefore, $\overline{y} \in A_{n_0} \subset B_X(x_{n_0}, \frac{1}{2^{n_0}})$. Since n_0 is arbitrary, $\overline{y} = \overline{x}$. □

Remark 15. *1. It is possible to* ***localize*** *the equilibrium points of the bifunction $f+g$ in Theorem 14. More precisely, under the assumption of Theorem 14, we have: for all $\varepsilon \in \mathbb{R}_0^+$, there exists $\varepsilon_1 \in \mathbb{R}_0^+$ such that if x_1 is an ε_1–approximatively equilibrium point of f in the direction of e, then there exist $g \in Z$ and $\overline{x} \in X$ such that $\|g\|_Z \leq \varepsilon$, $\|\overline{x} - x_1\| \leq \varepsilon$ and $\overline{x}$ is an equilibrium point of $f+g$.*
In order to prove this, it suffices, at the beginning of the proof, to define the first bump function b_1 as $b_1(x) := b(\frac{2r}{\varepsilon}x)$.

2. *Since $(f+g)(\overline{x},\overline{x}) = 0$, we have proved that $\overline{x}$ is an efficient solution of the function $(f+g)(\overline{x},\cdot) : X \to Y$. Using the ideas of [12], we can prove moreover that it is* ***strong*** *efficient:*

$$[\,(f+g)(\overline{x},u_m) \leq \omega_m \ \text{with}\ \omega_m \to 0\,] \quad \Rightarrow \quad [\,u_m \to \overline{x}\,]\,.$$

3. *In the proof, at every step, the function $g_k : X \times X \to Y$ is of the form $g_k(x,y) := \big(\tilde{g}_k(y) - \tilde{g}_k(x)\big)\varepsilon_k e$, where $\tilde{g}_k : X \to \mathbb{R}$ is a continuous bump function. So, if f is defined by $f(x,y) := h(y) - h(x)$, with $h : X \to Y$ o-lsc and bounded below, then we prove the existence of a continuous perturbation $\tilde{g}$, as small as desired, such that $h + \tilde{g}$ admits a (strong) efficient solution on X :*

$$\exists \overline{x} \in X, \ \{\overline{x}\} = \{x \in X : \ h + \tilde{g}(x) \leq h + \tilde{g}(\overline{x})\}\,.$$

In other words, we recover the Deville-Godefroy-Zizler variational principle obtained in [12].

Let us present now some examples of Banach spaces $(Z, \|\cdot\|_Z)$ of norm bounded, bounded below, continuous bifunctions from $X \times X$ to Y satisfying the conditions (i)-(iv) of Theorem 14. An easy way to construct such a Banach space of perturbations is to proceed as follows : let us consider $(\tilde{Z}, \|\cdot\|_{\tilde{Z}})$ a Banach space of *real-valued*, continuous and bounded functions defined on X which satisfies the corresponding conditions (i)-(iv) of the original Deville-Godefroy-Zizler variational principle (see Section 1). It suffices then to take Z as the space of all bifunctions $g : X \times X \to Y$ defined by $g(x, y) := (\tilde{g}(y) - \tilde{g}(x))e$, where $\tilde{g} \in \tilde{Z}$, endowed with the following norm :

$$\|g\|_Z := \|\tilde{g}\|_{\tilde{Z}} + \|g\|_\infty \ .$$

Examples of such Banach spaces $\tilde{Z}$ are given in [10] and [17].
The following example will give, as a corollary of Theorem 14, a variant of Ekeland principle for equilibrium problems. Let $\tilde{Z} = L$ be the space of all bounded real-valued Lipschitz continuous functions $\tilde{g}$ on X with $\|\tilde{g}\|_L := \|\tilde{g}\|_\infty + \|\tilde{g}\|_{\mathrm{Lip}}$ where

$$\|\tilde{g}\|_{\mathrm{Lip}} := \sup\Big\{\frac{|\tilde{g}(x) - \tilde{g}(y)|}{\|x - y\|} : x, y \in X,\ x \neq y\Big\}.$$

It is straightforward to prove that L is a Banach space which satisfies hypotheses (i)–(iii). Concerning hypothesis (iv), one can apply the construction exposed in [10, 17] to produce a bounded Lipschitzian bump function.

