# THE COHOMOLOGY OF SOME QUOTIENT NORM ONE TORI DEFINED OVER $\mathbb{Q}$

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Abstract. We compute the cohomology of certain quotient norm one algebraic tori defined over $\mathbb{Q}$.

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## Contents



## Introduction

This paper deals with the cohomology of some quotient norm one algebraic tori defined over $\mathbb{Q}$. They arise from a class of finite extensions of $\mathbb{Q}$ subject to certain local conditions at a given finite place and at the infinite one. These tori are anisotropic, and we compute their cohomology as well as their Tamagawa number.

The algebraic $\mathbb{Q}$-tori considered in this paper are introduced in section 1. Fix a prime number $p$ and let $\mathcal{F}^p_{\mathrm{CM}}(\mathbb{Q})$ be the class of finite extensions $F$ of $\mathbb{Q}$ satisfying the following local conditions at $p$ and $\infty$ : $F$ has complex multiplication and $\mathbb{Q}_p \otimes_{\mathbb{Q}} F$ is a field with complex multiplication too. Then complex conjugation induces an involution $\dagger$ on $F$. Now let $L/F$ be a nontrivial extension such that $L$ and $F$ are both in $\mathcal{F}^p_{\mathrm{CM}}(\mathbb{Q})$. They define the $\mathbb{Q}$-tori $T = \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$, $T^\dagger = \mathrm{Res}_{F^\dagger/\mathbb{Q}}(\mathbb{G}_m)$, $S = \mathrm{Res}_{L/\mathbb{Q}}(\mathbb{G}_m)$, and $S^\dagger = \mathrm{Res}_{L^\dagger/\mathbb{Q}}(\mathbb{G}_m)$. The norm maps $x \mapsto xx^\dagger$ on fields induce morphisms $T \to T^\dagger$ and $S \to S^\dagger$ on the associated tori. Letting $T_1$ and $S_1$ be their respective kernel, the quotient norm one $\mathbb{Q}$-torus we want to consider is $T_1/S_1$. Corollary 1.2 shows that $T_1/S_1$ is $\mathbb{Q}$-anisotropic.

In section 2 we compute the local cohomology of $T_1/S_1$ at $p$ and at $\infty$ (propositions 2.2 and 2.3). Section 3 is devoted to the computation of its global cohomology. It involves the kernels and cokernels of three fundamental morphisms arising from class field theory.

The first is the morphism $\iota : L^{\dagger\times}/N_\dagger(L^\times) \to F^{\dagger\times}/N_\dagger(F^\times)$ induced by inclusion. The two others are deduced from the commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(L) & \xrightarrow{\mathrm{Cor}_{L/L^\dagger}} & \mathrm{Br}(L^\dagger) \\ \downarrow{\scriptstyle \mathrm{Res}_{F/L}} & & \downarrow{\scriptstyle \mathrm{Res}_{F^\dagger/L^\dagger}} \\ \mathrm{Br}(F) & \xrightarrow{\mathrm{Cor}_{F/F^\dagger}} & \mathrm{Br}(F^\dagger) \end{array}$$

as the restrictions $\rho : \mathrm{Ker}\,\mathrm{Cor}_{L/L^\dagger} \to \mathrm{Ker}\,\mathrm{Cor}_{F/F^\dagger}$ and $\sigma : \mathrm{Coker}\,\mathrm{Cor}_{L/L^\dagger} \to \mathrm{Coker}\,\mathrm{Cor}_{F/F^\dagger}$ of $\mathrm{Res}_{F/L}$ and $\mathrm{Res}_{F^\dagger/L^\dagger}$ respectively. We show in proposition 3.8 that there are short exact sequences

$$1 \longrightarrow F_1/L_1 \longrightarrow H^0(\mathbb{Q}, T_1/S_1) \longrightarrow \mathrm{Ker}\,\iota \longrightarrow 1\,,$$

$$1 \longrightarrow \mathrm{Coker}\,\iota \longrightarrow H^1(\mathbb{Q}, T_1/S_1) \longrightarrow \mathrm{Ker}\,\rho \longrightarrow 1\,,$$

and that $H^2(\mathbb{Q}, T_1/S_1) \simeq \mathrm{Coker}\,\rho$. When $r \geq 2$ we find $H^{2r-1}(\mathbb{Q}, T_1/S_1) \simeq \mathrm{Coker}\,\sigma$ and $H^{2r}(\mathbb{Q}, T_1/S_1) = 1$.

Section 4 deals with local and global aspects. Let $C(T) = T(\mathbb{A})/T(\overline{\mathbb{Q}})$ be the adèle class group over $\overline{\mathbb{Q}}$ and $C_\mathbb{Q}(T) = T(\mathbb{A}_\mathbb{Q})/T(\mathbb{Q})$ be the one over $\mathbb{Q}$. Proposition 4.2 shows that $Ш^r(\mathbb{Q}, T_1/S_1)$ is trivial for all $r$. We have $H^0(\mathbb{Q}, C(T_1/S_1)) = C_\mathbb{Q}(T_1/S_1)$ and the same holds for $T_1$ and $S_1$ (lemma 4.3). Let $\iota_p : L_p^{\dagger\times}/N_\dagger(L_p^\times) \to F_p^{\dagger\times}/N_\dagger(F_p^\times)$ be the morphism induced by inclusion in the local setting. Theorem 4.4 shows that $C_\mathbb{Q}(T_1/S_1)$ has finite invariant volume $\#\,\mathrm{Ker}\,\iota_p$. Further theorem 4.5 shows that there are short exact sequences

$$1 \longrightarrow C_\mathbb{Q}(T_1)/C_\mathbb{Q}(S_1) \longrightarrow C_\mathbb{Q}(T_1/S_1) \longrightarrow \mathrm{Ker}\,\iota_p \longrightarrow 1\,,$$

$$1 \longrightarrow H^1(\mathbb{Q}, T_1/S_1) \longrightarrow H^1\big(\mathbb{Q}, T_1/S_1(\mathbb{A})\big) \longrightarrow \mathrm{Coker}\,\iota_p \longrightarrow 1\,,$$

and that $H^r(\mathbb{Q}, T_1/S_1) \simeq H^r(\mathbb{Q}, T_1/S_1(\mathbb{A}))$ for $r \geq 2$. Gathering these results together in corollary 4.6 we find that the Tamagawa number of $T_1/S_1$ is 1 when $[F : L]$ is odd and is 2 when $[F : L]$ is even, and that $H^1(\mathbb{Q}, T_1/S_1)$ has index 1 or 2 in $H^1(\mathbb{Q}, T_1/S_1(\mathbb{A}))$ accordingly.

## Notations

Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and for each prime number $\ell$ an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. For an extension $K$ of $\mathbb{Q}$ contained in $\overline{\mathbb{Q}}$ let $\mathbb{A}_K$ be its adèle ring, $I_K = \mathbb{A}_K^\times$ its idèle group, and $C_K = I_K/K^\times$ its idèle class group. When $K = \overline{\mathbb{Q}}$ we simply write $\mathbb{A}$, $I$, and $C$. When $K$ is a number field we let $S_K$ be the set of places of $K$, $S_K^f$ the subset of nonarchimedean ones, $S_K^\infty$ the archimedean ones. Write $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ and $G = G_\mathbb{Q}$ ; for $\ell \in S_\mathbb{Q}^f$ let $G_\ell = \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$.

For a topological abelian group $A$ let $A^\vee = \mathrm{Hom}_{\mathrm{cts}}(A, \mathbb{Q}/\mathbb{Z})$ be the group of continuous characters of finite order of $A$ and $A^\wedge$ the completion of $A$ with respect to the topology defined by the open subgroups of finite index. For a positive integer $n$ let $A^n$ be the direct sum of $n$ copies of $A$. For positive integers $n, m$ such that $n$ divides $m$ let

$$\mathrm{Diag} : A^n \hookrightarrow A^m = \big(A^{m/n}\big)^n$$

be the $n$-fold diagonal morphism $A \hookrightarrow A^{m/n}$, $a \mapsto (a, \dots, a)$. Finally write $\mathrm{KS}(A) \subseteq A \oplus A$ for the kernel of the sum $A \oplus A \to A$, $(a, b) \mapsto a + b$. Note that $\mathrm{KS}(A)$ is noncanonically isomorphic to $A$.

## 1. Tori

Fix a prime $p \in S^f_{\mathbb{Q}}$. We want to consider the class $\mathcal{F}^p_{\mathrm{CM}}(\mathbb{Q})$ of finite extensions $F$ of $\mathbb{Q}$ satisfying the following local conditions at $p$ and $\infty$ :

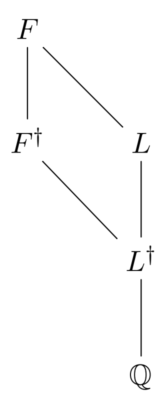
(i) $F$ is CM,
(ii) $F_p \underset{\mathrm{def}}{=} \mathbb{Q}_p \otimes_{\mathbb{Q}} F$ is a field,
(iii) $F_p$ is CM.

Let $F \in \mathcal{F}^p_{\mathrm{CM}}(\mathbb{Q})$ and let $\dagger$ be the involution on $F$ given by complex conjugation. Condition (iii) above means that the $\mathbb{Q}_p$-linear extension of $\dagger$ to $F_p$ is nontrivial. Write $\Gamma = \langle \dagger \rangle = \mathrm{Gal}(F/F^\dagger) \simeq \mathrm{Gal}(F_p/F_p^\dagger)$. Let

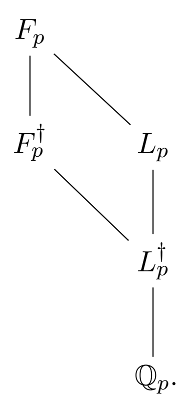
$$N_\dagger : F^\times \to F^{\dagger\times}$$

be the norm map $x \mapsto xx^\dagger$ and put $F_1 = \mathrm{Ker}\, N_\dagger \subset F^\times$. As no confusion should occur we also write $N_\dagger$ for the $p$-adic norm map $F_p^\times \to F_p^{\dagger\times}$ and set $F_{p,1} = F_1 \cap F_p^\times$.

