HENSELIAN RESIDUALLY p-ADICALLY CLOSED FIELDS

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Abstract. In *(Arch. Math. 57 (1991), pp. 446–455)*, R. Farré proved a positivstellensatz for real-series closed fields. Here we consider p-valued fields $\langle K, v_p\rangle$ with a non-trivial valuation v which satisfies a compatibility condition between v_p and v. We use this notion to establish the p-adic analogue of real-series closed fields; these fields are called *henselian residually p-adically closed fields*. First we solve a Hilbert's Seventeenth problem for these fields and then, we introduce the notions of residually p-adic ideal and residually p-adic radical of an ideal in the ring of polynomials in n indeterminates over a henselian residually p-adically closed field. Thanks to these two notions, we prove a Nullstellensatz theorem for this class of valued fields. We finish the paper with the study of the differential analogue of henselian residually p-adically closed fields. In particular, we give a solution to a Hilbert's Seventeenth problem in this setting.

Keywords: Henselian residually p-adically closed fields, model completeness, Hilbert's Seventeenth problem, residually p-adic ideal, Nullstellensatz, valued D-fields.

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1. Introduction

Let us recall that a valued field is a field K equipped with a surjective map $v : K \to \Gamma \cup \{\infty\}$, where $\Gamma := v(K^\times)$ is a totally ordered abelian group and v satisfies the following properties:

- $v(x) = \infty \iff x = 0$,
- $v(xy) = v(x) + v(y)$,
- $v(x + y) \geqslant \min\{v(x), v(y)\}$.

The subring $\mathcal{O}_K := \{x \in K | v(x) \geqslant 0\}$ of K is called the valuation ring of $\langle K, v\rangle$, the value group is $v(K^\times)$, the residue field of K is $k_K := \mathcal{O}_K/\mathcal{M}_K$ where $\mathcal{M}_K := \{x \in K | v(x) > 0\}$ is the maximal ideal of $\mathcal{O}_K$ and the canonical residue map is denoted by $\pi : \mathcal{O}_K \longmapsto k_K$. If K is a field equipped with two valuations v and w then we add a subscript v in order to distinguish the valuations rings, maximal ideals, residue fields and residue maps, respectively, of the valuation v with those of w (i.e. $\mathcal{O}_{K,v}$, $\mathcal{M}_{K,v}$, $k_{K,v}$ and π_v). Moreover if $\langle K, v\rangle$ is a valued field with an element of minimal positive value then that element is denoted by 1.

To each valuation defined on K we can associate a binary relation $\mathcal{D}$ which is interpreted by the set of 2-tuples (a, b) of K^2 such that $v(a) \leqslant v(b)$. So this relation $\mathcal{D}$ satisfies the following properties $(\star)$:

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- $\mathcal{D}$ is transitive, $\neg\mathcal{D}(0,1)$,
- $\mathcal{D}$ is compatible with $+$ and $.$,
- and either $\mathcal{D}(a,b)$ or $\mathcal{D}(b,a)$ for all $a, b \in K$.

Such a relation is called *a linear divisibility relation* (a l.d. relation).

If A is a domain with fraction field K and $\mathcal{D}$ is a relation which satisfies the properties $(\star)$ then, by extending $\mathcal{D}$ to K as follows:

$$\mathcal{D}(\frac{a}{b}, \frac{c}{d}) \iff \mathcal{D}(ad, bc) \text{ with } a, b, c, d \in A \text{ and } b, d \neq 0,$$

we get that the l.d. relation $\mathcal{D}$ on K induces a valuation v on K by defining $v(a) \leqslant v(b)$ if $\mathcal{D}(a,b)$. As for the valuation rings, we add a subscript v to its corresponding l.d. relation $\mathcal{D}_v$ if necessary.

If $\langle K, v\rangle$ is a valued field then we denote its Henselization by $\langle K^h, v^h\rangle$. For general valuation theory, the reader can be refer to [15].

In this paper, we are dealing with notions of p-valued fields, p-valuations and p-adically closed fields which are all assumed of p-rank 1 for some prime number p following the terminology of [14]. We are interested in henselian residually p-adically closed fields which is the p-adic counterpart of real-series closed fields (see [4] and [5, chapter 1] for a brief history of results about real-series closed fields).

First we define a theory analogous to the theory of real-series closed fields in a language including divisibility predicates $\mathcal{D}_{v_p}$ and $\mathcal{D}_v$. Each divisibility predicate corresponds to a valuation and these two valuations are connected with a compatibility condition as introduced in [7, Definition 2.2]. This theory is denoted by $HRpCF$ and its models are called henselian residually p-adically closed fields.

Then we prove an analogue of the Hilbert's Seventeenth problem for henselian residually p-adically closed fields by using the same ideas as in [4]. We introduce the field analogue of the notion of M-Kochen ring which was considered in Section 3 of [7] for valued domains. It allows us to characterize the intersection of the valuation rings of p-valuations which extend a fixed p-valuation v_p such that $v_p(M) \geqslant 0$ for some particular subset M. Since we want to use a model completeness result, we have to identify the subset M which is required in the solution of this problem for henselian residually p-adically closed fields.

In the third section, we follow the lines of the work [17] in order to prove a Nullstellensatz theorem for henselian residually p-adically closed fields. To this effect, we define the notions of residually p-adic ideal and residually p-adic radical of an ideal in the polynomial ring in n indeterminates over a model of $HRpCF$. Generally it suffices to adapt the proofs of [17] by replacing the role of the classical Kochen ring by our M-Kochen ring.

Finally, in the last section, we study a special class of D-henselian valued fields (first considered in [16]) which uses the results of [6] and [8]. In [6], we established and axiomatized the model-companion of the theory of differential p-valued fields which is denoted by $pCDF$ and whose models are called p-adically closed differential fields. It is a p-adic adaptation of the theory of closed ordered differential field (see [18]) which is denoted by $CODF$. In [8], we study D-henselian valued fields with residue differential field which is a model of $CODF$ and with a $\mathbb{Z}$-group as value group, i.e. a differential analogue of the theory of real-series closed fields. In particular, we prove a positivstellensatz result for these D-henselian valued fields.

Here we adapt these results to the p-adic case by using $pCDF$, i.e. we are interested in the valued D-field analogue of $HRpCF$. So we prove a Hilbert's Seventeenth problem for D-henselian valued fields whose residue field is a model of $pCDF$ and whose value group is a $\mathbb{Z}$-group. The model-theoretic tool that we need is a theorem of quantifier elimination in [8]; it enables us to prove the model completeness of the theory of these D-henselian valued fields in a suitable language by using linear divisibility predicates.

2. Hilbert's seventeenth problem for henselian residually p-adically closed fields

We begin this section with a notion which is the p-adic analogue of the convexity of a valuation in the case of real-series closed fields.

Definition 2.1. Let $\langle K, v_p, v\rangle$ be a p-valued field with v_p its p-valuation and v a non-trivial valuation on K. We say that v is *compatible with* v_p if the following holds

$$\forall x, y\,[v_p(x) \leqslant v_p(y) \Rightarrow v(x) \leqslant v(y)].$$

Let us recall a well-known fact on p-valued fields.

Lemma 2.2. *Let $\langle K, v_p\rangle$ be a p-valued field and let x be an element of K. If there exists an element y in K such that $y^\epsilon = 1 + px^\epsilon$, with $\epsilon = 2$ if $p \neq 2$ and $\epsilon = 3$ otherwise, then $v_p(x) \geqslant 0$. Conversely if $\langle K, v_p\rangle$ is henselian and $v_p(x) \geqslant 0$ then there exists an element y in K such that $y^\epsilon = 1 + px^\epsilon$ with ϵ as before.*

Proof. See Lemma 1.5 in [1]. □

Lemma 2.3. *Let $\langle K, v_p\rangle$ be a p-valued field and let v be a non-trivial henselian valuation on K with residue field $k_{K,v}$ of characteristic zero. Then v is compatible with v_p.*

Proof. Let x, y in K be such that $v(x) < v(y)$. Hence $\frac{y}{p.x} \in \mathcal{M}_{K,v}$ since the characteristic of $k_{K,v}$ is zero. Let us consider the polynomial $f(X) = X^\epsilon - (1 + p\cdot(\frac{y}{p.x})^\epsilon)$ with ϵ as in Lemma 2.2. So $f(X)$ has coefficients in $\mathcal{O}_{K,v}$. Moreover $\pi(f)(X)$ is equal to $X^\epsilon - 1$; hence 1 is a simple residue root of $f(X)$. By Hensel's Lemma applied to v, $f(X)$ has a root z such that $\pi(z) = 1$. So, by Lemma 2.2, we get that $v_p(p.x) \leqslant v_p(y)$, which implies $v_p(x) < v_p(y)$. □

Now we recall some definitions and results from [7], namely the notions of p-valued and p-convexly valued domains. It is useful in the next theorems for the following reasons:

- if $\langle K, v_p, v\rangle$ is a p-valued field with v a non-trivial valuation on K then $\mathcal{O}_{K,v}$ is a p-valued domain,
- moreover, if v is compatible with v_p and $char(k_{K,v}) = 0$ then $\mathcal{O}_{K,v}$ is a p-convexly valued domain.