Corollary 16 (Ekeland equilibrium principle). *Let $f : X \times X \to Y$ satisfying the following assumptions :*

(1) f is diagonal null;

(2) $f(x, .)$ is bounded below for all $x \in X$;

(3) f is lower transitive;

(4) f is cf-lsc.

Then, for every $\varepsilon \in \mathbb{R}_0^+$ and $e \in K \setminus \{0\}$, there exists $\varepsilon_1 \in \mathbb{R}_0^+$ such that if x_1 is an ε_1–approximatively equilibrium point of f in the direction of e, then there exists $\overline{x} \in X$ such that :

(a) $\forall y \in X, \quad f(\overline{x}, y) + \varepsilon e \notin -K$,

(b) $\|\overline{x} - x_1\| \leq \varepsilon$,

(c) $\overline{x}$ is an equilibrium point of $f + \varepsilon\|\cdot - \cdot\|e$, i. e . :

$$\forall y \in X \setminus \{\overline{x}\}, \quad f(\overline{x}, y) + \varepsilon\|\overline{x} - y\|e \notin -K \ .$$

PROOF. Let $\varepsilon \in \mathbb{R}_0^+$ and $e \in K \setminus \{0\}$ be fixed. We can suppose that $\|e\|_Y = 1$. By Theorem 14 and Remark 15 (1), there exists $\tilde{g} \in L$ and $\overline{x} \in X$ such that $\sup\{|\tilde{g}(x)-\tilde{g}(y)| : x, y \in X,\ x \neq y\} \leq \varepsilon$, $\sup\left\{\frac{|\tilde{g}(x)-\tilde{g}(y)|}{\|x-y\|} : x, y \in X,\ x \neq y\right\} \leq \varepsilon$, $\|\overline{x} - x_1\| \leq \varepsilon$ and $\overline{x}$ is an equilibrium point of $f+g$, where $g : X \times X \to Y$ is defined by $g(x,y) := (\tilde{g}(y)-\tilde{g}(x))e$:

$$\forall y \in X \setminus \{\overline{x}\}, \quad f(\overline{x}, y) + (\tilde{g}(y) - \tilde{g}(\overline{x}))e \notin -K \ .$$

Since, for all $y \in X$, $\tilde{g}(y) - \tilde{g}(\overline{x}) \leq \varepsilon$ and $\tilde{g}(y) - \tilde{g}(\overline{x}) \leq \varepsilon\|\overline{x} - y\|$, by transitivity of the order we thus have :

$$\forall y \in X \setminus \{\overline{x}\}, \quad f(\overline{x}, y) + \varepsilon e \notin -K \ ,$$

and

$$\forall y \in X \setminus \{\overline{x}\}, \quad f(\overline{x}, y) + \varepsilon\|\overline{x} - y\| e \notin -K \ .$$

□

Remark 17. *Let us recall the vector Ekeland principle obtained in [5] :*

Theorem ([5]). *Let (X, d) be a complete metric space. Assume that the function $f : X \times X \to Y$ satisfies the following assumptions:*

(1') f is diagonal null;

(2') $e^*(f(x,\cdot))$ is bounded below for all $x \in X$ (where e^* is in the dual cone of K and $e^*(e) = 1$ for a fixed point $e \in K \setminus \{0\}$);

(3') f is lower transitive;

(4') $f(x,\cdot)$ is q-lsc for all $x \in X$.

Let $\varepsilon \in \mathbb{R}_0^+$ and $\lambda \in \mathbb{R}_0^+$ be given and let $x_0 \in X$ be such that :

$$\text{(a')} \quad \forall y \in X,\ \ f(x_0, y) + \varepsilon e \notin -K.$$

Then, there exists $\overline{x} \in X$ such that :

(b') $d(\overline{x}, x_0) \leq \lambda$,

(c') $\forall y \in X \setminus \{\overline{x}\},\ \ f(\overline{x}, y) + \frac{\varepsilon}{\lambda} d(\overline{x}, y)e \notin -K$,

(d') $f(\overline{x}, x_0) \in K$.