Now let $L \in \mathcal{F}^p_{\mathrm{CM}}(\mathbb{Q})$ be a subfield of $F$, $L \neq F$. This is equivalent to $L$ being a $\dagger$-stable subfield of $F$ such that $L^\dagger \neq L$ and $L_p^\dagger \neq L_p = \mathbb{Q}_p \otimes_{\mathbb{Q}} L$. The restriction map identifies $\Gamma$ with $\mathrm{Gal}(L/L^\dagger) \simeq \mathrm{Gal}(L_p/L_p^\dagger)$, and we still denote by $N_\dagger$ the norm $L^\times \to L^{\dagger\times}$ (resp. $L_p^\times \to L_p^{\dagger\times}$) with kernel $L_1$ (resp. $L_{p,1}$). Thus we have the field extensions



and



Note that this situation yields some constraints on the arithmetic of the extensions involved. Namely if both $F_p/F_p^\dagger$ and $L_p/L_p^\dagger$ are unramified then the residue fields degree $f(F_p/L_p) = f(F_p^\dagger/L_p^\dagger)$ must be odd, because the residue field of $L_p^\dagger$ has a unique quadratic extension in $\overline{\mathbb{F}}_p$ ; if both $F_p/F_p^\dagger$, $L_p/L_p^\dagger$ are ramified and $p \neq 2$ then the ramification index $e(F_p/L_p) = e(F_p^\dagger/L_p^\dagger)$ must be odd, because the maximal unramified extension of $L_p^\dagger$ has a unique quadratic extension in $\overline{\mathbb{Q}}_p$ when $p \neq 2$.

We now define the $\mathbb{Q}$-tori associated to the class $\mathcal{F}^p_{\mathrm{CM}}(\mathbb{Q})$. Let $L, F \in \mathcal{F}^p_{\mathrm{CM}}(\mathbb{Q})$ with $L \subset F$ and consider the following $\mathbb{Q}$-tori

$$T \underset{\mathrm{def}}{=} \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m), \quad T^\dagger \underset{\mathrm{def}}{=} \mathrm{Res}_{F^\dagger/\mathbb{Q}}(\mathbb{G}_m), \quad S \underset{\mathrm{def}}{=} \mathrm{Res}_{L/\mathbb{Q}}(\mathbb{G}_m), \quad S^\dagger \underset{\mathrm{def}}{=} \mathrm{Res}_{L^\dagger/\mathbb{Q}}(\mathbb{G}_m).$$

We have $T(\mathbb{Q}) = F^\times$, $T(\mathbb{Q}_p) = F_p^\times$, and similarly for the three others. Again write $N_\dagger$ for the morphisms $T \to T^\dagger$ and $S \to S^\dagger$ induced by the norm and set

$$T_1 \underset{\text{def}}{=} \operatorname{Ker}\big(T \xrightarrow{N_\dagger} T^\dagger\big) \quad \text{and} \quad S_1 \underset{\text{def}}{=} \operatorname{Ker}\big(S \xrightarrow{N_\dagger} S^\dagger\big).$$

Thus $T_1$ is a $\mathbb{Q}$-torus with $T_1(\mathbb{Q}) = F_1$, $T_1(\mathbb{Q}_p) = F_{p,1}$, and similarly for $S_1$. We want to study the arithmetic of the quotient norm one $\mathbb{Q}$-torus $T_1/S_1$. This will be achieved by using the commutative diagram with exact rows ans columns

$$(\star) \qquad \begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & S_1 & \longrightarrow & S & \xrightarrow{N_\dagger} & S^\dagger & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & T_1 & \longrightarrow & T & \xrightarrow{N_\dagger} & T^\dagger & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & T_1/S_1 & \longrightarrow & T/S & \xrightarrow{N_\dagger} & T^\dagger/S^\dagger & \longrightarrow & 1. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

Let $\mathcal{T}$ be a $K$-torus and $X^*(\mathcal{T}) = \operatorname{Hom}(\mathcal{T}, \mathbb{G}_m)$ its $\mathbb{Z}[G_K]$-module of characters. Recall that the contravariant functor $\mathcal{T} \mapsto X^*(\mathcal{T})$ establishes an equivalence between the category of algebraic tori over $K$ and the category of finite free $\mathbb{Z}$-modules with discrete action of $G_K$ ([Pl-Ra] Thm.2.1). The torus $\mathcal{T}$ is said $K$-anisotropic if $X^*(\mathcal{T})^{G_K} = 0$.

**Lemma 1.1.** *The torus $T_1$ is $\mathbb{Q}$-anisotropic.*

*Proof.* Consider the algebraic $F^\dagger$-tori $T_0 = \operatorname{Res}_{F/F^\dagger}(\mathbb{G}_m)$ and $T_0^{(1)} = \operatorname{Ker}(T_0 \xrightarrow{N_\dagger} \mathbb{G}_m)$. We have $X^*(\mathbb{G}_m) = \mathbb{Z}$, $X^*(T_0) = \mathbb{Z}[\Gamma]$, and $X^*(T_0)^{G_{F^\dagger}} = \mathbb{Z}(1+\gamma)$ with $\gamma = \dagger$, the generator of $\Gamma$. The short exact sequence of $F^\dagger$-tori

$$1 \longrightarrow T_0^{(1)} \xrightarrow{\text{incl}} T_0 \xrightarrow{N_\dagger} \mathbb{G}_m \longrightarrow 1$$

yields a short exact sequence of $\mathbb{Z}[G_{F^\dagger}]$-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{X^*(N_\dagger)} \mathbb{Z}[\Gamma] \xrightarrow{\text{proj}} X^*\big(T_0^{(1)}\big) \longrightarrow 0$$

with $X^*(N_\dagger)(1) = 1+\gamma$. Hence $X^*\big(T_0^{(1)}\big) = \mathbb{Z}[\Gamma]/\mathbb{Z}(1+\gamma)$ and from the vanishing of $H^1(G_{F^\dagger}, \mathbb{Z}) = \operatorname{Hom}(G_{F^\dagger}, \mathbb{Z})$ it follows that $X^*\big(T_0^{(1)}\big)^{G_{F^\dagger}} = 0$. Now $T = \operatorname{Res}_{F^\dagger/\mathbb{Q}}(T_0)$ and applying the exact functor $\operatorname{Res}_{F^\dagger/\mathbb{Q}}$ to the above short exact sequence we find that $T_1 = \operatorname{Res}_{F^\dagger/\mathbb{Q}} T_0^{(1)}$. Thus $X^*(T_1) = \operatorname{Ind}_G^{G_{F^\dagger}}\big(X^*\big(T_0^{(1)}\big)\big)$ and $X^*(T_1)^G = X^*\big(T_0^{(1)}\big)^{G_{F^\dagger}} = 0$. $\square$

**Corollary 1.2.** *The torus $T_1/S_1$ is $\mathbb{Q}$-anisotropic.*

*Proof.* A quotient of an anisotropic torus is anisotropic : the projection $T_1 \to T_1/S_1$ yields an embedding $X^*(T_1/S_1) \hookrightarrow X^*(T_1)$ which injects $X^*(T_1/S_1)^G$ into $X^*(T_1)^G = 0$. $\square$

## 2. Local cohomology

We begin with the computation of the $\mathbb{Q}_p$-cohomology of the torus $T_1/S_1$. It involves the kernel and cokernel of the morphism

$$\iota_p : L_p^{\dagger\times}/N_\dagger(L_p^\times) \underset{\text{incl}}{\longrightarrow} F_p^{\dagger\times}/N_\dagger(F_p^\times)$$

induced by the inclusion of $L_p^{\dagger\times}$ in $F_p^{\dagger\times}$. The source and target having order 2 by assumption (and local class field theory), $\iota_p$ is either an isomorphism or is trivial, and both its kernel and cokernel are trivial or have order 2 accordingly.

**Lemma 2.1.** *If* $[F_p : L_p]$ *is odd then* $\operatorname{Ker}\iota_p = \operatorname{Coker}\iota_p = 1$. *If* $[F_p : L_p]$ *is even then* $\operatorname{Ker}\iota_p = L_p^{\dagger\times}/N_\dagger(L_p^\times)$ *and* $\operatorname{Coker}\iota_p = F_p^{\dagger\times}/N_\dagger(F_p^\times)$.

Note that our asumptions imply that $[F_p : L_p] = [F : L]$.

*Proof.* By local class field theory we have a commutative diagram

$$\begin{array}{ccc} F_p^{\dagger\times}/N_\dagger(F_p^\times) & \xrightarrow{N_{F_p^\dagger/L_p^\dagger}} & L_p^{\dagger\times}/N_\dagger(L_p^\times) \\ \wr\downarrow \operatorname{rec}_{F_p/F_p^\dagger} & & \wr\downarrow \operatorname{rec}_{L_p/L_p^\dagger} \\ \operatorname{Gal}(F_p/F_p^\dagger) & \underset{\operatorname{Res}_{F_p/L_p}}{\xrightarrow{\sim}} & \operatorname{Gal}(L_p/L_p^\dagger). \end{array}$$

By assumption the restriction $\operatorname{Res}_{F_p/L_p}$ is an isomorphism so the norm $N_{F_p^\dagger/L_p^\dagger}$ as well. The compositum $N_{F_p^\dagger/L_p^\dagger}\circ\iota_p : L_p^{\dagger\times}/N_\dagger(L_p^\times) \to L_p^{\dagger\times}/N_\dagger(L_p^\times)$ is the map raising to the power $[F_p^\dagger : L_p^\dagger]$, hence $\iota_p$ is an isomorphism if and only if $[F_p^\dagger : L_p^\dagger] = [F_p : L_p]$ is odd. □

For an algebraic torus $\mathcal{T}$ over $\mathbb{Q}_p$ and for all $r \geq 0$ let

$$H^r(\mathbb{Q}_p, \mathcal{T}) \underset{\text{def}}{=} H^r_{\text{cts}}\big(G_p, \mathcal{T}(\overline{\mathbb{Q}}_p)\big).$$

We have $H^0(\mathbb{Q}_p, \mathcal{T}) = \mathcal{T}(\mathbb{Q}_p)$. It is known that $H^1(\mathbb{Q}_p, \mathcal{T})$ is finite ([Pl-Ra] Corollary of Prop.6.9). For $r \geq 3$ we have $H^r(\mathbb{Q}_p, \mathcal{T}) = 1$ because $G_p$ has cohomological dimension 2 ([Se] II.4.3,Cor. and I.3.1,Cor.).

**Proposition 2.2.** *The commutative diagram* $(\star)$ *induces*

(i) *a short exact sequence*

$$1 \longrightarrow F_{p,1}/L_{p,1} \longrightarrow H^0(\mathbb{Q}_p, T_1/S_1) \longrightarrow \operatorname{Ker}\iota_p \longrightarrow 1,$$

(ii) *an isomorphism* $H^1(\mathbb{Q}_p, T_1/S_1) \simeq \operatorname{Coker}\iota_p$,

(iii) $H^r(\mathbb{Q}_p, T_1/S_1) = 1$ *for* $r \geq 2$.

*Proof.* The torus $T_1$ is $\mathbb{Q}_p$-anisotropic as $T_1(\mathbb{Q}_p) = F_{p,1}$ is compact ([Pl-Ra] Thm.3.1), hence the quotient $T_1/S_1$ is $\mathbb{Q}_p$-anisotropic as well (corollary 1.2). The local Nakayama-Tate theorem ([Mi] I.2.4) then gives $H^2(\mathbb{Q}_p, T_1/S_1) \simeq H^0\big(\mathbb{Q}_p, X^*(T_1/S_1)\big)^{\wedge\vee} = 1$. Of course the same holds for $S_1$ and $T_1$.