Definition 2.4. Let A be a domain containing $\mathbb{Q}$. We say that A is *a p-valued domain* if A is not a field and its fraction field $Q(A)$ is p-valued.

Definition 2.5. Let F be a p-valued field with v_p its p-valuation and let $A \subseteq B$ be two subsets of F. We say that A is *p-convex in B* if for all $a \in A$ and $b \in B$, $v_p(a) \leqslant v_p(b)$ implies $b \in A$.

With our terminology, we can state easy results.

Lemma 2.6. *Let $\langle F, v_p \rangle$ be a p-valued field and let A be a p-valued domain which is p-convex in F. Then A is a valuation ring and $F = Q(A)$.*

Proof. See Lemma 2.3 in [7]. □

Notation 2.7. In the sequel, if A is a valuation ring then we denote the maximal ideal and the residue field of A by $\mathcal{M}_A$ and k_A respectively. The previous lemma shows that any p-convex subdomain A of a p-valued field F supports a valuation v which corresponds to a l.d. relation $\mathcal{D}_v$ on the domain A. So the notations $\mathcal{M}_A$ and k_A are always relative to this valuation v. If A is a ring then we denote by $A^\times$ the set of units of A and if B is a subset of A then we denote by $B^\bullet$ the set $B \setminus \{0\}$.

Definition 2.8. A *p-convexly valued domain* A is a p-valued domain such that A is a valuation ring and $\mathcal{M}_A$ is p-convex in A.

Remark 2.9. Equivalent properties characterize p-convexly valued domains A (see Lemma 2.5 of [7]); for example,

$$A \models \forall x, y \left(v_p(x) \leqslant v_p(y) \rightarrow \exists z(xz = y)\right),$$

which motivates Definition 2.1.

Another equivalent property is that A is a valuation ring and for every $a \in \mathcal{M}_A$, $v_p(a) > 0$.

Let $\mathcal{L}_p$ be an expansion of the language of rings $\mathcal{L}_{\text{rings}} \cup \{\mathcal{D}_{v_p}, \mathcal{D}_v\}$ such that $\mathcal{D}_{v_p}$ will be interpreted as a l.d. relation with respect to a p-valuation v_p and $\mathcal{D}_v$ as a l.d. relation with respect to a valuation v. The $\mathcal{L}_p$-theory of p-convexly valued domains is denoted by $pCVR$. An axiomatization of $pCVR$ in $\mathcal{L}_p$ can be found in Section 2 of [7].

Now we recall a part of Lemma 2.9 in [7].

Lemma 2.10. *Let $\mathcal{A}$, $\mathcal{B}$ be two $\mathcal{L}_p$-structures which are models of $pCVR$ and B is a p-convexly valued domain extension of A (i.e. $\langle A, \mathcal{D}_{v_p} \rangle \subseteq \langle B, \mathcal{D}_{v_p} \rangle$ or $Q(A) \subseteq Q(B)$ as p-valued fields). Then the following are equivalent:*

(1) $\mathcal{A} \subseteq_{\mathcal{L}_p} \mathcal{B}$*;*
(2) $A \cap \mathcal{M}_B = \mathcal{M}_A$*;*

Remark 2.11. By Lemma 2.10 in [7], we know that if A is a p-convexly valued domain then $v_p(A^\times)$ is a convex subgroup of $v_p(Q(A)^\times)$. Hence if A is a p-convexly valued domain then, by p-convexity of $\mathcal{M}_A$ in A, we have $v_p(A^\times) < v_p(\mathcal{M}_A)$.

So we can define a p-valuation on the residue field k_A of A, denoted by $\widetilde{v}_p$, as follows:

- if $x = 0$ in k_A then $\widetilde{v}_p(x) = \infty$;
- otherwise if $x \neq 0$ in k_A, we take $y \in A^\times$ such that $\pi_v(y) = x$ and define $\widetilde{v}_p(x)$ as $v_p(y)$ (where v is the valuation with respect to A).

Since $v_p(A^\times) < v_p(\mathcal{M}_A)$, $\widetilde{v}_p$ is well-defined and $\langle k_A, \widetilde{v}_p\rangle$ is a p-valued field.

The two next lemmas will allow us to extend p-convexly valued domains in the most natural way as possible.

Lemma 2.12. *Let A be a p-valued domain and let $\langle K, v_p\rangle$ be a p-valued field extension of $Q(A)$ such that there exists an element of K of value lower than $v_p(A^\bullet)$.*

Then there exists a minimal p-convexly valued domain $pcH(A,K)$ containing A whose fraction field is K. Furthermore, if A is a p-convexly valued domain then $A \subseteq_{\mathcal{L}_p} pcH(A,K)$.

Proof. See Lemma 2.14 in [7] where $pcH(A,K)$ is defined as follows

$$\big\{k \in K : \quad \exists a \in A \text{ such that } K \models v_p(a) \leqslant v_p(k)\big\}.$$

□

Lemma 2.13. *Let A be a p-convexly valued domain and let $\widetilde{Q(A)}$ be a p-adic closure of $Q(A)$ for the p-valuation v_p on $Q(A)$.*

Then there exists a p-convexly valued domain $\widetilde{A}$ such that

- *$A \subseteq_{\mathcal{L}_p} \widetilde{A}$, the valuation v with respect to $\widetilde{A}$ is henselian,*
- *its residue field $k_{\widetilde{A}}$ is p-adically closed, its value group is divisible*
- *and its fraction field is $\widetilde{Q(A)}$.*

Proof. See Lemma 2.15 in [7]. □

Now we recall the definition of the Kochen's operator which plays an important role in the characterization of p-valued field extensions (see Chapter 6 in [14]).

Definition 2.14. The following operator $\gamma_p(X)$ is called the *Kochen's operator*:

$$\gamma_p(X) = \frac{1}{p} \cdot \frac{X^p - X}{(X^p - X)^2 - 1}\ .$$

Let us introduce the notion of M-Kochen ring defined in Definition 3.6 in [7]. It yields, in Theorem 2.21, a characterization of the intersection of the valuation rings of p-valuations which extend a given p-valuation v_p such that $v_p(M) \geqslant 0$ for some particular subset M.

Definition 2.15. For any field extension L of a p-valued $\langle K, v_p\rangle$ and any subset M of L, the *M-Kochen ring* $R^M_{\gamma_p}(L)$ is defined as the subring of L consisting of quotients of the form

$$a = \frac{b}{1+pd} \text{ with } b, d \in \mathcal{O}_{K,v_p}[\gamma_p(L), M] \text{ and } 1 + pd \neq 0$$

where $\mathcal{O}_{K,v_p}[\gamma_p(L), M]$ denotes the subring of L generated by $\gamma_p(L) \setminus \{\infty\}$ and M over the ring $\mathcal{O}_{K,v_p}$.

Remark 2.16. If $\langle K, v_p\rangle$ is a henselian p-valued field then $\mathcal{O}_{K,v_p}$ is equal to $\gamma_p(K)$ (see Remark 1 in [11]). In this case, the elements of the M-Kochen ring $R^M_{\gamma_p}(L)$ (for a field extension L of K) have the following form $a = \frac{b}{1+pd}$ with $b, d \in \mathbb{Z}[\gamma_p(L), M]$ and $1 + pd \neq 0$. Let us note that the fraction field of $R^M_{\gamma_p}(L)$ is L (see Merckel's Lemma in [14, Appendix]).

Definition 2.17. Let $\mathcal{L}_{p,a}$ be the following language $\mathcal{L}_p \cup \{a\}$. Let $\langle K, v_p, v, a\rangle$ be a p-valued field with v_p its p-valuation, a non-trivial valuation v on K and a distinguished element a of K.

We say that K is *a henselian residually p-adically closed field* if $v(K^\times)$ is a $\mathbb{Z}$-group with $v(a) = 1$, v is henselian and its residue field $\langle k_{K,v}, \widetilde{v}_p\rangle$ is p-adically closed (see Remark 2.11).

We will denote this $\mathcal{L}_{p,a}$-theory $Th(K)$ by $HRpCF$.

Clearly, a canonical model of $HRpCF$ is the field of Laurent series over $\mathbb{Q}_p$, denoted by $\mathbb{Q}_p((t))$ (t plays the role of the distinguished element a).

Remark 2.18. More generally if we consider a p-adically closed field K with its p-valuation v_p then we can obtain a henselian p-adically closed field by considering the field of Laurent series $K((t))$ over K with its t-adic valuation compatible with the following natural p-valuation w_p: for any $f := \sum_{i\geqslant z} f_i t^i$ with $f_z \neq 0$, we define $w_p(f) := (z, v_p(f_z)) \in \mathbb{Z} \times v_p(K^\times)$, lexicographically ordered.