In that paper [5], the authors mentioned that if f is lower transitive and if there exists $\hat{y} \in X$ such that the function $e^*(f(\cdot, \hat{y}))$ is upper bounded, then there exists $x_0 \in X$ satisfying (a'). Let us compare their Theorem with Corollary 16 :

- Condition (2') is weaker than (2), see [5].
- If $\text{int}\, K \neq \varnothing$ (resp. if $\dim Y$ is finite and f is bounded below) then, by Proposition 5 (resp. Proposition 6), assumption (4) is weaker than (4').
- Of course, an ε-approximatively equilibrium point x_1 of f in the direction of e satisfies (a'). And Proposition 13 shows the existence of such an ε-approximatively equilibrium point under assumptions (2) and (3).
- In Corollary 16, starting with an ε-approximatively equilibrium point x_1, we find a point satisfying (a) (see (a')) and we get (b) (similar to (b') with x_1 instead of x_0) and (c) (similar to (c')). We don't get $f(\overline{x}, x_1) \in K$.

Let us recall that a bornology on X, denoted by β, is any family of bounded sets whose union is all X, which is closed under reflection through the origin (that is $S \in \beta$ implies $-S \in \beta$), under multiplication by positive scalars and is directed upwards (that is the union of any two members of β is contained in some member of β). There are many possibilities. Let us describe the smallest and the largest ones : the *Gâteaux* bornology $\beta = G$ consisting of all finite symmetric sets and the *Fréchet* bornology $\beta = F$ consisting of all bounded symmetric sets. A function $f : X \to Y$ is said to be *β-differentiable* at x and $T \in \mathcal{L}(X, Y)$ is called its *β-derivative* at x, if for each $S \in \beta$,

$$\lim_{t \xrightarrow{>} 0} \frac{f(x+ty) - f(x)}{t} = T(x) \quad \text{uniformly for } y \in S.$$

We denote the β-derivative of f at x by $\partial_\beta f(x)$. It is clear that we find again the well-known Gâteaux (resp. Fréchet) derivative with $\beta = G$ (resp. $\beta = F$). We can take for $\tilde{Z}$ the Banach space D_β of all real-valued functions defined on X that are bounded, Lipschitz continuous and β-differentiable equipped with the norm

$$\|\tilde{g}\|_{D_\beta} := \|\tilde{g}\|_\infty + \|\partial_\beta \tilde{g}\|_\infty$$

(cfr. [17] for a proof that this space is complete and verifies hypotheses (i)–(iv) of the scalar version of the Deville-Godefroy-Zizler variational principle.) This gives the following variant of Borwein-Preiss smooth principle for equilibrium problem :

Corollary 18 (Borwein-Preiss smooth equilibrium principle). *Let X be a Banach space that admits a Lipschitz continuous bump function which is β-differentiable. Then, for every cf-lsc, diagonal null and lower transitive bifunction $f : X \times X \to Y$ such that $f(x, \cdot)$ is bounded below for all $x \in X$, and for every $\varepsilon \in \mathbb{R}_0^+$, there exists a bifunction $g : X \times X \to Y$ which is Lipschitz continuous and β-differentiable such that $\|g\|_\infty \leq \varepsilon$, $\|\partial_\beta g\|_\infty \leq \varepsilon$ and $f + g$ admits an equilibrium point.*

5.2 Existence results for vector equilibria

Here we are interested in the existence of exact solutions of vector equilibrium problems. First, in the compact case, using Proposition 13 (which is clearly true for bifunctions defined on $C \times C$ where C is a subset of X), and then using the existence of approximate solutions, we prove the same existence results for the problem (WVEP) than in [5] but under weaker assumptions.

Theorem 19. *Let C be a compact subset of X. If the bifunction $f : C \times C \to Y$ satisfies :*

(i) f is diagonal null;

(ii) $f(x, \cdot)$ is bounded below for all $x \in C$;

(iii) f is lower transitive;

(iv) f is cf-lsc;

then, the solution set for the problem (WVEP) for f on $C \times C$ is non-empty.

In [5, Theorem 3], assumption (iv) is replaced (in a stronger way, cfr. Section 3) by asking $f(x, \cdot)$ q-lsc for all x and $f(\cdot, y)$ usc for all y.