The short exact sequence $1 \to T_1 \to T \xrightarrow{N_\dagger} T^\dagger \to 1$ yields the exact sequence

$$F_p^\times \xrightarrow{N_\dagger} F_p^{\dagger\times} \longrightarrow H^1(\mathbb{Q}_p, T_1) \longrightarrow H^1(\mathbb{Q}_p, T).$$

By Shapiro's Lemma and Hilbert 90 we have $H^1(\mathbb{Q}_p, T) \simeq H^1(F_p, \mathbb{G}_m) = 1$, therefore $H^1(\mathbb{Q}_p, T_1) \simeq F_p^{\dagger\times}/N_\dagger(F_p^\times)$. Similarly $H^1(\mathbb{Q}_p, S_1) \simeq L_p^{\dagger\times}/N_\dagger(L_p^\times)$. Now from the short exact sequence $1 \to S_1 \to T_1 \to T_1/S_1 \to 1$ and $H^2(\mathbb{Q}_p, S_1) = 1$ we obtain the exact sequence

$$\begin{array}{c} 1 \longrightarrow F_{p,1}/L_{p,1} \longrightarrow H^0(\mathbb{Q}_p, T_1/S_1) \longrightarrow L_p^{\dagger\times}/N_\dagger(L_p^\times) \\ \downarrow \iota_p \\ F_p^{\dagger\times}/N_\dagger(F_p^\times) \longrightarrow H^1(\mathbb{Q}_p, T_1/S_1) \longrightarrow 1 \end{array}$$

from which the statements on $H^r(\mathbb{Q}_p, T_1/S_1)$ for $r = 0, 1$ follow. □

We now compute the $\mathbb{R}$-cohomology of $T_1/S_1$. Let $\mathbb{C}_1$ be the subgroup of norm one elements in $\mathbb{C}^\times$ and $\mathbb{R}_+^\times$ the subgroup of positive elements in $\mathbb{R}^\times$.

**Proposition 2.3.** *The commutative diagram* $(\star)$ *induces*

(i) *an isomorphism* $H^0(\mathbb{R}, T_1/S_1) \simeq \mathrm{Coker}\Big({\mathbb{C}_1}^{[L^\dagger:\mathbb{Q}]} \xrightarrow{\mathrm{Diag}} {\mathbb{C}_1}^{[F^\dagger:\mathbb{Q}]}\Big)$,

(ii) *an isomorphism* $H^1(\mathbb{R}, T_1/S_1) \simeq \mathrm{Coker}\Big(\big(\mathbb{R}^\times/\mathbb{R}_+^\times\big)^{[L^\dagger:\mathbb{Q}]} \xrightarrow{\mathrm{Diag}} \big(\mathbb{R}^\times/\mathbb{R}_+^\times\big)^{[F^\dagger:\mathbb{Q}]}\Big)$,

(iii) $H^{2r}(\mathbb{R}, T_1/S_1) = 1$ *for all* $r \geq 1$,

(iv) *isomorphisms* $H^{2r+1}(\mathbb{R}, T_1/S_1) \simeq \mathrm{Coker}\Big(\big(\tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\big)^{[L^\dagger:\mathbb{Q}]} \xrightarrow{\mathrm{Diag}} \big(\tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\big)^{[F^\dagger:\mathbb{Q}]}\Big)$ *for all* $r \geq 1$*, via the local* inv *isomorphism* $\mathrm{Br}(\mathbb{R}) \simeq \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.

*Proof.* Put $n = [L^\dagger : \mathbb{Q}]$ and $m = [F^\dagger : \mathbb{Q}]$. Since $L$ is totally imaginary and $L^\dagger$ totally real we have $H^0(\mathbb{R}, S) \simeq (\mathbb{C}^\times)^n$ and $H^0(\mathbb{R}, S^\dagger) \simeq (\mathbb{R}^\times)^n$. Further for all $r \geq 1$ we have $H^r(\mathbb{R}, S) = 1$, $H^{2r-1}(\mathbb{R}, S^\dagger) = 1$, and $H^{2r}(\mathbb{R}, S^\dagger) \simeq \mathrm{Br}(\mathbb{R})^n \simeq \big(\frac{1}{2}\mathbb{Z}/\mathbb{Z}\big)^n$, the latter isomorphism being the $n$-fold local inv. From the short exact sequence

$$1 \longrightarrow S_1 \longrightarrow S \xrightarrow{N_\dagger} S^\dagger \longrightarrow 1$$

and $N_\dagger(\mathbb{C}^\times) = \mathbb{R}_+^\times$ we find that $H^0(\mathbb{R}, S_1) \simeq {\mathbb{C}_1}^n$, $H^1(\mathbb{R}, S_1) \simeq \big(\mathbb{R}^\times/\mathbb{R}_+^\times\big)^n$, and for $r \geq 1$, $H^{2r}(\mathbb{R}, S_1) = 1$, $H^{2r+1}(\mathbb{R}, S_1) \simeq \big(\frac{1}{2}\mathbb{Z}/\mathbb{Z}\big)^n$. Similarly, the same holds with $S_1$ and $n$ replaced by $T_1$ and $m$. The result then follows from the short exact sequence

$$1 \longrightarrow S_1 \longrightarrow T_1 \longrightarrow T_1/S_1 \longrightarrow 1$$

and the injectivity of the map Diag. □

**Corollary 2.4.** *There are noncanonical isomorphisms*

$$H^0(\mathbb{R}, T_1/S_1) \simeq \mathbb{C}_1^{[F^\dagger:\mathbb{Q}]-[L^\dagger:\mathbb{Q}]}, \quad H^{2r+1}(\mathbb{R}, T_1/S_1) \simeq \big(\tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\big)^{[F^\dagger:\mathbb{Q}]-[L^\dagger:\mathbb{Q}]} \text{ for all } r \geq 1,$$

*and* $H^{2r}(\mathbb{R}, T_1/S_1) = 1$ *for all* $r \geq 1$.

*Proof.* Obvious from proposition 2.3 and the isomorphism $\mathbb{R}^\times/\mathbb{R}_+^\times \simeq \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. □

## 3. Global cohomology

We now compute the $\mathbb{Q}$-cohomology of the torus $T_1/S_1$. It involves the kernels and cokernels of three morphisms (see proposition 3.8), the first of which is

$$\iota : L^{\dagger\times}/N_\dagger(L^\times) \underset{\text{incl}}{\longrightarrow} F^{\dagger\times}/N_\dagger(F^\times)$$

the morphism induced by the inclusion of $L^{\dagger\times}$ in $F^{\dagger\times}$. For a place $v \in S_{L^\dagger}$ and an algebraic extension $K$ of $L^\dagger$ set

$$S_K(v) \underset{\text{def}}{=} \{\, w \in S_K \mid w \mid v \,\}.$$

We have a commutative diagram

$$\begin{array}{ccc} (L_v^\dagger \otimes_{L^\dagger} L)^\times & \hookrightarrow & (L_v^\dagger \otimes_{L^\dagger} F)^\times = \bigoplus_{w\in S_{F^\dagger}(v)} (F_w^\dagger \otimes_{F^\dagger} F)^\times \\ \Big\downarrow N_v & & \Big\downarrow \oplus_{w\in S_{F^\dagger}(v)} N_w \\ L_v^{\dagger\times} & \hookrightarrow & (L_v^\dagger \otimes_{L^\dagger} F^\dagger)^\times = \bigoplus_{w\in S_{F^\dagger}(v)} F_w^{\dagger\times} \end{array}$$

where the maps $N_v$, $N_w$ are induced by $N_\dagger$ and the horizontal ones by inclusions, hence localising $\iota$ at $v \in S_{L^\dagger}$ yields a morphism

$$\iota_v : L_v^{\dagger\times}/N_v(L_v^\dagger \otimes_{L^\dagger} L)^\times \longrightarrow \bigoplus_{w\in S_{F^\dagger}(v)} F_w^{\dagger\times}/N_w(F_w^\dagger \otimes_{F^\dagger} F)^\times.$$

When $v = p$ (the unique place lying above $p$) we recover the morphism $\iota_p$ introduced in section 2.

**Lemma 3.1.** *The localisation maps induce isomorphisms*

$$\operatorname{Ker}\iota \simeq \bigoplus_{\substack{v\in S_{L^\dagger}\\ v\neq p}} \operatorname{Ker}\iota_v \quad \textit{and} \quad \operatorname{Coker}\iota \simeq \bigoplus_{\substack{v\in S_{L^\dagger}\\ v\neq p}} \operatorname{Coker}\iota_v.$$

**Remark 3.2.** Recall that $\iota_p$ is either an isomorphism or is trivial according to the parity of $[F_p : L_p] = [F : L]$ (lemma 2.1). When $[F : L]$ is odd then $\operatorname{Ker}\iota_p = \operatorname{Coker}\iota_p = 1$, consequently $\operatorname{Ker}\iota \simeq \bigoplus_{v\in S_{L^\dagger}} \operatorname{Ker}\iota_v$ and $\operatorname{Coker}\iota \simeq \bigoplus_{v\in S_{L^\dagger}} \operatorname{Coker}\iota_v$.