Let us consider a p-valued field $\langle K, v_p\rangle$ with its p-valuation v_p henselian and let assume moreover that its value group contains a non-trivial smallest convex subgroup G such that $v(K^\times)/G$ (equipped with its induced ordering) has a smallest positive element. Then $\langle K, v_p, w\rangle$ can be extended to a model of $HRpCF$ where w is the coarse valuation with respect to G. It suffices for this to apply Lemma 2.23 like in Theorem 2.28.

In the theory of henselian residually p-adically closed fields, another operator $\gamma(X)$ (defined in the following lemma) will play an important role as the one of Kochen's operator in the p-adic field case (see [11]). It enables us to determine whenever an element of the maximal ideal of a valued field $\langle K, v\rangle$ has the least positive value.

Lemma 2.19. *Let $\langle K, v\rangle$ be a valued field and let a be a non-zero element of K. Let γ be the operator defined by $\gamma(X) = \frac{X}{X^2-a}$. Then the following are equivalent:*

(1) $v(a) = 1$,
(2) $\gamma(K) \subseteq \mathcal{O}_K$ *and* $a \in \mathcal{M}_K$.

Proof. See Lemma 2.3 in [4]. □

Lemma 2.20. *Let $\langle K, v\rangle$ be a henselian valued field such that $v(a) = 1$. Then $\mathcal{O}_{K,v} = \gamma(K)$.*

Proof. By Lemma 2.19, we have that $\gamma(K) \subseteq \mathcal{O}_K$ and $a \in \mathcal{M}_K$. Let y be in $\mathcal{O}_K$ and let us consider the polynomial $f(X) = X - y(X^2 - a)$. Then 0 is a simple residue root of f and by Hensel's Lemma, there exists an element x of $\mathcal{O}_{K,v}$ such that $f(x) = 0$; hence $y = \gamma(x)$. □

This is the content of Theorem 3.11 in [7].

Theorem 2.21. *Let L be a field extension of a p-valued field $\langle K, v_p\rangle$ and let M be a subset of L such that $v_p((M \cap K)^\bullet) \geqslant 0$. Assume that there exists a p-valuation w_p on L such that $M \subseteq \mathcal{O}_{L,w_p}$.*

Then the subring $R^M_{\gamma_p}(L)$ of L is the intersection of the valuation rings $\mathcal{O}_{L,v}$ where v ranges over the p-valuations of L which extend the one of K such that M belongs to $\mathcal{O}_{L,v}$.

The two next lemmas allow us to extend the $\mathcal{L}_p$-structure of a p-valued field $\langle K, v_p, v\rangle$ with a valuation v compatible with v_p to particular valued field extensions $\langle L, w\rangle$.

In the following proofs, we use the notations of Remark 2.11: if $\langle K, v_p, v\rangle$ is a p-valued field such that the non-trivial valuation v is compatible with v_p and $char(k_{K,v}) = 0$ then $\mathcal{O}_{K,v}$ is a p-convexly valued domain and $\langle k_{\mathcal{O}_{K,v}}, \widetilde{v}_p\rangle$ is p-valued. Let us note that $k_{\mathcal{O}_{K,v}} = k_{K,v}$.

Lemma 2.22. *Let $\langle K, v_p, v\rangle$ be a p-valued field such that v is a non-trivial valuation compatible with v_p and $char(k_{K,v}) = 0$, let $\langle L, w\rangle$ be a valued field extension of $\langle K, v\rangle$ with $v(K^\times) = w(L^\times)$ and let $\overline{w}_p$ be a p-valuation on $k_{L,w}$ such that $\langle k_{L,w}, \overline{w}_p\rangle$ is a p-valued field extension of $\langle k_{K,v}, \widetilde{v}_p\rangle$. Then there exists a p-valuation w_p on L such that w is compatible with w_p and $\widetilde{w}_p = \overline{w}_p$.*

Proof. First we define a subring $\mathcal{O}_{L,w_p}$ of $\mathcal{O}_{L,w}$ and then we show that it is a valuation ring and that the corresponding valuation w_p is a p-valuation satisfying the required properties.

We define the subring $\mathcal{O}_{L,w_p}$ of $\mathcal{O}_{L,w}$ as follows: let $x \in L$, we say that $x \in \mathcal{O}_{L,w_p}$ iff the value $w(x)$ is strictly positive or $w(x) = 0$ and $\overline{w}_p(\pi_w(x)) \geqslant 0$. Clearly, $\mathcal{O}_{L,w_p}$ is a valuation ring of L, the corresponding valuation w_p is a p-valuation on L since $\overline{w}_p$ is a p-valuation on $k_{L,w}$; and the compatibility of w with w_p comes from the definition.

If y is an element of $k_{L,w}$ such that $y = \pi_w(x) \neq 0$ for some x in $\mathcal{O}^\times_{L,w}$ then $\widetilde{w}_p(y)$ is defined as $w_p(x)$. By definition, $w_p(x) \geqslant 0$ iff $\overline{w}_p(\pi_w(x)) = \overline{w}_p(y) \geqslant 0$. So we get that $\overline{w}_p$ coincides with $\widetilde{w}_p$. □

The next lemma is based on the previous one and a construction used by R. Farré in Proposition 1.3 of [4].

Lemma 2.23. *Let $\langle K, v_p, v\rangle$ be a p-valued field such that v is a non-trivial valuation compatible with v_p and $char(k_{K,v}) = 0$ and let H be an ordered abelian group such that $v(K^\times) \subseteq H \subseteq \widehat{v(K^\times)}$, the divisible hull of $v(K^\times)$.*

Then there exists an algebraic valued field extension $\langle L, w_p, w\rangle$ of $\langle K, v_p, v\rangle$ such that $\langle L, w\rangle$ is henselian, $\langle k_{L,w}, \widetilde{w}_p\rangle$ is p-adically closed and $w(L^\times) = H$.

Proof. First, we take a henselian valued field extension $\langle L, w\rangle$ of $\langle K, v\rangle$ such that its residue field is a p-adic closure $\langle \widehat{k_K}, \widehat{\widetilde{v}}_p\rangle$ of $\langle k_{K,v}, \widetilde{v}_p\rangle$ and $v(K^\times) = w(L^\times)$ (see [15, p. 164]). By applying Lemma 2.22, we take a p-valuation w_p on L extending v_p such that w is compatible with w_p and $\widetilde{w}_p = \widehat{\widetilde{v}}_p$. Let $\langle \widehat{L}, \widehat{w}_p\rangle$ be a p-adic closure of $\langle L, w_p\rangle$.

Since $\mathcal{O}_{L,w}$ is a p-convexly domain (with respect to w_p), we can apply Lemma 2.13 to find a p-convexly valued domain $\widehat{\mathcal{O}}$ with fraction field $\widehat{L}$ such that $\mathcal{O}_{L,w} \subseteq_{\mathcal{L}_p} \widehat{\mathcal{O}}$.

So $\langle \widehat{L}, \widehat{w}_p, \widehat{w} \rangle$ is an $\mathcal{L}_p$-extension of $\langle K, v_p, v \rangle$ where $\widehat{w}$ is the valuation corresponding to the valuation ring $\widehat{\mathcal{O}}$. Moreover the valuation $\widehat{w}$ on $\widehat{L}$ is henselian and so, by Lemma 2.3, $\widehat{w}$ is compatible with $\widehat{w_p}$. By construction, the value group of $\langle \widehat{L}, \widehat{w} \rangle$ is the divisible hull $\widehat{v(K^{\times})}$ of $v(K^{\times})$ and $\langle k_{\widehat{L},\widehat{w}}, \widetilde{\widehat{w}_p} \rangle = \langle \widehat{k_K}, \widehat{\widetilde{v}_p} \rangle$. We finally take a field extension L_0 of L into $\widehat{L}$ maximal with the property $v(L_0^{\times}) \subseteq H$. We will have finished if we prove $v(L_0^{\times}) = H$. Otherwise let h be an element of $H \setminus v(L_0^{\times})$ and n its order into $H/v(L_0^{\times})$. Taking $b \in L_0$ with $v(b) = n \cdot h$ and $c = \sqrt[n]{b} \in \widehat{L}$ we have $v(c) = h$. We then note that the following natural inequalities

$$n \leqslant (v(L_0^{\times}) + (h) : v(L_0^{\times})) \leqslant (v(L_0(c)^{\times}) : v(L_0^{\times})) \leqslant [L_0(c) : L_0]$$

are in fact equalities and therefore $v(L_0(c)^{\times}) = v(L_0^{\times}) + (h) \subseteq H$, contradicting the maximality of L_0. □

Now we show the definability of the p-valuation v_p in henselian residually p-adically closed fields.