PROOF OF THEOREM 19. Let $e \in K \setminus \{0\}$. By Proposition 13, we construct a sequence $(x_n)_{n \geq 1} \subset C$ such that for all $n \geq 1$:

$$\exists \rho_n \in \mathbb{R}_0^+,\ \forall x \in C,\ \forall \xi \in B_Y(0, \rho_n) : \quad f(x_n, x) + \frac{1}{n}e + \xi \notin -K, \tag{13}$$

and such that for all $n \geq 2$:

$$f(x_{n-1}, x_n) \in -K + B_Y(0, \frac{1}{2^{n-1}}). \tag{14}$$

Since f is lower transitive we have $f(x_n, x_{n+l}) \leq \sum_{i=n}^{n+l-1} f(x_i, x_{i+1})$ for all $n \geq 2$ and all $l \geq 1$, and then $f(x_n, x_{n+l}) \in -K + B_Y(0, \frac{1}{2^{n-1}})$. Up to a subsequence, we can assume (since C is compact) that $x_n \to \overline{x} \in C$. Since f is cf-lsc, there exists a sequence $\omega_n \to 0$ such that :

$$f(x_n, \overline{x}) \in -K + B_Y(0, \omega_n). \tag{15}$$

By contradiction, let us suppose that :

$$\exists \overline{y} \in C, \quad f(\overline{x}, \overline{y}) \in -\operatorname{int} K. \tag{16}$$

By lower transitivity of f and (15), we have :

$$\begin{aligned} f(x_n, \overline{y}) &\leq f(x_n, \overline{x}) + f(\overline{x}, \overline{y}) \\ &\in -K + B_Y(f(\overline{x}, \overline{y}), \omega_n). \end{aligned}$$

On the other hand, by (16), for n big enough: $B_Y(f(\overline{x}, \overline{y}), \omega_n) + \frac{1}{n}e \subset -\operatorname{int} K$. So, $f(x_n, \overline{y}) + \frac{1}{n}e \in -\operatorname{int} K$ for n big enough. This contradicts (13) and completes the proof. □

Now, by following the proof of [5, Theorem 4] and using Theorem 19, we recover an existence result in the non-compact case for cf-lsc bifunctions.

Theorem 20. *Let X be a reflexive Banach space. If the bifunction $f : X \times X \to Y$ satisfies :*

(i) f is diagonal null;

(ii) $f(x, \cdot)$ is bounded below for all $x \in X$;

(iii) f is lower transitive;

(iv) f is cf-lsc;

(v) for all $x \in X$, the level set $L(x) := \{y \in X : f(x, y) \in -K\}$ is weakly closed;

(vi) (coercivity condition) there exists a compact set $C \subset X$ such that :

$$\exists x_0 \in X,\ \forall x \in X \setminus C,\ \exists y \in X,\ \|y - x_0\| < \|x - x_0\| : \ f(x, y) \in -K.$$

then, the solution set for the problem (WVEP) is non-empty.

In [6], Bianchi and Pini introduced coercivity conditions (as condition (vi) above) as weak as possible, exploiting the generalized monotonicity of the function f defining the scalar equilibrium problem. By using other coercivity conditions, we get finally existence results in the non-compact case, without any assumptions on the level sets $L(x)$ (hypothesis (v) in Theorem 20 is satisfied under some convexity assumptions on f, for example $f(x, \cdot)$ quasi-convex for all $x \in X$, see [3]). The coercivity condition in the following result is stronger than the previous one, but natural in our context. It allows to bound sequences of approximatively equilibrium points.

Theorem 21. *Let X be a reflexive Banach space. If the bifunction $f : X \times X \to Y$ satisfies :*

(i) f is diagonal null;

(ii) $f(x, \cdot)$ is bounded below for all $x \in X$;

(iii) f is lower transitive;

(iv) f is weakly cf-lsc;

(v) (coercivity condition) for each sequence $(x_n)_{n \geq 1}$ in X such that $\|x_n\| \to +\infty$, there exist $\delta \in \mathbb{R}_0^+$ and a subsequence $(x_{n_k})_{k \geq 1} \subset (x_n)_{n \geq 1}$ such that :

$$\forall k \geq 1,\ \exists l \geq 1 : \ f(x_{n_k}, x_{n_{k+l}}) \notin -K + B_Y(0, \delta).$$

then, the solution set for the problem (WVEP) is non-empty.