*Proof.* Since $H^1(L, \mathbb{G}_m) = 1$ by Hilbert 90 we have a short exact sequence of $\Gamma$-modules

$$1 \longrightarrow L^\times \longrightarrow I_L \longrightarrow C_L \longrightarrow 1.$$

As $\Gamma$ is cyclic we have $\hat{H}^{-1}(\Gamma, C_L) = 1$, and again by Hilbert 90 we have $\hat{H}^1(\Gamma, L^\times) = 1$. Thus Tate cohomology yields a short exact sequence

$$1 \longrightarrow L^{\dagger\times}/N_\dagger(L^\times) \longrightarrow I_{L^\dagger}/N_\dagger(I_L) \longrightarrow C_{L^\dagger}/N_\dagger(C_L) \longrightarrow 1.$$

By class field theory both $L_p^{\dagger\times}/N_\dagger(L_p^\times)$ and $C_{L^\dagger}/N_\dagger(C_L)$ have order 2, and the former embeds in the latter ([Ne] Prop.5.6). Thus

$$L_p^{\dagger\times}/N_\dagger(L_p^\times) \simeq C_{L^\dagger}/N_\dagger(C_L)$$

which provides via the inclusion $L_p^{\dagger\times}/N_\dagger(L_p^\times) \hookrightarrow I_{L^\dagger}/N_\dagger(I_L)$ a section to the above short exact sequence. Therefore we have an isomorphism

$$I_{L^\dagger}/N_\dagger(I_L) \simeq L^{\dagger\times}/N_\dagger(L^\times) \oplus L_p^{\dagger\times}/N_\dagger(L_p^\times).$$

The same holds with $L, L^\dagger$ replaced by $F, F^\dagger$ and the isomorphisms involved are compatible with the inclusions $L \subset F$, $L^\dagger \subset F^\dagger$. Hence

$$\mathrm{Ker}\Big(I_{L^\dagger}/N_\dagger(I_L) \xrightarrow{\text{incl}} I_{F^\dagger}/N_\dagger(I_F)\Big) = \bigoplus_{v\in S_{L^\dagger}} \mathrm{Ker}\,\iota_v \simeq \mathrm{Ker}\,\iota \oplus \mathrm{Ker}\,\iota_p \quad \text{and}$$

$$\mathrm{Coker}\Big(I_{L^\dagger}/N_\dagger(I_L) \xrightarrow{\text{incl}} I_{F^\dagger}/N_\dagger(I_F)\Big) = \bigoplus_{v\in S_{L^\dagger}} \mathrm{Coker}\,\iota_v \simeq \mathrm{Coker}\,\iota \oplus \mathrm{Coker}\,\iota_p$$

from which the result follows. □

For $v \in S_{L^\dagger}$ the $L_v^\dagger$-algebra $L_v^\dagger \otimes_{L^\dagger} L$ is either a field or isomorphic to $L_v^\dagger \times L_v^\dagger$ according to $\#S_L(v) = 1$ or 2 respectively. Define the following subsets of $S_{L^\dagger}$ and $S_{F^\dagger}$:

$$S_{L^\dagger}(L) \underset{\text{def}}{=} \{v \in S_{L^\dagger} \mid L_v^\dagger \otimes_{L^\dagger} L \simeq L_v^\dagger \times L_v^\dagger\}$$
$$\text{and} \quad S_{F^\dagger}(F) \underset{\text{def}}{=} \{w \in S_{F^\dagger} \mid F_w^\dagger \otimes_{F^\dagger} F \simeq F_w^\dagger \times F_w^\dagger\}.$$

Our assumptions imply that the places lying above $p$ or $\infty$ do not belong to $S_{L^\dagger}(L)$, nor to $S_{F^\dagger}(F)$. Note that $v \in S_{L^\dagger}(L)$ (resp. $w \in S_{F^\dagger}(F)$) if and only if $L_v^{\dagger\times}/N_v(L_v^\dagger \otimes_{L^\dagger} L)^\times = 1$ (resp. $F_w^{\dagger\times}/N_w(F_w^\dagger \otimes_{F^\dagger} F)^\times = 1$). We also set for each place $v \in S_{L^\dagger}$

$$S_{F^\dagger}(F, v) \underset{\text{def}}{=} S_{F^\dagger}(F) \cap S_{F^\dagger}(v).$$

Thus we have

$$\iota_v : L_v^{\dagger\times}/N_v(L_v^\dagger \otimes_{L^\dagger} L)^\times \longrightarrow \bigoplus_{\substack{w\in S_{F^\dagger}(v)\\ w\notin S_{F^\dagger}(F,v)}} F_w^{\dagger\times}/N_w(F_w^\dagger \otimes_{F^\dagger} F)^\times$$

the right-hand side being 1 when $S_{F^\dagger}(F, v) = S_{F^\dagger}(v)$.

**Lemma 3.3.** *Let $v \in S_{L^\dagger}$. When $v \in S_{L^\dagger}(L)$ we have $\mathrm{Ker}\,\iota_v = \mathrm{Coker}\,\iota_v = 1$. When $v \notin S_{L^\dagger}(L)$ the local reciprocity maps together with the identification $\Gamma \simeq \mathbb{Z}/2\mathbb{Z}$ induce isomorphisms*

$$\mathrm{Ker}\,\iota_v \simeq \mathrm{Ker}\begin{pmatrix} \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \displaystyle\bigoplus_{\substack{w\in S_{F^\dagger}(v)\\ w\notin S_{F^\dagger}(F,v)}} \mathbb{Z}/2\mathbb{Z} \\ 1 & \longmapsto & \big([F_w^\dagger : L_v^\dagger] \bmod 2\mathbb{Z}\big)_w \end{pmatrix}$$

*and*

$$\mathrm{Coker}\,\iota_v \simeq \mathrm{Coker}\begin{pmatrix} \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \displaystyle\bigoplus_{\substack{w\in S_{F^\dagger}(v)\\ w\notin S_{F^\dagger}(F,v)}} \mathbb{Z}/2\mathbb{Z} \\ 1 & \longmapsto & \big([F_w^\dagger : L_v^\dagger] \bmod 2\mathbb{Z}\big)_w \end{pmatrix}.$$

Thus when $v \notin S_{L^\dagger}(L)$ and $S_{F^\dagger}(F,v) = S_{F^\dagger}(v)$ we have $\operatorname{Ker}\iota_v = L_v^{\dagger\times}/N_v(L_v^\dagger \otimes_{L^\dagger} L)^\times \simeq \mathbb{Z}/2\mathbb{Z}$ and $\operatorname{Coker}\iota_v = 1$; when $v \notin S_{L^\dagger}(L)$ and $S_{F^\dagger}(F,v) \neq S_{F^\dagger}(v)$ the kernel and cokernel of $\iota_v$ depend only on the parity of the local extension degrees $[F_w^\dagger : L_v^\dagger]$ (for $v$ lying above $p$ we recover the statement of lemma 2.1).

*Proof.* When $v \in S_{L^\dagger}(L)$ every $w \in S_{F^\dagger}$ lying above $v$ belongs to $S_{F^\dagger}(F)$, hence the source and target of $\iota_v$ are both trivial, so $\operatorname{Ker}\iota_v = \operatorname{Coker}\iota_v = 1$.

When $v \notin S_{L^\dagger}(L)$ assume that $\{w \mid v,\, w \notin S_{F^\dagger}(F)\}$ is not empty and let $w$ be an element thereof. By local class field theory we have a commutative diagram

$$\begin{array}{ccccc}
 & & \overset{[F_w^\dagger:L_v^\dagger]}{\frown} & & \\
L_v^{\dagger\times}/N_\dagger(L_v^\times) & \xrightarrow{\iota_{v,w}} & F_w^{\dagger\times}/N_\dagger(F_w^\times) & \xrightarrow{N_{F_w^\dagger/L_v^\dagger}} & L_v^{\dagger\times}/N_\dagger(L_v^\times) \\
\wr\big\downarrow \mathrm{rec}_{L_v/L_v^\dagger} & & \wr\big\downarrow \mathrm{rec}_{F_w/F_w^\dagger} & & \wr\big\downarrow \mathrm{rec}_{L_v/L_v^\dagger} \\
\operatorname{Gal}(L_v/L_v^\dagger) & \dashrightarrow & \operatorname{Gal}(F_w/F_w^\dagger) & \xrightarrow[\mathrm{Res}_{F_w/L_v}]{\sim} & \operatorname{Gal}(L_v/L_v^\dagger)
\end{array}$$

where $\iota_{v,w}$ is induced by the inclusion, so that $\iota_v = (\iota_{v,w})_{w|v}$. The result then follows from the canonical identifications $\mathbb{Z}/2\mathbb{Z} \simeq \Gamma \simeq \operatorname{Gal}(F_w/F_w^\dagger) \simeq \operatorname{Gal}(L_v/L_v^\dagger)$. □

We introduce the following subsets of $S_{F^\dagger}$. Set

$$S_{F^\dagger}(L) \underset{\mathrm{def}}{=} \{\, w \in S_{F^\dagger} \mid w_{|L^\dagger} \in S_{L^\dagger}(L)\} = \bigsqcup_{v \in S_{L^\dagger}(L)} S_{F^\dagger}(v).$$

Note that $S_{F^\dagger}(L) \subseteq S_{F^\dagger}(F)$. Also define the "odd" and "even" part of $S_{F^\dagger}$ relative to $L^\dagger$

$$S_{F^\dagger}^{\mathrm{odd}} \underset{\mathrm{def}}{=} \{\, w \in S_{F^\dagger} \mid [F_w^\dagger : L_v^\dagger] \text{ is odd, } v = w_{|L^\dagger}\},$$
$$S_{F^\dagger}^{\mathrm{even}} \underset{\mathrm{def}}{=} \{\, w \in S_{F^\dagger} \mid [F_w^\dagger : L_v^\dagger] \text{ is even, } v = w_{|L^\dagger}\}.$$

Of course we have $S_{F^\dagger} = S_{F^\dagger}^{\mathrm{odd}} \sqcup S_{F^\dagger}^{\mathrm{even}}$. Note that $v \notin S_{L^\dagger}(L)$ implies $S_{F^\dagger}(F,v) \subset S_{F^\dagger}^{\mathrm{even}}$. Further we introduce the subsets of $S_{L^\dagger}$

$$S_{L^\dagger}^{\mathrm{odd}} \underset{\mathrm{def}}{=} \{v \in S_{L^\dagger} \mid S_{F^\dagger}(v) \cap S_{F^\dagger}^{\mathrm{odd}} \neq \varnothing\} \quad \text{and} \quad S_{L^\dagger}^{\mathrm{even}} \underset{\mathrm{def}}{=} \{v \in S_{L^\dagger} \mid S_{F^\dagger}(v) \subset S_{F^\dagger}^{\mathrm{even}}\}.$$

**Corollary 3.4.** *The local reciprocity maps induce an isomorphism*

$$\operatorname{Ker}\iota \simeq \bigoplus_{\substack{v \notin S_{L^\dagger}(L) \\ v \neq p \\ v \in S_{L^\dagger}^{\mathrm{even}}}} \mathbb{Z}/2\mathbb{Z}$$

*and there is a noncanonical isomorphism*

$$\operatorname{Coker}\iota \simeq \bigoplus_{\substack{v \notin S_{L^\dagger}(L) \\ v \neq p \\ v \in S_{L^\dagger}^{\mathrm{odd}}}} (\mathbb{Z}/2\mathbb{Z})^{\#S_{F^\dagger}(v) - \#S_{F^\dagger}(F,v) - 1} \oplus \bigoplus_{\substack{w \notin S_{F^\dagger}(F) \\ w \neq p \\ w \in S_{F^\dagger}^{\mathrm{even}}}} \mathbb{Z}/2\mathbb{Z}\,.$$

*Proof.* According to lemmas 3.1 and 3.3 the localisation maps yield isomorphisms

$$\operatorname{Ker}\iota \simeq \bigoplus_{\substack{v\notin S_{L^\dagger}(L)\\ v\neq p}} \operatorname{Ker}\iota_v \quad \text{and} \quad \operatorname{Coker}\iota \simeq \bigoplus_{\substack{v\notin S_{L^\dagger}(L)\\ v\neq p}} \operatorname{Coker}\iota_v\,.$$