Lemma 2.24. *Let $\langle K, v_p, v \rangle$ be a henselian residually p-adically closed field. Then the membership to the valuation ring $\mathcal{O}_{K,v_p}$ is existentially definable in the language $\mathcal{L}_{\mathcal{D}} := \mathcal{L}_{rings} \cup \{\mathcal{D}\}$.*

Proof. By definition of $HRpCF$, $\langle k_{K,v}, \widetilde{v}_p \rangle$ is p-adically closed with respect to $\widetilde{v}_p$ and $\langle k_{K,v}, \widetilde{v}_p \rangle \models \forall z\, [\widetilde{v}_p(z) \geqslant 0 \iff \exists y\, (y^{\epsilon} = 1 + pz^{\epsilon})]$ with ϵ choosen as in the statement of Lemma 2.3. Since v is compatible with v_p, the equivalent properties of p-convexly valued domains give us $v_p(\mathcal{M}_{K,v}) > 0$ (see Remark 2.9).

If $v(x) = 0$ then $\langle k_{K,v}, \widetilde{v}_p \rangle \models \exists y\, [y^{\epsilon} = 1 + p\pi_v(x)^{\epsilon}] \vee \exists w\, [w^{\epsilon} = 1 + p\pi_v(x^{-1})^{\epsilon}]$. If $\widetilde{v}_p(\pi_v(x)) \geqslant 0$ then $\widetilde{v}_p(y) = 0$ (otherwise we deal with x^{-1} and w); hence if z is an element of K such that $\pi_v(z) = y$ then z is a simple residue root of $f(Y) = Y^{\epsilon} - (1 + px^{\epsilon})$. By Hensel's Lemma applied to v, we get that $K \models \exists w\, [w^{\epsilon} = 1 + px^{\epsilon}]$; i.e. $v_p(x) \geqslant 0$.

So we conclude that $v_p(x) \geqslant 0$ iff

$$v(x) > 0 \vee [v(x) = 0 \wedge \exists y\, (y^{\epsilon} = 1 + px^{\epsilon})] \vee [v(x) = 0 \wedge \exists z\, (z^{\epsilon} = x^{\epsilon} + p)].$$

□

Remark 2.25. Since the theory of p-adically closed fields pCF is model complete in the language of fields and the theory of $\mathbb{Z}$-groups is model complete in the language of abelian totally ordered groups $\{+, -, \leqslant, 0, 1\}$, we get that the theory $HRpCF$ is model complete in $\mathcal{L}_{\mathcal{D}} \cup \{\underline{a}\} := \mathcal{L}_{\mathcal{D},a}$ by classical Ax-Kochen-Ersov principle for valued fields of equicharacteristic zero (see, for example, the results from [3]).

Moreover, for henselian residually p-adically closed fields, we conclude that the p-valuation v_p is henselian since it holds for $\mathbb{Q}^h(t)^h$ (with $\mathbb{Q}^h$ is the Henselization of $\mathbb{Q}$ with respect to its natural p-valuation v_p) and $\mathcal{D}_p$ is existentially definable in $\mathcal{L}_{\mathcal{D},a}$ (see Lemma 2.24).

Lemma 2.26. *In the $\mathcal{L}_{\mathcal{D},a}$-theory of henselian residually p-adically closed fields, the negations of nth power predicates P_n are existentially definable in the language of rings with the distinguished element a.*

Proof. Let K be a model of $HRpCF$. We consider a non-zero element x in K such that $v(x) \geqslant 0$ (otherwise if $v(x) < 0$ then we use that $K \models P_n(x) \iff P_n(x^{-n+1})$). Then for each natural number, we get that

$$K \models \exists y\,[\bigvee_{i=0}^{n-1} v(x) = v(a^i y^n)] \text{ since } v(K^\times) \text{ is a } \mathbb{Z}\text{-group with } v(a) = 1.$$

Since $\mathcal{O}_{K,v}$ satisfies Hensel's Lemma and $k_{K,v}$ is p-adically closed, this is equivalent to

$$k_{K,v} \models \exists z\,[\bigvee_{i=0}^{n-1} \bigvee_{q\in\Delta_n} z^n = q \cdot \pi_v(x \cdot a^{-i} \cdot y^{-n})]$$

where $\Delta_n = \{q \in \mathbb{N} | q = \lambda p^r,\, 0 \leqslant r < n,\, \lambda \in \Lambda_n\}$ and $\Lambda_n = \{\lambda \in \mathbb{N} | 1 \leqslant \lambda \leqslant p^{v_p(n)+1},\, p \not| \lambda\}$ (see [1]). So we get that $K = \bigcup_{i=0}^{n-1} \bigcup_{q\in\Delta_n} qa^i K^n$ (and the union is disjoint). □

Now we state and prove the analogue of the Hilbert's Seventeenth problem for a henselian residually p-adically closed field K. In the sequel, we denote the ring of polynomials in n indeterminates over K by $K[\underline{X}]$ and its fraction field by $K(\underline{X})$.

Before stating the theorem, we recall a lemma from [11].

Lemma 2.27. *Let D be a divisible totally ordered abelian group with d a positive element in D. Let H be a subgroup of D which is maximal with respect to the property that $d = 1$ in H. Then H forms a $\mathbb{Z}$-group.*

Now we define the following subsets of $K(\underline{X})$: the subring $A := \langle\gamma(K(\underline{X}))\rangle$ of $K(\underline{X})$ generated by $\gamma(K(\underline{X}))$ and $M := A \cdot \mathcal{M}_{K,v}$.

Theorem 2.28. *Let $\langle K, v_p, v\rangle$ be a henselian residually p-adically closed field and let f be in $K(\underline{X})$. Assume that $v_p(f(\bar{x})) \geqslant 0$ for every $\bar{x} \in K^n$ such that $f(\bar{x})$ is defined (*).*

Then f belongs to the M-Kochen ring $R^M_{\gamma_p}(K(\underline{X}))$ of $K(\underline{X})$.

Proof. Suppose that f does not belong to $R^M_{\gamma_p}(K(\underline{X}))$. Since there exists a p-valuation v_p on $K(\underline{X})$ which extends the one of K such that $v_p(M) \geqslant 0$ (see Remark 2.29), we can extend the p-valuation v_p on K to a p-valuation w_p on $K(\underline{X})$ such that $w_p(M) \geqslant 0$ and $w_p(f) < 0$ by applying Lemma 2.21.

Let us consider $B = pcH(A, K(\underline{X}))$. Since B is not a field, Lemma 2.12 yields that B is a p-convexly valued domain whose fraction field is $K(\underline{X})$. In the following, we denote by w the valuation on $K(\underline{X})$ corresponding to the valuation ring B. Since $a \in \mathcal{M}_{K,v}$ and $v_p(A \cdot \mathcal{M}_{K,v}) \geqslant 0$, we get that $v_p(a^{-1}) < v_p(A)$; hence $a \in \mathcal{M}_B$. Since $\gamma(K(\underline{X})) \subseteq B$, $\langle K(\underline{X}), w\rangle$ is a valued field such that $w(a) = 1$ (see Lemma 2.19). The following statement of Lemma 2.10 shows us that $\mathcal{O}_{K,v} \subseteq_{\mathcal{L}_{\mathcal{D},a}} B$:

$$\mathcal{M}_B \cap \mathcal{O}_{K,v} = \mathcal{M}_{K,v}.$$

Indeed, the inclusion $\subseteq$ is trivial and for the other one, we know that B satisfies $v_p(m^{-1}) < v_p(h)$ for any $m \in \mathcal{M}_{K,v}$ and any $h \in A$ and by definition of B, it implies that $m^{-1} \notin pcH(A, K(\underline{X})) = B$; so the conclusion follows.

Since $\mathcal{O}_{K,v} = \gamma(K) \subseteq \gamma(K(\underline{X})) \subseteq B = \mathcal{O}_{K(\underline{X}),w}$ (see Lemma 2.20) and $\mathcal{M}_{K,v} = a \cdot \mathcal{O}_{K,v} \subseteq a \cdot \mathcal{O}_{K(\underline{X}),w} = \mathcal{M}_B$ by Lemma 2.19, we conclude

$$\langle K, \mathcal{D}_{v_p}, \mathcal{D}_v, a\rangle \subseteq_{\mathcal{L}_{p,a}} \langle K(\underline{X}), \mathcal{D}_{w_p}, \mathcal{D}_w, a\rangle.$$

Now we use Proposition 2.23 applied to Lemma 2.27 in order to obtain an extension $\langle L, \overline{w}_p, \overline{w}\rangle$ of $\langle K(\underline{X}), w_p, w\rangle$ such that $\langle L, \overline{w}\rangle$ henselian, $\langle k_{L,\overline{w}}, \widetilde{\overline{w}_p}\rangle$ is p-adically closed, $\langle L, \overline{w}_p\rangle$ is a p-valued extension of $\langle K(\underline{X}), w_p\rangle$ and $\overline{w}(L^\times)$ is a $\mathbb{Z}$-group with $\overline{w}(a) = 1_{\overline{w}(L^\times)}$.