PROOF. As in the proof of Theorem 19, let us consider a sequence $(x_n)_{n\geq 1} \subset X$ satisfying relations (13) and (14). The coercivity condition implies that this sequence is bounded. Indeed, if not, there exist $\delta \in \mathbb{R}_0^+$ and a subsequence $(x_{n_k})_{k\geq 1} \subset (x_n)_{n\geq 1}$ such that : $\forall k \geq 1,\ \exists l \geq 1 :\quad f(x_{n_k}, x_{n_{k+l}}) \notin -K + B_Y(0, \delta)$. For $k \geq 1$ such that $\frac{1}{2^{n_k - 1}} < \delta$, there exists $l \geq 1$ such that $f(x_{n_k}, x_{n_{k+l}}) \notin -K + B_Y(0, \frac{1}{2^{n_k-1}})$ and this is a contradiction. So, up to a subsequence, we can assume that $(x_n)_{n\geq 1}$ weakly converges to $\overline{x} \in X$. Since f is weakly cf-lsc, relation (15) is satisfied and we then conclude as in the proof of Theorem 19. □

Remark 22. *Let us note that the following (strong) coercivity condition implies hypothesis (v) in Theorem 21 :*

$$\forall x \in X,\ \forall k \in K \setminus \{0\},\ \exists \alpha \in \mathbb{R}_0^+,\ \forall y \in X,\ \|y\| \geq \alpha :\ \ f(x, y) \geq k.$$

The same result can also be obtained if hypothesis (v) is replaced by the following one : there exist a bounded set $C \subset X$ and $k_0 \in K \setminus \{0\}$ such that :

$$\forall x \in X \setminus C,\ \exists y \in X :\ \ f(x, y) \in -k_0 - K.$$

If the subset C is assumed to be compact, we get the result by asking the bifunction f cf-lsc instead of weakly cf-lsc. Let us note that, if $f(x, \cdot)$ is q-lsc and convex for all $x \in X$, then $f(x, \cdot)$ is weakly q-lsc for all $x \in X$ and then f is weakly cf-lsc.

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References

[1] Q. H. Ansari, W. Oettli, D. Schläger, *A generalization of vectorial equilibria*, Math. Methods Oper. Res. **46** (1997), 147–152.

[2] Q. H. Ansari, S. Schaible, J. C. Yao, *Systems of vector equilibrium problems and its applications*, J. Optim. Theory Appl. **107** (2000), 547–557.

[3] M. Bianchi, N. Hadjisavvas, S. Schaible, *Vector equilibrium problems with generalized monotone bifunctions*, J. Optim. Theory Appl. **92** (1997), 527–542.

[4] M. Bianchi, G. Kassay, R. Pini, *Existence of equilibria via Ekeland's principle*, J. Math. Anal. Appl. **305** (2005), 502–512.

[5] M. Bianchi, G. Kassay, R. Pini, *Ekeland's principle for vector equilibrium problems*, to appear in Nonlinear Analysis.

[6] M. Bianchi, R. Pini, *Coercivity conditions for equilibrium problems*, J. Optim. Theory Appl. **124** (2005), 79–92.

[7] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63**, 1994, 123–145.

[8] J. Borwein, J. Penot and M. Théra, *Conjugate convex operators*, J. Math. Anal. Appl. **102**, 1984, 399–414.

[9] J. Borwein and D. Preiss, *Smooth variational principle with applications to subdifferentiability of convex functions*, Trans. Amer. Math. Soc. **303**, 1987, 517–527.

[10] R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman monographs and Surveys in Pure and Appl. Math. Longman Scientific & Technical, 1993.

[11] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. **47**, 1974, 324–353.

[12] C. Finet, L. Quarta, C. Troestler, *Vector-valued variational principles*, Nonlinear Anal. **52**(1), 2003, 197–218.

[13] F. Giannessi (ed.), *Vector variational inequalities and vector equilibria. Mathematical theories*, Nonconvex Optimization and Its Applications **38**, Kluwer Academic Publishers, 2000.

[14] X. H. Gong, *Efficiency and Heinig efficiency for vector equilibrium problems*, J. Optim. Theory Appl. **108**, 2001, 139–154.

[15] A. Göpfert, H. Riahi, C. Tammer, C. Zălinescu, *Variational methods in partially ordered spaces*, CMS Books in Mathematics **17**, Springer, 2003.

[16] N. Hadjisavvas, S. Schaible, *From scalar to vector equilibrium problems in the quasimonotone case*, J. Optim. Theory Appl. **96** (1998), 297–309.

[17] R. R. Phelps, *Convex functions, monotone operators and differentiability*, Lecture Notes in Mathematics **1364**, second edition, Springer-Verlag, Berlin, 1993.

[18] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1970.

[19] M. Théra, *Études des fonctions convexes vectorielles semi-continues*, Thèse de 3e cycle, Université de Pau, 1978.