Let $v \in S_{L^\dagger}$, $v \notin S_{L^\dagger}(L)$. Then $S_{F^\dagger}(v) - S_{F^\dagger}(F,v) \subset S_{F^\dagger}^{\text{even}}$ is equivalent to $S_{F^\dagger}(v) \subset S_{F^\dagger}^{\text{even}}$ and $S_{F^\dagger}(v) \cap S_{F^\dagger}^{\text{odd}} \neq \varnothing$ implies $S_{F^\dagger}(F,v) \neq S_{F^\dagger}(F)$. Thus lemma 3.3 shows that

$$\begin{cases} \operatorname{Ker}\iota_v \simeq \mathbb{Z}/2\mathbb{Z} \;\text{ and }\; \operatorname{Coker}\iota_v \simeq \big(\mathbb{Z}/2\mathbb{Z}\big)^{\#S_{F^\dagger}(v)-\#S_{F^\dagger}(F,v)} & \text{if } S_{F^\dagger}(v) \subset S_{F^\dagger}^{\text{even}} \\ \operatorname{Ker}\iota_v \simeq 0 \;\text{ and }\; \operatorname{Coker}\iota_v \simeq \big(\mathbb{Z}/2\mathbb{Z}\big)^{\#S_{F^\dagger}(v)-\#S_{F^\dagger}(F,v)-1} & \text{if } S_{F^\dagger}(v) \cap S_{F^\dagger}^{\text{odd}} \neq \varnothing \end{cases}$$

where all the isomorphisms are induced by the localisation maps except for the last one which is not canonical. The result follows by summing over the places $v \in S_{L^\dagger}$ such that $v \notin S_{L^\dagger}(L)$ and $v \neq p$, and replacing the sum of $\#S_{F^\dagger}(v) - \#S_{F^\dagger}(F,v)$ copies of $\mathbb{Z}/2\mathbb{Z}$ over those $v$ such that $S_{F^\dagger}(v) \subset S_{F^\dagger}^{\text{even}}$ by the sum of $\mathbb{Z}/2\mathbb{Z}$ over the $w \in S_{F^\dagger}$ such that $w \notin S_{F^\dagger}(F)$, $w \neq p$, and $w \in S_{F^\dagger}^{\text{even}}$. □

The two other morphisms involved in the $\mathbb{Q}$-cohomology of $T_1/S_1$ are of similar nature and we treat them simultanuously. They both arise from the commutative diagram

$$\begin{array}{ccc} \operatorname{Br}(L) & \xrightarrow{\operatorname{Cor}_{L/L^\dagger}} & \operatorname{Br}(L^\dagger) \\ {\scriptstyle \operatorname{Res}_{F/L}}\downarrow & & \downarrow{\scriptstyle \operatorname{Res}_{F^\dagger/L^\dagger}} \\ \operatorname{Br}(F) & \xrightarrow{\operatorname{Cor}_{F/F^\dagger}} & \operatorname{Br}(F^\dagger) \end{array}$$

as

$$\rho : \operatorname{Ker}\operatorname{Cor}_{L/L^\dagger} \xrightarrow{\operatorname{Res}_{F/L}} \operatorname{Ker}\operatorname{Cor}_{F/F^\dagger}$$

$$\text{and} \qquad \sigma : \operatorname{Coker}\operatorname{Cor}_{L/L^\dagger} \xrightarrow{\operatorname{Res}_{F^\dagger/L^\dagger}} \operatorname{Coker}\operatorname{Cor}_{F/F^\dagger}\,.$$

We have $\operatorname{Ker}\rho = \operatorname{Br}(F/L) \cap \operatorname{Ker}\operatorname{Cor}_{L/L^\dagger}$. For an extension $K'/K$ of number fields and $v \in S_K$ we define

$$\operatorname{Br}(K',v) = \bigoplus_{w\in S_{K'}(v)} \operatorname{Br}(K'_w)$$

where as usual $S_{K'}(v)$ is the set of places of $K'$ lying above $v$. We have $\operatorname{Br}(K,v) = \operatorname{Br}(K_v)$ and $\bigoplus_{w\in S_{K'}} \operatorname{Br}(K'_w) = \bigoplus_{v\in S_K} \operatorname{Br}(K',v)$. We also let

$$\operatorname{Res}_{K'/K}(v) \;:\; \begin{cases} \operatorname{Br}(K,v) & \longrightarrow \quad \operatorname{Br}(K',v) \\ \alpha & \longmapsto \quad \big(\operatorname{Res}_{K'_w/K_v}(\alpha)\big)_{w\in S_{K'}(v)} \end{cases}$$

$$\text{and} \qquad \operatorname{Cor}_{K'/K}(v) \;:\; \begin{cases} \operatorname{Br}(K',v) & \longrightarrow \quad \operatorname{Br}(K,v) \\ (\beta_w)_{w\in S_{K'}(v)} & \longmapsto \quad \otimes_{w\in S_{K'}(v)} \operatorname{Cor}_{K'_w/K_v}(\beta_w). \end{cases}$$

For each $v \in S_{L^\dagger}$ we have $\operatorname{Br}(F,v) = \bigoplus_{w \in S_{F^\dagger}(v)} \operatorname{Br}(F,w) = \bigoplus_{u \in S_L(v)} \operatorname{Br}(F,u)$, and a commutative diagram

$$\begin{array}{ccc} \operatorname{Br}(L,v) & \xrightarrow{\operatorname{Cor}_{L/L^\dagger}(v)} & \operatorname{Br}(L^\dagger,v) \\ \downarrow{\scriptstyle \operatorname{Res}_{F/L}(v)} & & \downarrow{\scriptstyle \operatorname{Res}_{F^\dagger/L^\dagger}(v)} \\ \operatorname{Br}(F,v) & \xrightarrow{\operatorname{Cor}_{F/F^\dagger}(v)} & \operatorname{Br}(F^\dagger,v) \end{array}$$

with $\operatorname{Cor}_{F/F^\dagger}(v) = \oplus_{w \in S_{F^\dagger}(v)} \operatorname{Cor}_{F/F^\dagger}(w)$ and $\operatorname{Res}_{F/L}(v) = \oplus_{u \in S_L(v)} \operatorname{Res}_{F/L}(u)$. Hence localising $\rho$ and $\sigma$ at $v \in S_{L^\dagger}$ yields morphisms

$$\rho_v : \operatorname{Ker}\operatorname{Cor}_{L/L^\dagger}(v) \xrightarrow{\operatorname{Res}_{F/L}(v)} \operatorname{Ker}\operatorname{Cor}_{F/F^\dagger}(v)$$

$$\text{and} \quad \sigma_v : \operatorname{Coker}\operatorname{Cor}_{L/L^\dagger}(v) \xrightarrow{\operatorname{Res}_{F^\dagger/L^\dagger}(v)} \operatorname{Coker}\operatorname{Cor}_{F/F^\dagger}(v).$$

**Lemma 3.5.** *The local restriction maps induce isomorphisms*

$$\operatorname{Ker}\rho \simeq \bigoplus_{v \in S_{L^\dagger}} \operatorname{Ker}\rho_v\,, \quad \operatorname{Coker}\rho \simeq \bigoplus_{v \in S_{L^\dagger}} \operatorname{Coker}\rho_v$$

$$\textit{and} \quad \operatorname{Ker}\sigma \simeq \bigoplus_{v \in S_{L^\dagger}} \operatorname{Ker}\sigma_v\,, \quad \operatorname{Coker}\sigma \simeq \bigoplus_{v \in S_{L^\dagger}} \operatorname{Coker}\sigma_v.$$

*Proof.* By class field theory we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Br}(L) & \longrightarrow & \bigoplus_{v \in S_{L^\dagger}} \operatorname{Br}(L,v) & \xrightarrow{\operatorname{inv}_L} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \operatorname{Cor}_{L/L^\dagger}} & & \downarrow{\scriptstyle \oplus_v \operatorname{Cor}_{L/L^\dagger}(v)} & & \| & & \\ 0 & \longrightarrow & \operatorname{Br}(L^\dagger) & \longrightarrow & \bigoplus_{v \in S_{L^\dagger}} \operatorname{Br}(L^\dagger,v) & \xrightarrow{\operatorname{inv}_{L^\dagger}} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \end{array}$$

so $\operatorname{Ker}\operatorname{Cor}_{L/L^\dagger} \simeq \bigoplus_{v \in S_{L^\dagger}} \operatorname{Ker}\operatorname{Cor}_{L/L^\dagger}(v)$ and $\operatorname{Coker}\operatorname{Cor}_{L/L^\dagger} \simeq \bigoplus_{v \in S_{L^\dagger}} \operatorname{Coker}\operatorname{Cor}_{L/L^\dagger}(v)$. The same holds with $L, L^\dagger$ replaced by $F, F^\dagger$, and the result follows from the commutativity of the diagrams

$$\begin{array}{ccc} \operatorname{Ker}\operatorname{Cor}_{L/L^\dagger} & \xrightarrow{\quad\rho\quad} & \operatorname{Ker}\operatorname{Cor}_{F/F^\dagger} \\ \downarrow\wr & & \downarrow\wr \\ \bigoplus_{v \in S_{L^\dagger}} \operatorname{Ker}\operatorname{Cor}_{L/L^\dagger}(v) & \xrightarrow{\oplus_v \rho_v} & \bigoplus_{v \in S_{L^\dagger}} \operatorname{Ker}\operatorname{Cor}_{F/F^\dagger}(v) \end{array}$$

$$\text{and} \quad \begin{array}{ccc} \operatorname{Coker}\operatorname{Cor}_{L/L^\dagger} & \xrightarrow{\quad\sigma\quad} & \operatorname{Coker}\operatorname{Cor}_{F/F^\dagger} \\ \downarrow\wr & & \downarrow\wr \\ \bigoplus_{v \in S_{L^\dagger}} \operatorname{Coker}\operatorname{Cor}_{L/L^\dagger}(v) & \xrightarrow{\oplus_v \sigma_v} & \bigoplus_{v \in S_{L^\dagger}} \operatorname{Coker}\operatorname{Cor}_{F/F^\dagger}(v). \end{array}$$

□

**Lemma 3.6.** *Let* $v \in S_{L^\dagger}$.

(i) *When* $v \in S_{L^\dagger}(L)$ *the local* inv *maps induce isomorphisms*

$$\operatorname{Ker}\rho_v \simeq \operatorname{Ker}\begin{pmatrix} \mathrm{KS}(\mathbb{Q}/\mathbb{Z}) & \longrightarrow & \bigoplus_{w \in S_{F^\dagger}(v)} \mathrm{KS}(\mathbb{Q}/\mathbb{Z}) \\ \lambda & \longmapsto & \big( [F_w^\dagger : L_v^\dagger]\, \lambda \big)_{w \in S_{F^\dagger}(v)} \end{pmatrix}$$