By applying Ax-Kochen-Ersov transfer theorem for henselian valued fields of equicharacteristic zero, we deduce that $\langle K, \mathcal{D}_v, a\rangle \prec \langle L, \mathcal{D}_{\overline{w}}, a\rangle$ in the language $\mathcal{L}_{\mathcal{D},a}$. Keeping in mind that, as well in K as in L, the p-valuations are existentially definable in the language $\mathcal{L}_{\text{rings}} \cup \{\mathcal{D}\}$, we have that $\langle K, \mathcal{D}_{v_p}, \mathcal{D}_v, a\rangle \prec_{\mathcal{L}_{p,a}} \langle L, \mathcal{D}_{\overline{w}_p}, \mathcal{D}_{\overline{w}}, a\rangle$. But $\overline{w}_p(f) < 0$ in L implies $\overline{w}_p(f(\underline{X})) < 0$ and hence the formula ϕ expressing $\exists \bar{x}\,(f(\bar{x}))$ is defined and $\overline{w}_p(f(\bar{x})) < 0$ holds in L. By the elementary inclusion, ϕ holds in $\langle K, v_p, v\rangle$ showing that (*) is false. $\square$

Remark 2.29. In the previous theorem, we have to find a p-valuation v_p on $K(\underline{X})$ which extends the one of K such that $v_p(M) \geqslant 0$, i.e. $v_p(A \cdot \mathcal{M}_{K,v}) \geqslant 0$. We take a $|K|^+$-saturated $\mathcal{L}_{p,a}$-elementary extension L of K and so, we satisfy in L the n-type required for $X_1, \cdots, X_n$. This n-type is consistent since in L, we have that $\gamma(L) \subseteq \mathcal{O}_{L,v}$ and so $A(L) \cdot \mathcal{M}_{L,v} \subseteq \mathcal{M}_{L,v} \subseteq \mathcal{O}_{L,v_p}$ where the subring $A(L)$ of L generated by $\gamma(L)$ is equal to $\mathcal{O}_{L,v}$.

3. Nullstellensatz for henselian residually p-adically closed fields

In this section, we introduce the notion of residually p-adic ideal and the one of residually p-adic radical of an ideal in $K[\underline{X}]$ over a henselian residually p-adically closed field K, by analogy with these notions in the p-adic case (see [17]). These two notions are related to the M-Kochen ring with the previous subset M of $K(\underline{X})$, i.e. $A \cdot \mathcal{M}_{K,v}$ where $A := \langle \gamma(K(\underline{X}))\rangle$ is the subring of $K(\underline{X})$ generated by $\gamma(K(\underline{X}))$. We will closely follow the work of A. Srhir in order to prove a Nullstellensatz theorem for henselian residually p-adically closed fields.

Definition 3.1. Let $\langle K, v_p, v, a\rangle$ be a p-valued field with v a non-trivial valuation and let a be a non-zero element of K.

We call such a field *residually p-valued* if v is compatible with v_p, $k_{K,v}$ is of characteristic zero and $v(a) = 1$.

Definition 3.2. Let $\langle K, v_p, v, a\rangle$ be a residually p-valued field and let L be a field extension of K.

We say that L is *a formally residually p-valued field over* K if L admits a p-valuation w_p which extends the given p-valuation v_p on K and a valuation w such that $\langle L, w_p, w\rangle$ is residually p-valued and $K \subseteq_{\mathcal{L}_p} L$; i.e. $\langle L, w_p, w\rangle$ is a *residually p-valued field* extension.

Remark 3.3. If $\langle K, v_p, v, a\rangle$ is a residually p-valued field then $K(X)$ is formally residually p-valued over K. It suffices to extend the two valuations v_p and v as follows.

Let f be an element of $K[X]$, i.e. $f = \sum_{i=k}^{N} f_i X^i$ for some natural numbers $0 \leqslant k \leqslant N$ with $f_k \neq 0$; k is called *the initial degree* of f. Then we let $w(f) := (k, v(f_k)) \in \mathbb{N} \times v(K^\times)$ and so, we extend w to the field of rational functions $K(X)$ by letting $w(g/h) := w(g) - w(h) \in \mathbb{Z} \times v(K^\times)$ (lexicographically ordered) where g, $h \in K[X]$ and $h \neq 0$. We proceed similarly for w_p which is a p-valuation on $K(X)$ extending the one of K. Let us show that w is compatible with w_p on $K(X)$. So we consider elements $f/g, s/t \in K(X)$ such that $w_p(f/g) \leqslant w_p(s/t)$. We have to distinguish two cases:

- the difference of the initial degrees of (f, g) and (s, t) is the same and so, we conclude by using the compatibility of v with v_p;
- the difference of the initial degrees of (f, g) is strictly less than the one of (s, t) and the conclusion follows from the definition of w and the lexicographic order of $\mathbb{Z} \times v(K^\times)$.

By induction, we get the same result for $K(\underline{X})$.

In [7, Theorem 3.4], we showed the following

Theorem 3.4. *Let L be a field extension of the p-valued field $\langle K, v_p\rangle$ and let M be a subset of L such that $v_p((M \cap K)^\bullet) \geqslant 0$.*

A necessary and sufficient condition for L to be a p-valued field extension of K such that $v_p(M^\bullet) \geqslant 0$ is that

$$\frac{1}{p} \notin \mathcal{O}_{K,v_p}[\gamma_p(L), M].$$

So we can deduce the following

Proposition 3.5. *Let L be a field extension of a residually p-valued field $\langle K, v_p, v, a\rangle$. Then L is formally residually p-valued over K iff $\frac{1}{p} \notin \mathcal{O}_{K,v_p}[\gamma_p(L), M]$ where M is equal to $A.\mathcal{M}_{K,v}$ and $A := \langle \gamma(L)\rangle$ is the subring of L generated by $\gamma(L)$.*

Proof. The implication ($\Rightarrow$) is trivial. Indeed, if we assume that $\langle L, w_p, w, a\rangle$ is a residually p-valued field extension of K then we get that $v_p(\mathcal{O}_{K,v_p}[\gamma_p(L), M]) \geqslant 0$ since $w_p(\gamma_p(L)) \geqslant 0$ (see Lemma 6.2 in [14]), $\gamma(L) \subseteq \mathcal{O}_{L,w}$ and so, $A \cdot \mathcal{M}_{L,w} \subseteq \mathcal{M}_{L,w} \subseteq \mathcal{O}_{L,w_p}$ (because w is compatible with w_p).

For the other one, there exists a p-valuation w_p on L such that $w_p(M) \geqslant 0$ by Theorem 3.4. It suffices to follow the same proof as the one of Theorem 2.28 in order to build a valuation w on L such that w is compatible with w_p and $w(a) = 1$. □

In Section 2, we have already defined the notion of M-Kochen ring $R^M_{\gamma_p}(L)$ for a field extension L of a p-valued field $\langle K, v_p\rangle$.

For the rest of the section, we assume that K is a henselian residually p-adically closed field and that M is the subset of any field extension L as in the previous proposition. Hence we have that the elements of the M-Kochen ring $R^M_{\gamma_p}(L)$ over L have the following form $a = \frac{b}{1+pd}$ with $b, d \in \mathbb{Z}[\gamma_p(L), M]$ and $1 + pd \neq 0$ since the p-valuation v_p is henselian (see Remark 2.16).

Proposition 3.6. *Let L be a field extension of K. Then L is a formally residually p-valued field over K iff $\frac{1}{p} \notin R^M_{\gamma_p}(L)$.*

Proof. We assume that L is formally residually p-valued over K. If $\frac{1}{p} \in R^M_{\gamma_p}(L)$ then there exist $t, s \in \mathbb{Z}[\gamma_p(L), M]$ such that $\frac{1}{p} = \frac{t}{1+ps}$. Thus we have $p(t-s) = 1$. This contradicts Proposition 3.5.

Conversely assume that $\frac{1}{p} \notin R^M_{\gamma_p}(L)$. Since $\mathbb{Z}[\gamma_p(L), M] \subseteq R^M_{\gamma_p}(L)$, one has $\frac{1}{p} \notin \mathbb{Z}[\gamma_p(L), M]$. □

Now we prove the analogue of Corollary 1.6 in [17].

Corollary 3.7. *Let L be a henselian residually p-adically closed field such that $K \subseteq_{\mathcal{L}_{\mathcal{D},a}} L$. Let I be an ideal of $K[\underline{X}]$ generated by $f_1, \cdots, f_r$ and let g be a polynomial not in I. Let $\Phi : K[\underline{X}]/I \longmapsto L$ be a K-homomorphism such that $\Phi(\bar{g}) \neq 0$. Then there exists a K-homomorphism $\Psi : K[\underline{X}]/I \longmapsto K$ such that $\Psi(\bar{g}) \neq 0$.*

Proof. We put $x_1 = \Phi(X_1 + I), \cdots, x_n = \Phi(X_n + I)$ and $\bar{x} := (x_1, \cdots, x_n)$. Then $\bar{x} \in L^n$, $f_1(\bar{x}) = \cdots = f_r(\bar{x}) = 0$ and $g(\bar{x}) \neq 0$. This statement can be expressed by an elementary $\mathcal{L}_{\mathcal{D},a}$-sentence with parameters from K which holds in L. Since the $\mathcal{L}_{\mathcal{D},a}$-theory $HRpCF$ is model complete, we infer that this statement also holds in K. Thus there exists $\bar{y} \in K^n$ such that $f_1(\bar{y}) = \cdots = f_r(\bar{y}) = 0$ and $g(\bar{y}) \neq 0$. □

Now Definition 3.1 of [17] motivates the following definition of a residually p-adic ideal in $K[\underline{X}]$.