*and*

$$\operatorname{Coker}\rho_v \simeq \operatorname{Coker}\begin{pmatrix} \mathrm{KS}(\mathbb{Q}/\mathbb{Z}) & \longrightarrow & \bigoplus_{w \in S_{F^\dagger}(v)} \mathrm{KS}(\mathbb{Q}/\mathbb{Z}) \\ \lambda & \longmapsto & \big( [F_w^\dagger : L_v^\dagger]\, \lambda \big)_{w \in S_{F^\dagger}(v)} \end{pmatrix}.$$

*When* $v \notin S_{L^\dagger}(L)$ *we have* $\operatorname{Ker}\rho_v = 0$ *and the local* inv *maps induce an isomorphism*

$$\operatorname{Coker}\rho_v \simeq \bigoplus_{w \in S_{F^\dagger}(F,v)} \mathrm{KS}(\mathbb{Q}/\mathbb{Z}).$$

(ii) *When* $v \in S^f_{L^\dagger}$ *we have* $\operatorname{Ker}\sigma_v = \operatorname{Coker}\sigma_v = 0$. *When* $v \in S^\infty_{L^\dagger}$ *we have* $\operatorname{Ker}\sigma_v = 0$ *and the local* inv *maps induce an isomorphism*

$$\operatorname{Coker}\sigma_v \simeq \operatorname{Coker}\Big( \tfrac{1}{2}\mathbb{Z}/\mathbb{Z} \xrightarrow{\text{Diag}} \bigoplus_{w \in S_{F^\dagger}(v)} \tfrac{1}{2}\mathbb{Z}/\mathbb{Z} \Big).$$

*Proof.* Let $v \in S_{L^\dagger}$ and $\operatorname{inv}_{L^\dagger}(v) = \operatorname{inv}_v$, $\operatorname{inv}_L(v) = \oplus_{u \in S_L(v)} \operatorname{inv}_u$. We have a commutative diagram

$$\begin{CD} \operatorname{Br}(L,v) @>{\operatorname{Cor}_{L/L^\dagger}(v)}>> \operatorname{Br}(L^\dagger,v) \\ @V{\operatorname{inv}_L(v)}VV @VV{\operatorname{inv}_{L^\dagger}(v)}V \\ \bigoplus_{u \in S_L(v)} \mathbb{Q}/\mathbb{Z} @>{\sum_{u|v}}>> \mathbb{Q}/\mathbb{Z}\,. \end{CD}$$

(i) When $v \in S_{L^\dagger}(L) \subset S^f_{L^\dagger}$ the maps $\operatorname{inv}_L(v)$ and $\operatorname{inv}_{L^\dagger}(v)$ are isomorphisms thus

$$\operatorname{Ker}\operatorname{Cor}_{L/L^\dagger}(v) \underset{\operatorname{inv}_L(v)}{\simeq} \operatorname{Ker}\begin{pmatrix} \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z} & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ (\lambda,\mu) & \longmapsto & \lambda+\mu \end{pmatrix} = \mathrm{KS}(\mathbb{Q}/\mathbb{Z}).$$

When $v \notin S_{L^\dagger}(L)$ we have $\operatorname{Ker}\operatorname{Cor}_{L/L^\dagger}(v) = 0$. Similarly $\operatorname{inv}_F(v) = \oplus_{w \in S_{F^\dagger}(v)} \operatorname{inv}_w$ induces an isomorphism

$$\operatorname{Ker}\operatorname{Cor}_{F/F^\dagger}(v) \underset{\operatorname{inv}_F(v)}{\simeq} \bigoplus_{w \in S_{F^\dagger}(F,v)} \mathrm{KS}(\mathbb{Q}/\mathbb{Z})$$

the right-hand side being 0 if the set $S_{F^\dagger}(F,v)$ is empty (e.g. if $v \in S^\infty_{L^\dagger}$). Hence the statement is clear when $v \notin S_{L^\dagger}(L)$, and otherwise it follows from the commutativity of the diagram

$$\begin{CD} \operatorname{Ker}\operatorname{Cor}_{L/L^\dagger}(v) @>{\rho_v}>> \operatorname{Ker}\operatorname{Cor}_{F/F^\dagger}(v) \\ @V{\wr}V{\operatorname{inv}_L(v)}V @V{\wr}V{\operatorname{inv}_F(v)}V \\ \mathrm{KS}(\mathbb{Q}/\mathbb{Z}) @>{\xi_v}>> \bigoplus_{w \in S_{F^\dagger}(v)} \mathrm{KS}(\mathbb{Q}/\mathbb{Z}) \end{CD}$$

where $\xi_v(\lambda) = \big( [F_w^\dagger : L_v^\dagger]\, \lambda \big)_{w \in S_{F^\dagger}(v)}$ for $\lambda \in \mathrm{KS}(\mathbb{Q}/\mathbb{Z})$.

(ii) When $v \in S^f_{L^\dagger}$ we have $\operatorname{Coker}\operatorname{Cor}_{L/L^\dagger}(v) = \operatorname{Coker}\operatorname{Cor}_{F/F^\dagger}(v) = 0$, thus $\operatorname{Ker}\sigma_v = \operatorname{Coker}\sigma_v = 0$. When $v \in S^\infty_{L^\dagger}$ we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Coker}\operatorname{Cor}_{L/L^\dagger}(v) & \xrightarrow{\sigma_v} & \operatorname{Coker}\operatorname{Cor}_{F/F^\dagger}(v) \\ \wr\downarrow \operatorname{inv}_{L^\dagger}(v) & & \wr\downarrow \operatorname{inv}_{F^\dagger}(v) \\ \tfrac{1}{2}\mathbb{Z}/\mathbb{Z} & \xrightarrow{\text{Diag}} & \bigoplus_{w\in S_{F^\dagger}(v)} \tfrac{1}{2}\mathbb{Z}/\mathbb{Z} \end{array}$$

since $F^\dagger_w = L^\dagger_v = \mathbb{R}$, and the result follows. $\square$

**Corollary 3.7.** *For $v \in S_{L^\dagger}$ let $d_v = \gcd\{\,[F^\dagger_w : L^\dagger_v],\, w \in S_{F^\dagger}(v)\,\}$.*

(i) *There are noncanonical isomorphisms*

$$\operatorname{Ker}\rho \simeq \bigoplus_{v\in S_{L^\dagger}(L)} \tfrac{1}{d_v}\mathbb{Z}/\mathbb{Z} \quad \textit{and} \quad \operatorname{Coker}\rho \simeq \bigoplus_{v\in S_{L^\dagger}(L)} \big(\mathbb{Q}/\mathbb{Z}\big)^{\#S_{F^\dagger}(v)-1} \oplus \bigoplus_{\substack{w\in S_{F^\dagger}(F)\\ w\notin S_{F^\dagger}(L)}} \mathbb{Q}/\mathbb{Z}\,.$$

(ii) $\operatorname{Ker}\sigma = 0$ *and there is a noncanonical isomorphism*

$$\operatorname{Coker}\sigma \simeq \Big(\tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\Big)^{[F^\dagger:\mathbb{Q}]-[L^\dagger:\mathbb{Q}]}.$$

*Proof.* (i) According to lemma 3.5 the local restriction maps yield isomorphisms

$$\operatorname{Ker}\rho \simeq \bigoplus_{v\in S_{L^\dagger}} \operatorname{Ker}\rho_v \quad \text{and} \quad \operatorname{Coker}\rho \simeq \bigoplus_{v\in S_{L^\dagger}} \operatorname{Coker}\rho_v\,.$$

Let $v \in S_{L^\dagger}$ and $\delta_v : \mathbb{Q}/\mathbb{Z} \to \bigoplus_{w\in S_{F^\dagger}(v)}\mathbb{Q}/\mathbb{Z}$ be the morphism $\lambda \mapsto \big([F^\dagger_w : L^\dagger_v]\,\lambda\big)_{w\in S_{F^\dagger}(v)}$. The kernel of $\delta_v$ is $\frac{1}{d_v}\mathbb{Z}/\mathbb{Z}$. Further $\operatorname{Coker}\delta_v = (\operatorname{Ker}\delta_v^\vee)^\vee$ is noncanonically isomorphic to $\#S_{F^\dagger}(v) - 1$ copies of $\mathbb{Q}/\mathbb{Z}$ since the kernel of the dual morphism $\delta_v^\vee : \bigoplus_{w\in S_{F^\dagger}(v)}\mathbb{Z} \to \mathbb{Z}$, $(n_w)_w \mapsto \sum_{w\in S_{F^\dagger}(v)}[F^\dagger_w : L^\dagger_v]\, n_w$ is noncanonically isomorphic to $\#S_{F^\dagger}(v) - 1$ copies of $\mathbb{Z}$. Pick an isomorphism $\operatorname{KS}\big(\mathbb{Q}/\mathbb{Z}\big) \simeq \mathbb{Q}/\mathbb{Z}$. Then lemma 3.6(i) shows that

$$\begin{cases} \operatorname{Ker}\rho_v \simeq \frac{1}{d_v}\mathbb{Z}/\mathbb{Z} \text{ and } \operatorname{Coker}\rho_v \simeq \big(\mathbb{Q}/\mathbb{Z}\big)^{\#S_{F^\dagger}(v)-1} & \text{if } v \in S_{L^\dagger}(L) \\ \operatorname{Ker}\rho_v = 0 \text{ and } \operatorname{Coker}\rho_v \simeq \big(\mathbb{Q}/\mathbb{Z}\big)^{\#S_{F^\dagger}(F,v)} & \text{if } v \notin S_{L^\dagger}(L) \end{cases}$$

where all the isomorphisms are noncanonical. The result follows by summing over the places $v \in S_{L^\dagger}$ and replacing the sum of $\#S_{F^\dagger}(F, v)$ copies of $\mathbb{Q}/\mathbb{Z}$ over those $v$ such that $v \notin S_{L^\dagger}(L)$ by the sum of $\mathbb{Q}/\mathbb{Z}$ over the $w \in S_{F^\dagger}$ such that $w \notin S_{F^\dagger}(L)$.

(ii) According to lemmas 3.5 and 3.6(ii) we have $\operatorname{Ker}\sigma = 0$ and the local restriction maps yield an isomorphism

$$\operatorname{Coker}\sigma \simeq \bigoplus_{v\in S^\infty_{L^\dagger}} \operatorname{Coker}\sigma_v\,.$$

Let $v \in S^\infty_{L^\dagger}$. We have $\#S_{F^\dagger}(v) = [F^\dagger : L^\dagger]$ since $F^\dagger$ is totally real, thus lemma 3.6(ii) shows that there is a noncanonical isomorphism

$$\operatorname{Coker}\sigma_v \simeq \Big(\tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\Big)^{[F^\dagger:L^\dagger]-1}.$$

The result follows by summing over the places $v \in S_{L^\dagger}^\infty$ and writing $\#S_{L^\dagger}^\infty\big([F^\dagger : L^\dagger] - 1\big) = [L^\dagger : \mathbb{Q}]\big([F^\dagger : L^\dagger] - 1\big) = [F^\dagger : \mathbb{Q}] - [L^\dagger : \mathbb{Q}]$. □

**Proposition 3.8.** *The commutative diagram* $(\star)$ *induces*

(i) *a short exact sequence*

$$1 \longrightarrow F_1/L_1 \longrightarrow H^0(\mathbb{Q}, T_1/S_1) \longrightarrow \operatorname{Ker}\iota \longrightarrow 1,$$

(ii) *a short exact sequence*

$$1 \longrightarrow \operatorname{Coker}\iota \longrightarrow H^1(\mathbb{Q}, T_1/S_1) \longrightarrow \operatorname{Ker}\rho \longrightarrow 1,$$

(iii) *an isomorphism* $H^2(\mathbb{Q}, T_1/S_1) \simeq \operatorname{Coker}\rho$*,*

(iv) *isomorphisms* $H^{2r-1}(\mathbb{Q}, T_1/S_1) \simeq \operatorname{Coker}\sigma$ *for all* $r \geq 2$*,*

(v) $H^{2r}(\mathbb{Q}, T_1/S_1) = 1$ *for all* $r \geq 2$*.*

*Proof.* When $r \geq 3$ localising at $\infty$ yields an isomorphism $H^r(\mathbb{Q}, T_1/S_1) \simeq H^r(\mathbb{R}, T_1/S_1)$ ([Mi] I.4.21) and proposition 2.3 together with lemmas 3.5 and 3.6(ii) give the result.

We have $H^1(\mathbb{Q}, T) \simeq H^1(\mathbb{Q}, \mathbb{G}_m) = 1$ by Shapiro's Lemma and Hilbert 90, and from the short exact sequence

$$1 \longrightarrow T_1 \longrightarrow T \xrightarrow{N_\dagger} T^\dagger \longrightarrow 1$$

we find that $H^1(\mathbb{Q}, T_1) \simeq F^{\dagger\times}/N_\dagger(F^\times)$. We further have $H^1(\mathbb{Q}, T^\dagger) = 1$ and $H^3(\mathbb{Q}, T) \simeq H^3(\mathbb{R}, T) \simeq H^1(\mathbb{R}, T) = 1$. Thus we also get the exact sequence

$$1 \longrightarrow H^2(\mathbb{Q}, T_1) \longrightarrow H^2(\mathbb{Q}, T) \longrightarrow H^2(\mathbb{Q}, T^\dagger) \longrightarrow H^3(\mathbb{Q}, T_1) \longrightarrow 1.$$

The canonical isomorphisms $H^2(\mathbb{Q}, T) \simeq H^2(F, \mathbb{G}_m) \simeq \operatorname{Br}(F)$ and $H^2(\mathbb{Q}, T^\dagger) \simeq \operatorname{Br}(F^\dagger)$ therefore induce isomorphisms $H^2(\mathbb{Q}, T_1) \simeq \operatorname{Ker}\operatorname{Cor}_{F/F^\dagger}$ and $H^3(\mathbb{Q}, T_1) \simeq \operatorname{Coker}\operatorname{Cor}_{F/F^\dagger}$. Similarly we find $H^1(\mathbb{Q}, S_1) \simeq L^{\dagger\times}/N_\dagger(L^\times)$, $H^2(\mathbb{Q}, S_1) \simeq \operatorname{Ker}\operatorname{Cor}_{L/L^\dagger}$ and $H^3(\mathbb{Q}, S_1) \simeq \operatorname{Coker}\operatorname{Cor}_{L/L^\dagger}$. Now from the short exact sequence $1 \to S_1 \to T_1 \to T_1/S_1 \to 1$ we obtain the exact sequence

$$\begin{array}{ccccccc}
1 & \longrightarrow & F_1/L_1 & \longrightarrow & H^0(\mathbb{Q}, T_1/S_1) & \longrightarrow & L^{\dagger\times}/N_\dagger(L^\times) \\
 & & & & & & \downarrow{\scriptstyle \iota} \\
 & & \operatorname{Ker}\operatorname{Cor}_{L/L^\dagger} & \longleftarrow & H^1(\mathbb{Q}, T_1/S_1) & \longleftarrow & F^{\dagger\times}/N_\dagger(F^\times) \\
 & & \downarrow{\scriptstyle \rho} & & & & \\
 & & \operatorname{Ker}\operatorname{Cor}_{F/F^\dagger} & \longrightarrow & H^2(\mathbb{Q}, T_1/S_1) & \longrightarrow & \operatorname{Coker}\operatorname{Cor}_{L/L^\dagger} \xrightarrow{\sigma} \operatorname{Coker}\operatorname{Cor}_{F/F^\dagger}.
\end{array}$$

As $\sigma$ is injective (corollary 3.7(ii)) the statements on $H^r(\mathbb{Q}, T_1/S_1)$ for $r = 0, 1, 2$ follow. □

## 4. Local and global

For an algebraic torus $\mathcal{T}$ over $\mathbb{Q}$ we have $H^r\big(\mathbb{Q},\mathcal{T}(\mathbb{A})\big)=\bigoplus_{\ell\in S_{\mathbb{Q}}} H^r(\mathbb{Q}_\ell,\mathcal{T})$ when $r\geq 1$, and for all $r\geq 0$ we let

$$Ш^r(\mathbb{Q},\mathcal{T}) \underset{\text{def}}{=} \operatorname{Ker}\Big(H^r(\mathbb{Q},\mathcal{T})\to H^r\big(\mathbb{Q},\mathcal{T}(\mathbb{A})\big)\Big).$$

Clearly $Ш^0(\mathbb{Q},\mathcal{T})=1$. It is known that $Ш^1(\mathbb{Q},\mathcal{T})$ is finite ([Pl-Ra] Corollary to Prop.6.9), and applying [Mi] I.4.20(a) to $X^*(\mathcal{T})$ we see that $Ш^2(\mathbb{Q},\mathcal{T})$ is finite too. For $r\geq 3$ we have $H^r\big(\mathbb{Q},\mathcal{T}(\mathbb{A})\big)=H^r\big(\mathbb{R},\mathcal{T}\big)$ since $G_\ell$ has cohomological dimension 2 when $\ell\neq\infty$, and the local restriction map $H^r\big(\mathbb{Q},\mathcal{T}\big)\to H^r\big(\mathbb{R},\mathcal{T}\big)$ is an isomorphism ([Mi] I.4.21). Thus $Ш^r(\mathbb{Q},\mathcal{T})=1$ when $r\geq 3$.

**Remark 4.1.** Let $K$ be a finite Galois extension of $\mathbb{Q}$ and let

$$Ш^r(K/\mathbb{Q},\mathcal{T}) \underset{\text{def}}{=} \operatorname{Ker}\Big(H^r(K/\mathbb{Q},\mathcal{T})\to H^r\big(K/\mathbb{Q},\mathcal{T}(\mathbb{A})\big)\Big).$$

Assume that $K$ is a splitting field for $\mathcal{T}$. Then $H^1(K,\mathcal{T})=H^1\big(K,\mathcal{T}(\mathbb{A})\big)=1$ by Hilbert 90, so the initial segment of the Hochschild-Serre exact sequence gives isomorphisms $H^1(K/\mathbb{Q},\mathcal{T})\simeq H^1(\mathbb{Q},\mathcal{T})$ and $H^1\big(K/\mathbb{Q},\mathcal{T}(\mathbb{A})\big)\simeq H^1\big(\mathbb{Q},\mathcal{T}(\mathbb{A})\big)$. Hence

$$Ш^1(K/\mathbb{Q},\mathcal{T})=Ш^1(\mathbb{Q},\mathcal{T}).$$

Again by $H^1(K,\mathcal{T})=H^1\big(K,\mathcal{T}(\mathbb{A})\big)=1$, Hochschild-Serre gives the commutative diagram with exact rows

$$\begin{array}{ccccccc}
1 \longrightarrow & H^2(K/\mathbb{Q},\mathcal{T}) & \longrightarrow & H^2(\mathbb{Q},\mathcal{T}) & \longrightarrow & H^2(K,\mathcal{T}) \\
& \downarrow & & \downarrow & & \downarrow \\
1 \longrightarrow & H^2\big(K/\mathbb{Q},\mathcal{T}(\mathbb{A})\big) & \longrightarrow & H^2\big(\mathbb{Q},\mathcal{T}(\mathbb{A})\big) & \longrightarrow & H^2\big(K,\mathcal{T}(\mathbb{A})\big)
\end{array}$$

which yields the exact sequence

$$1\longrightarrow Ш^2(K/\mathbb{Q},\mathcal{T})\longrightarrow Ш^2(\mathbb{Q},\mathcal{T})\longrightarrow Ш^2(K,\mathcal{T}).$$

We have isomorphisms $H^2(K,\mathcal{T})\simeq \operatorname{Br}(K)^d$ and $H^2\big(K,\mathcal{T}(\mathbb{A})\big)\simeq\bigoplus_{v\in S_K}\operatorname{Br}(K_v)^d$ with $d=\dim\mathcal{T}$, so by global class field theory $Ш^2(K,\mathcal{T})=1$. Hence

$$Ш^2(K/\mathbb{Q},\mathcal{T})=Ш^2(\mathbb{Q},\mathcal{T}).$$

**Proposition 4.2.** *We have* $Ш^r(\mathbb{Q},T_1/S_1)=1$ *for all* $r$.

*Proof.* By remark 4.1 and [Pl-Ra] Prop.6.12 we have $Ш^2(\mathbb{Q},T_1/S_1)=1$ since $T_1/S_1$ is $\mathbb{Q}_p$-anisotropic. The short exact sequence $1\to S_1\to T_1\to T_1/S_1\to 1$ induces the commutative diagram with exact rows

$$\begin{array}{ccccccc}
H^1(\mathbb{Q},S_1) & \longrightarrow & H^1(\mathbb{Q},T_1) & \longrightarrow & H^1(\mathbb{Q},T_1/S_1) & \longrightarrow & H^2(\mathbb{Q},S_1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1\big(\mathbb{Q},S_1(\mathbb{A})\big) & \longrightarrow & H^1\big(\mathbb{Q},T_1(\mathbb{A})\big) & \longrightarrow & H^1\big(\mathbb{Q},T_1/S_1(\mathbb{A})\big) & \longrightarrow & H^2\big(\mathbb{Q},S_1(\mathbb{A})\big).
\end{array}$$

As in the proof of proposition 3.8 we have $H^1(\mathbb{Q},T_1) \simeq F^{\dagger\times}/N_\dagger(F^\times)$, $H^1\big(\mathbb{Q},T_1(\mathbb{A})\big) \simeq T^\dagger(\mathbb{A})/N_\dagger T(\mathbb{A})$, and similarly for $S_1$. Thus the above diagram yields

$$\begin{array}{ccccccc} 1 & \longrightarrow & \operatorname{Coker}\iota & \longrightarrow & H^1(\mathbb{Q},T_1/S_1) & \longrightarrow & H^2(\mathbb{Q},S_1) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \bigoplus_{v\in S_{L^\dagger}}\operatorname{Coker}\iota_v & \longrightarrow & H^1\big(\mathbb{Q},T_1/S_1(\mathbb{A})\big) & \longrightarrow & H^2\big(\mathbb{Q},S_1(\mathbb{A})\big). \end{array}$$

As $S_1$ is $\mathbb{Q}_p$-anisotropic we have $Ш^2(\mathbb{Q},S_1)=1$, and the map $\operatorname{Coker}\iota \to \bigoplus_{v\in S_{L^\dagger}}\operatorname{Coker}\iota_v$ is injective by lemma 3.1. Therefore $Ш^1(\mathbb{Q},T_1/S_1)=1$. □

For a $\mathbb{Q}$-torus $\mathcal{T}$ let $C(\mathcal{T})=\mathcal{T}(\mathbb{A})/\mathcal{T}(\overline{\mathbb{Q}})$ be its adèle class group over $\overline{\mathbb{Q}}$ and $C_{\mathbb{Q}}(\mathcal{T}) = \mathcal{T}(\mathbb{A}_{\mathbb{Q}})/\mathcal{T}(\mathbb{Q})$ the one over $\mathbb{Q}$.