Definition 3.8. Let I be an ideal of $K[\underline{X}]$ generated by the polynomials $f_1, \cdots, f_r$. We say that I is *a residually p-adic ideal* of $K[\underline{X}]$ if for any $g \in K[\underline{X}]$, for any $m \in \mathbb{N}^\bullet$ and for any $\lambda_1, \cdots, \lambda_r \in R^M_{\gamma_p}(K(\underline{X})).K[\underline{X}]$ such that $g^m = \lambda_1 f_1 + \cdots + \lambda_r f_r$ then we have $g \in I$, where $R^M_{\gamma_p}(K(\underline{X})).K[\underline{X}]$ is the subring of $K(\underline{X})$ generated by $R^M_{\gamma_p}(K(\underline{X}))$ and $K[\underline{X}]$.

Remark 3.9. As in Remark 3.2 in [17], this definition does not depend on the choice of the basis $f_1, \cdots, f_r$ of the ideal I. If $\bar{a}$ is an element of K^n then the maximal ideal $K[\underline{X}]$ defined by $\mathcal{M}_{\bar{a}} := \{f \in K[\underline{X}] | f(\bar{a}) = 0\}$ is a residually p-adic ideal of $K[\underline{X}]$.

Notation 3.10. If I is an ideal of $K[\underline{X}]$, we will denote by $\mathcal{Z}(I)$ the algebraic set of K^n defined by $\mathcal{Z}(I) := \{\bar{x} \in K^n | f(\bar{x}) = 0 \quad \forall f \in I\}$ and by $\mathcal{I}(\mathcal{Z}(I)) := \{f \in K[\underline{X}] | f(\bar{x}) = 0 \quad \forall \bar{x} \in \mathcal{Z}(I)\}$.

If, in addition, I is a prime ideal of $K[\underline{X}]$, then we shall denote by $\bar{\cdot}$ the residue map with respect to I and by $K(I)$ the residue field of I, i.e. the fraction field of the domain $K[\underline{X}]/I$.

Proposition 3.11. *Let I be an ideal of $K[\underline{X}]$ generated by the polynomials $f_1, \cdots, f_r$. Then the ideal $\mathcal{I}(\mathcal{Z}(I))$ is a residually p-adic ideal.*

Proof. Let g be a polynomial in $K[\underline{X}]$, $m \in \mathbb{N}^\bullet$ and $\lambda_1, \cdots, \lambda_r \in R^M_{\gamma_p}(K(\underline{X})) \cdot K[\underline{X}]$ such that $g^m = \lambda_1 f_1 + \cdots + \lambda_r f_r$. We have to show that $g \in \mathcal{I}(\mathcal{Z}(I))$. Let $\bar{x}$ be in $\mathcal{Z}(I)$. We consider the following K-rational place $\phi : K(\underline{X}) \longmapsto K \cup \{\infty\}$ such that $\phi(X_i) = x_i$ for $1 \leqslant i \leqslant n$. Since $f_j \in I$, we have $\phi(f_j) = 0$ for all $1 \leqslant i \leqslant r$.

Claim: for any $\lambda \in R^M_{\gamma_p}(K(\underline{X})) \cdot K[\underline{X}]$, we have $\phi(\lambda) \neq \infty$.

By Lemma 2.19, we have that for any $h \in K(\underline{X})$, $\phi(\gamma(h)) \neq \infty$ and by Lemma 2.1 in [10], for any $\lambda \in R^{\emptyset}_{\gamma_p}(K(\underline{X}))$, $\phi(\lambda) \neq \infty$. So by definition of $R^M_{\gamma_p}(K(\underline{X}))$ and the fact that $\phi(X_i) \neq \infty$, we get the claim.

Now from the Claim, we deduce that $\phi(g) = 0$, i.e. $g(\bar{x}) = 0$. It follows that $g \in \mathcal{I}(\mathcal{Z}(I))$. Hence $\mathcal{I}(\mathcal{Z}(I))$ is a residually p-adic ideal. □

The next proposition gives a characterization of residually p-adic ideals in terms of formally residually p-valued field over K. So we get the analogue of Proposition 3.6 in [17].

Proposition 3.12. *Let I be a prime ideal of $K[\underline{X}]$ generated by the polynomials $f_1, \cdots, f_r$. Then I is a residually p-adic ideal if and only if its residue field $K(I)$ is formally residually p-valued over K.*

Proof. We assume that the residue field $K(I)$ of I is not formally residually p-valued over K. By Theorem 3.4, one has $\frac{1}{p} \in R^{M'}_{\gamma_p}(K(I))$ where $A' := \langle \gamma(K(I)) \rangle$ is the subring of $K(I)$ generated by $\gamma(K(I))$ and M' is equal to $A'.\mathcal{M}_{K,v}$.

More precisely there exist $\bar{f}/\bar{g}$ and $\bar{h}/\bar{l}$ in $\mathbb{Z}[\gamma_p(K(I)), M']$ such that $\frac{1}{p} = \frac{\bar{f}/\bar{g}}{1+p\bar{h}/\bar{l}}$. One can choose f/g and h/l such that $f/g, h/l \in \mathbb{Z}[\gamma_p(K(\underline{X})), M]$ where M is equal to $A \cdot \mathcal{M}_{K,v}$ with A the subring of $K(\underline{X})$ generated by $\gamma(K(\underline{X}))$. We obtain the equality $\overline{gl + p(gh - fl)} = 0$, i.e. $gl + p(gh - fl) \in I$. It follows that there exist $\alpha_1, \cdots, \alpha_r \in K[\underline{X}]$ such that $gl + p(gh - fl) = \sum_{i=1}^r \alpha_i f_i$. By Remark 3.3 and Proposition 3.5, we have $1 + p(h/l - f/g) \neq 0$. So we can write $gl = \sum_{i=1}^r \lambda_i f_i$ with $\lambda_i := \frac{\alpha_i}{1+p(h/l-f/g)}$ for $1 \leqslant i \leqslant r$. Since f/g, $h/l \in \mathbb{Z}[\gamma_p(K(\underline{X})), M]$, we have $\lambda_i \in R^M_{\gamma_p}(K(\underline{X})) \cdot K[\underline{X}]$ for all $1 \leqslant i \leqslant r$. Hence we have $gl = \lambda_1 f_1 + \cdots + \lambda_r f_r$. Since I is a residually p-adic ideal, we get $gl \in I$. On the other hand, $g \notin I$ and $l \notin I$ imply $gl \notin I$. This is a contradiction.

Conversely assume that the residue field $K(I)$ is formally residually p-valued over K. We first prove $I = \mathcal{I}(\mathcal{Z}(I))$ and then we conclude from Proposition 3.11 that I is residually p-adic.

Let $f \notin I$. As in Theorem 2.28, we can take an extension $\langle L, \overline{w}_p, \overline{w} \rangle$ of $K(I)$ which is a model of $HRpCF$ such that $f \neq 0$ in L. By using Corollary 3.7, there exists a K-homomorphism $\Psi : K[\underline{X}]/I \longmapsto K$ such that $\Psi(f) \neq 0$. We put $x_1 := \Psi(\bar{X}_1), \cdots, x_n := \Psi(\bar{X}_n)$ and $\bar{x} := (x_1, \cdots, x_n) \in K^n$. Then we have $\bar{x} \in \mathcal{Z}(I)$ and $f(\bar{x}) \neq 0$. Thus $f \notin \mathcal{I}(\mathcal{Z}(I))$. Hence $I = \mathcal{I}(\mathcal{Z}(I))$. □

As in Example 3.7 in [17], for any integer i such that $1 \leqslant i \leqslant n$, the prime ideal $(X_1, \cdots, X_i)$ of $K[X_1, \cdots, X_n]$ is a residually p-adic ideal. The next proposition may be considered as the residually p-adic counterpart of Proposition 3.8 in [17].

Proposition 3.13. *Let I be a residually p-adic ideal of $K[\underline{X}]$ generated by the polynomials $f_1, \cdots, f_r$. Then one has the following properties:*

- *I is a radical ideal of $K[\underline{X}]$,*
- *All the minimal prime ideals of $K[\underline{X}]$ containing I are residually p-adic ideals.*

Proof. The proof is the same as the one in [17] with Λ replaced by $R^M_{\gamma_p}(K(\underline{X}))$. □

Now we give the geometric characterization of residually p-adic ideals which is the analogue of Theorem 3.9 in [17].