**Lemma 4.3.** *The commutative diagram* $(\star)$ *induces*

(i) $H^0\big(\mathbb{Q},C(T_1)\big)=C_{\mathbb{Q}}(T_1)$ *and* $H^1\big(\mathbb{Q},C(T_1)\big)\simeq F_p^{\dagger\times}/N_\dagger(F_p^\times)$,
(ii) $H^0\big(\mathbb{Q},C(T_1/S_1)\big)=C_{\mathbb{Q}}(T_1/S_1)$ *and* $H^1\big(\mathbb{Q},C(T_1/S_1)\big)\simeq \operatorname{Coker}\iota_p$.

*Proof.* The isomorphisms $H^1(\mathbb{Q},T_1)\simeq F^{\dagger\times}/N_\dagger(F^\times)=\hat{H}^0(\Gamma,F^\times)$ and $H^1\big(\mathbb{Q},T_1(\mathbb{A})\big)\simeq I_{F^\dagger}/N_\dagger(I_F)=\hat{H}^0(\Gamma,I_F)$ together with $\hat{H}^{-1}(\Gamma,C_F)=1$ show that $Ш^1(\mathbb{Q},T_1)=1$. Hence there is a short exact sequence

$$1 \longrightarrow T_1(\mathbb{Q}) \longrightarrow T_1(\mathbb{A}_{\mathbb{Q}}) \longrightarrow H^0\big(\mathbb{Q},C(T_1)\big) \longrightarrow 1$$

so $H^0\big(\mathbb{Q},C(T_1)\big)=T_1(\mathbb{A}_{\mathbb{Q}})/T_1(\mathbb{Q})=C_{\mathbb{Q}}(T_1)$. Similarly $H^0\big(\mathbb{Q},C(T_1/S_1)\big)=C_{\mathbb{Q}}(T_1/S_1)$ since $Ш^1(\mathbb{Q},T_1/S_1)=1$ by proposition 4.2.

From $1\to C(T_1)\to C(T)\xrightarrow{N_\dagger} C(T^\dagger)\to 1$ and $H^1\big(\mathbb{Q},C(T)\big)=H^1(F,C)=1$ we obtain an isomorphism $H^1\big(\mathbb{Q},C(T_1)\big)\simeq C_{\mathbb{Q}}(T^\dagger)/N_\dagger C_{\mathbb{Q}}(T)=C_{F^\dagger}/N_\dagger(C_F)$, and as in the proof of lemma 3.1 we have $C_{F^\dagger}/N_\dagger(C_F)\simeq F_p^{\dagger\times}/N_\dagger(F_p^\times)$. Similarly $H^1\big(\mathbb{Q},C(S_1)\big)\simeq L_p^{\dagger\times}/N_\dagger(L_p^\times)$, so from $1\to C(S_1)\to C(T_1)\to C(T_1/S_1)\to 1$ we deduce the exact sequence

$$L_p^{\dagger\times}/N_\dagger(L_p^\times) \xrightarrow{\iota_p} F_p^{\dagger\times}/N_\dagger(F_p^\times) \longrightarrow H^1\big(\mathbb{Q},C(T_1/S_1)\big) \longrightarrow H^2\big(\mathbb{Q},C(S_1)\big).$$

By the global Nakayama-Tate theorem $H^2\big(\mathbb{Q},C(S_1)\big)\simeq H^0\big(\mathbb{Q},X^*(S_1)\big)^{\wedge\vee}$ ([Mi] I.4.7) and $H^0\big(\mathbb{Q},X^*(S_1)\big)=0$ since $S_1$ is $\mathbb{Q}$-anisotropic (lemma 1.1), hence $H^2\big(\mathbb{Q},C(S_1)\big)=1$. Therefore $H^1\big(\mathbb{Q},C(T_1/S_1)\big)\simeq\operatorname{Coker}\iota_p$. □

For a $\mathbb{Q}$-torus $\mathcal{T}$ recall that there is a Haar measure $\tau$ on $\mathcal{T}(\mathbb{A}_{\mathbb{Q}})$ called the Tamagawa measure (see [Pl-Ra] 3.5 and 5.3). When it exists, the invariant volume of $C_{\mathbb{Q}}(\mathcal{T}) = \mathcal{T}(\mathbb{A}_{\mathbb{Q}})/\mathcal{T}(\mathbb{Q})$ with respect to $\tau$ is called the Tamagawa number of $\mathcal{T}$ and is denoted $\tau(\mathcal{T})$.

**Theorem 4.4.** *The rational class group $C_{\mathbb{Q}}(T_1/S_1)$ is compact and has finite invariant volume*

$$\tau(T_1/S_1)=\#\operatorname{Ker}\iota_p\,.$$

*Proof.* Since $T_1/S_1$ is $\mathbb{Q}$-anisotropic (corollary 1.2) $C_{\mathbb{Q}}(T_1/S_1)$ is compact and has finite invariant volume ([Pl-Ra] Thm.5.5). Ono's theorem [On] gives the formula

$$\tau(T_1/S_1) = \frac{\# H^1\big(\mathbb{Q}, X^*(T_1/S_1)\big)}{\# Ш^1(\mathbb{Q}, T_1/S_1)}.$$

There are isomorphisms $H^1\big(\mathbb{Q}, X^*(T_1/S_1)\big) \simeq H^1\big(\mathbb{Q}, C(T_1/S_1)\big)^\vee \simeq \operatorname{Coker} \iota_p{}^\vee$ by the global Nakayama-Tate theorem and lemma 4.3(ii), and $Ш^1(\mathbb{Q}, T_1/S_1) = 1$ by proposition 4.2. Hence $\tau(T_1/S_1) = \# \operatorname{Coker} \iota_p = \# \operatorname{Ker} \iota_p$. □

**Theorem 4.5.** *The commutative diagram* $(\star)$ *induces*

(i) *a short exact sequence*

$$1 \longrightarrow C_{\mathbb{Q}}(T_1)/C_{\mathbb{Q}}(S_1) \longrightarrow C_{\mathbb{Q}}(T_1/S_1) \longrightarrow \operatorname{Ker} \iota_p \longrightarrow 1,$$

(ii) *a short exact sequence*

$$1 \longrightarrow H^1(\mathbb{Q}, T_1/S_1) \longrightarrow H^1\big(\mathbb{Q}, T_1/S_1(\mathbb{A})\big) \longrightarrow \operatorname{Coker} \iota_p \longrightarrow 1,$$

(iii) *isomorphisms* $H^r(\mathbb{Q}, T_1/S_1) \simeq H^r\big(\mathbb{Q}, T_1/S_1(\mathbb{A})\big)$ *for all* $r \geq 2$.

*Proof.* From the short exact sequence $1 \to C(S_1) \to C(T_1) \to C(T_1/S_1) \to 1$ and lemma 4.3(i) we deduce the exact sequence

$$1 \longrightarrow C_{\mathbb{Q}}(S_1) \longrightarrow C_{\mathbb{Q}}(T_1) \longrightarrow C_{\mathbb{Q}}(T_1/S_1) \longrightarrow L_p^{\dagger\times}/N_\dagger(L_p^\times) \xrightarrow{\iota_p} F_p^{\dagger\times}/N_\dagger(F_p^\times)$$

from which (i) follows.

By the Poitou-Tate theorem as in [Mi] I.4.20, for $r \geq 3$ the localisation maps yield isomorphisms $H^r(\mathbb{Q}, T_1/S_1) \simeq H^r\big(\mathbb{Q}, T_1/S_1(\mathbb{A})\big) = H^r(\mathbb{R}, T_1/S_1)$, and we have an exact sequence

$$\begin{array}{ccccc} H^1\big(\mathbb{Q}, X^*(T_1/S_1)\big)^\vee & \longleftarrow & H^1\big(\mathbb{Q}, T_1/S_1(\mathbb{A})\big) & \longleftarrow & H^1(\mathbb{Q}, T_1/S_1) \\ \downarrow & & & & \\ H^2(\mathbb{Q}, T_1/S_1) & \longrightarrow & H^2\big(\mathbb{Q}, T_1/S_1(\mathbb{A})\big) & \longrightarrow & H^0\big(\mathbb{Q}, X^*(T_1/S_1)\big)^\vee \longrightarrow 1. \end{array}$$

Proposition 4.2 shows that $Ш^1(\mathbb{Q}, T_1/S_1) = Ш^2(\mathbb{Q}, T_1/S_1) = 1$, the global Nakayama-Tate theorem and lemma 4.3(ii) that $H^1\big(\mathbb{Q}, X^*(T_1/S_1)\big)^\vee \simeq \operatorname{Coker} \iota_p$, and corollary 1.2 that $H^0\big(\mathbb{Q}, X^*(T_1/S_1)\big) = 0$. The statements in (ii) and (iii) follow. □

**Corollary 4.6.** *We have*

$$\begin{cases} \tau(T_1/S_1) = 1 \text{ and } H^1(\mathbb{Q}, T_1/S_1) \simeq H^1\big(\mathbb{Q}, T_1/S_1(\mathbb{A})\big) & \text{when } [F:L] \text{ is odd,} \\ \tau(T_1/S_1) = 2 \text{ and } \#\Big(H^1\big(\mathbb{Q}, T_1/S_1(\mathbb{A})\big)/H^1(\mathbb{Q}, T_1/S_1)\Big) = 2 & \text{when } [F:L] \text{ is even.} \end{cases}$$

*Proof.* Combine theorems 4.4 and 4.5 with lemma 2.1. □

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