Theorem 3.14. *Let I be an ideal of $K[\underline{X}]$ generated by the polynomials $f_1, \cdots, f_r$. Then I is a residually p-adic ideal if and only if $I = \mathcal{I}(\mathcal{Z}(I))$.*

Proof. If $I = \mathcal{I}(\mathcal{Z}(I))$ then, by Proposition 3.11, I is a residually p-adic ideal.

Conversely suppose that I is a residually p-adic ideal. First assume that I is prime. Then, by Lemma 3.12, the residue field $K(I)$ of I is formally residually p-valued over K. Therefore $I = \mathcal{I}(\mathcal{Z}(I))$ (see the second part of the proof in Proposition 3.12). Second, if I is any residually p-adic ideal then I is clearly a radical ideal of $K[\underline{X}]$. Thus $I = \bigcap_{i=1}^{k} I_i$ where I_i are the minimal prime ideals of I in $K[\underline{X}]$. So we know, by Proposition 3.13, that $I_1, \cdots, I_k$ are residually p-adic ideals of $K[\underline{X}]$. Hence $I = \bigcap_{i=1}^{k} \mathcal{I}(\mathcal{Z}(I_i)) = \mathcal{I}(\mathcal{Z}(I))$. □

The next result provides a residually p-adic analogue of Corollary 3.10 in [17].

Corollary 3.15. *Let I be an ideal of $K[\underline{X}]$ generated by the polynomials $f_1, \cdots, f_r$. Then the ideal $\mathcal{I}(\mathcal{Z}(I))$ is the smallest residually p-adic ideal of $K[\underline{X}]$ containing I.*

Proof. We know, from Proposition 3.11, that $\mathcal{I}(\mathcal{Z}(I))$ is a residually p-adic ideal of $K[\underline{X}]$ containing I. Moreover, if I_1 is a residually p-adic ideal of $K[\underline{X}]$ such that $I \subseteq I_1$, then we have that $\mathcal{I}(\mathcal{Z}(I)) \subseteq \mathcal{I}(\mathcal{Z}(I_1))$. Since I_1 is a residually p-adic ideal, we conclude from Theorem 3.14 that $I_1 = \mathcal{I}(\mathcal{Z}(I_1))$. Thus $\mathcal{I}(\mathcal{Z}(I)) \subseteq I_1$. Hence the ideal $\mathcal{I}(\mathcal{Z}(I))$ is the smallest residually p-adic ideal of $K[\underline{X}]$ containing I. □

Now we give the definition of the residually p-adic radical of an ideal $I \subseteq K[\underline{X}]$ and some of its algebraic properties.

Definition 3.16. Let I be an ideal of $K[\underline{X}]$ generated by the polynomials $f_1, \cdots, f_r$. The *residually p-adic radical* of I is the subset of $K[\underline{X}]$ defined by

$$\sqrt[p]{I} := \{g \in K[\underline{X}] | \exists m \in \mathbb{N}^{\bullet} \text{ and } \exists \lambda_1, \cdots, \lambda_r \in R^M_{\gamma_p}(K(\underline{X})).K[\underline{X}] : g^m = \sum_{i=1}^{r} \lambda_i f_i\}.$$

As in the definition of residually p-adic ideal, the residually p-adic radical of a polynomial ideal is independent of the choice of the basis of the ideal. By replacing the ring Λ by $R^M_{\gamma_p}(K(\underline{X}))$ in the proof of the Proposition 4.3 in [17], we see that $\sqrt[p]{I}$ is the smallest residually p-adic ideal of $K[\underline{X}]$ containing I. Let us remark that an ideal I of $K[\underline{X}]$ is a residually p-adic ideal if and only if $I = \sqrt[p]{I}$.

Proposition 3.17. *Let I be an ideal of $K[\underline{X}]$. Then $\sqrt[p]{I}$ is the intersection of all the residually p-adic prime ideals of $K[\underline{X}]$ containing I.*

Proof. It suffices to replace $\Lambda.K[\underline{X}]$ by $R^M_{\gamma_p}(K(\underline{X})).K[\underline{X}]$ in the proof of Proposition 4.5 in [17]. □

Now we are able to prove the Nullstellensatz for henselian residually p-adically closed fields.

Theorem 3.18. *Let I be an ideal of $K[\underline{X}]$. Then $\sqrt[p]{I} = \mathcal{I}(Z(I))$.*

Proof. Immediate consequence of Corollary 3.15 and the fact that $\sqrt[p]{I}$ is the smallest residually p-adic ideal of $K[\underline{X}]$ containing I. □

The following result gives a correspondence between algebraic sets of K^n and residually p-adic ideals of $K[\underline{X}]$. Thus we provide a residually p-adic analogue of Proposition 5.2 in [17].

Proposition 3.19. *There exists a one to one correspondence between algebraic sets of K^n and residually p-adic ideals of $K[\underline{X}]$.*

Proof. It suffices to use, in the proof of [17], Theorem 3.14 instead of Theorem 3.9 in [17]. □

As an immediate consequence of this proposition, we obtain the following corollary.

Corollary 3.20. *There exists a one to one correspondence between irreducible algebraic sets of K^n and residually p-adic prime ideals of $K[\underline{X}]$.*

Corollary 3.21. *There exists a one to one correspondence between points of K^n and residually p-adic maximal ideals of $K[\underline{X}]$.*

Proof. Let $\mathcal{M}$ be a residually p-adic maximal ideal of $K[\underline{X}]$. Then, according to Proposition 3.12, the field $K(\mathcal{M})$ is formally residually p-valued over K. As in Theorem 2.28, we can take an extension $\langle L, \overline{w}_p, \overline{w}, a\rangle$ of this field which is a model of $HRpCF$. Hence we have a K-homomorphism $\Phi : K[\underline{X}]/\mathcal{M} \longmapsto L$. Then, by model completeness of the $\mathcal{L}_{\mathcal{D},a}$-theory of henselian residually p-adically closed fields or more precisely, by Corollary 3.7, we obtain a K-homomorphism $\Psi : K[\underline{X}]/\mathcal{M} \longmapsto K$. We put $x_i = \Psi(\overline{X_i})$ for $1 \leqslant i \leqslant n$ and $\bar{x} = (x_1, \cdots, x_n)$. If $f \in \mathcal{M}$ then $f(\bar{x}) = \Psi(\overline{f}) = 0$ i.e. $\bar{x} \in \mathcal{Z}(\mathcal{M})$. Therefore $\mathcal{M} \subseteq \mathcal{I}(\{\bar{x}\})$. Hence $\mathcal{M} = \mathcal{I}(\{\bar{x}\})$ since $\mathcal{M}$ is a maximal ideal.

Conversely, let $\bar{a} \in K^n$. By Remark 3.9, the maximal ideal $\mathcal{M}_{\bar{a}}$ defined by $\mathcal{M}_{\bar{a}} := \{f \in K[\underline{X}] | f(\bar{a}) = 0\}$ is a residually p-adic maximal ideal of $K[\underline{X}]$. □

Now we define in a similar way as in [2] the model-theoretic radical ideal of an ideal in $K[\underline{X}]$. Our goal is to show by using the arguments of the previous results that the algebraic and model-theoretic notions of radical coincide.

Definition 3.22. Let I be an ideal of $K[\underline{X}]$. The *model-theoretic radical ideal* of I is defined as the following polynomial ideal, denoted by ${}_{\mathrm{HRpCF}}\mathrm{Rad}(I)$

$$ {}_{\mathrm{HRpCF}}\mathrm{Rad}(I) := \bigcap_{p \in \mathcal{P}} P $$

where $\mathcal{P}$ is the following set

$\{P$ ideal of $K[\underline{X}]$ containing I such that $K[\underline{X}]/P$ can be $\mathcal{L}_p$-embedded over K in a model L of $HRpCF\}$.

Note that if P is in $\mathcal{P}$ then P is prime.

Now we prove the theorem which was previously announced.

Theorem 3.23. *Under the previous assumptions and notations, ${}_{HRpCF}Rad(I) = \sqrt[p]{I}$.*

Proof. Let $f_1, \dots, f_r$ be generators of the ideal I in $K[\underline{X}]$.

(1) First we show that $\sqrt[p]{I} \subseteq {}_{\mathrm{HRpCF}}\mathrm{Rad}(I)$. Let $g \in K[\underline{X}]$ such that $g \notin {}_{\mathrm{HRpCF}}\mathrm{Rad}(I)$. Thus there exists a prime ideal J in $K[\underline{X}]$ containing I but not g such that

$$K \subseteq_{\mathcal{L}_p} L$$

where $L \models HRpCF$ and $K[\underline{X}]/J \subseteq L$. By model completeness of the $\mathcal{L}_p$-theory $HRpCF$, we get that $g \notin \mathcal{I}(\mathcal{Z}(I))$. Furthermore, by Theorem 3.18, we get that $g \notin \sqrt[p]{I}$.

(2) Second we prove the other inclusion and we assume that $g \notin \sqrt[p]{I}$. Now it suffices to follow the ideas in the proof of Theorem 4.4 in [7].

Let S be the following multiplicative subset of $K[\underline{X}]$

$$\{g^m : m \in \mathbb{N}\}.$$

We consider the following set $\mathcal{J}$ of ideals in $K[\underline{X}]$

$$\mathcal{J} = \{J \supseteq I \text{ proper residually } p\text{-adic ideal of } K[\underline{X}] \text{ such that } J \text{ is disjoint of } S\}.$$

Clearly $\mathcal{J}$ is non-emty since $\sqrt[p]{I}$ belongs to $\mathcal{J}$. By Zorn's Lemma, there exists a maximal element J in $\mathcal{J}$. So J is a proper residually p-adic ideal in $K[\underline{X}]$ containing I such that $J \cap S = \emptyset$. Let us show that J is prime. Assume that $f \cdot h \in J$ for some $f, h \in K[\underline{X}] \setminus J$. By maximality of $J \in \mathcal{J}$, we get that $\sqrt[p]{\langle f, J\rangle} \cap S \neq \emptyset$ and $\sqrt[p]{\langle h, J\rangle} \cap S \neq \emptyset$. So we have that

$$g^{k_1} = \lambda f + \sum_{i=1}^{l} \lambda_i \cdot j_i \text{ and } g^{k_2} = \lambda' h + \sum_{i=1}^{l} \lambda'_i \cdot j_i$$

where $j_1, \dots, j_l$ are generators of J, $\lambda, \lambda', \lambda_i, \lambda'_i$ belongs to $R^M_{\gamma_p(K(\underline{X}))} \cdot K[\underline{X}]$ and $k_1, k_2 \in \mathbb{N}$. So we obtain that $g^{k_1+k_2}$ belongs to J since J is residually p-adic.

By Proposition 3.12, $K(I)$ is formally residually p-valued over K. As in the proof of Proposition 3.12, we can take an extension $\langle L, \overline{w}_p, \overline{w}\rangle$ of $K(I)$ which is a model of $HRpCF$ and $K \subseteq_{\mathcal{L}_p} L$ with $g \neq 0$ in L. So by definition of ${}_{\mathrm{HRpCF}}\mathrm{Rad}(I)$, we have that $g \notin {}_{\mathrm{HRpCF}}\mathrm{Rad}(I)$. □

4. Hilbert's seventeenth problem for a class of 0-D-henselian fields

In this section, we keep previous notations and conventions; the usual terminology in differential algebra can be found in [13].

In Section 5 of [6], we introduce the theory of p-adically closed differential fields which is the model-companion of the universal theory of differential p-valued fields in the differential Macintyre's language (see [12]), i.e. $\mathcal{L}^D_{\mathcal{D}_p,p_\omega} := \mathcal{L}_{\text{fields}} \cup \{D, \mathcal{D}_p, p_n : n \in \mathbb{N} \setminus \{0,1\}\}$ where $\mathcal{D}_p$ will be interpreted as a l.d. relation with respect to a p-valuation v_p, the p_n are predicates for nth powers and D is a unary function interpreted as a derivation. This $\mathcal{L}^D_{\mathcal{D}_p,p_\omega}$-theory admits quantifier elimination and is denoted by $pCDF$.

Let us recall an axiomatization of $pCDF$.

(1) Axioms for differential p-valued fields where $\mathcal{D}_p$ is the l.d. relation with respect to the p-valuation v_p and D is a derivation,
(2) Hensel's Lemma with respect to the p-valuation v_p and the value group is a $\mathbb{Z}$-group,
(3) $\forall x\,[p_n(x) \iff \exists y\,(y^n = x)]$,
(4) (DL)-scheme of axioms (following the terminology in Section 3 of [6]): for any positive integer n, for any differential polynomial $f(X, \cdots, X^{(n)})$ of order n with coefficients in the valuation ring $\mathcal{O}_{v_p}$ $(:= \{x|\mathcal{D}_p(1,x)\})$,

$$\forall \epsilon\, \forall b_0, \cdots, b_n \,\{ \bigwedge_{i=0}^{n} \mathcal{D}_p(1, b_i) \wedge f^*(b_0, \cdots, b_n) = 0 \wedge (\frac{\partial}{\partial X^{(n)}} f^*)(b_0, \cdots, b_n) \neq 0$$

$$\Rightarrow \exists y\,[\mathcal{D}_p(1, y) \wedge f(y) = 0 \wedge \bigwedge_{i=0}^{n} \mathcal{D}_p(\epsilon, y^{(i)} - b_i)]\}$$

where f^* is the differential polynomial f seen as an ordinary polynomial in the differential indeterminates $X, \cdots, X^{(n)}$.

By using $pCDF$ as differential residue field theory and the theory of $\mathbb{Z}$-groups as value group theory, we can introduce the valued D-field analogue of the theory of henselian residually p-adically closed fields. For this purpose, we adapt the setting of the work [16] to our p-adic case.

First we recall the structure of the canonical example of valued D-field whose the theory will be studied in a residually p-adic setting (see also Section 6 in [16]).

We consider a differential field $\langle \mathbf{k}, \delta \rangle$ which is a model of $pCDF$ -hence it is linearly differentially closed and admits quantifier elimination in the language $\mathcal{L}^{D}_{\mathcal{D}_p, p_\omega}$ (see [6])- and a $\mathbb{Z}$-group $\mathbf{G}$. It is a well-known fact that $Th(\mathbf{G})$ admits quantifier elimination in the language of abelian totally ordered groups with additional unary predicates of divisibility $\{n|.\}_{n\in\omega}$ which means:

$$\forall g \in \mathbf{G}\,[n|g \iff \exists g' \in \mathbf{G}\,(\underbrace{g' + \cdots + g'}_{n \text{ times}} = g)].$$

We are interested in the field $\mathbf{k}((t^{\mathbf{G}}))$ of generalized power series. The set $\mathbf{k}((t^{\mathbf{G}}))$ is defined by $\{f : \mathbf{G} \longmapsto \mathbf{k} : \mathrm{supp}(f) := \{g \in \mathbf{G} : f(g) \neq 0\}$ is well-ordered in the ordering induced by $\mathbf{G}\}$. Each element of $\mathbf{k}((t^{\mathbf{G}}))$ can be viewed as a formal power series $\sum_{g\in\mathbf{G}} f(g)t^g$ with the addition and the multiplication defined as follows: $(f + h)(g) := f(g) + h(g)$ and $(f.h)(g) := \sum_{g'+g''=g} f(g')h(g'')$ for any $g \in \mathbf{G}$.

The canonical valuation v on $\mathbf{k}((t^{\mathbf{G}}))$ is defined as $\min \mathrm{supp}(f)$ for any $f \in \mathbf{k}((t^{\mathbf{G}}))$ and the canonical derivation D is defined as follows: $(Df)(g) := \delta(f(g))$.

Moreover, the three-sorted theory of this valued D-field in the corresponding three-sorted language is called the theory of ($\mathbf{k}$,$\mathbb{Z}$)-D-henselian valued fields. Now we give an axiomatization of this theory, for a model $\langle K, k, \Gamma \rangle$:

Axiom 1. K and k are differential fields of characteristic zero and $\forall \eta\,[p_n(\eta) \iff \exists \delta\,(\delta^n = \eta)]$.

Axiom 2. K is a valued field whose value group $v(K^\times)$ is equal to Γ via the valuation map v and whose residue field $\pi(\mathcal{O}_K)$ is equal to k via the residue map π.

Axiom 3. $\forall x \in K\ \{[v(Dx) \geqslant v(x)] \wedge [\pi(Dx) = D\pi(x)]\}$ and
$\forall x \exists y\, [Dy = 0 \wedge v(y) = v(x)]$.

Axiom 4 (D-Hensel's Lemma). If $P \in \mathcal{O}_K\{X\}$ is a differential polynomial over $\mathcal{O}_K$, $b \in \mathcal{O}_K$ and $v(P(b)) > 0 = v(\frac{\partial}{\partial X^{(i)}} P(b))$ for some i, then there is some $c \in K$ with $P(c) = 0$ and $v(b - c) \geqslant v(P(b))$.

Axiom 5. $\Gamma \equiv \mathbf{G}$ and $k \equiv \mathbf{k}$.

If $\langle K, D, v\rangle$ is a valued field $\langle K, v\rangle$ with a derivation D which satisfies $\forall x\, [v(Dx) \geqslant v(x)]$ then we say that K is a valued D-field. Moreover, if K satisfies Axiom 4 then the valuation v is said D-henselian.

Now we define the theory of henselian residually p-adically closed D-fields.

Definition 4.1. We will call $\langle K, D, v_p, v, a\rangle$ *a henselian residually p-adically closed D-field* if $\langle K, D, v_p\rangle$ is a p-valued differential field with a D-henselian valuation v such that its differential residue field $\langle k_{K,v}, \widetilde{v}_p\rangle$ is a model of $pCDF$ and its value group is a $\mathbb{Z}$-group with $v(a) = 1$ and $D(a) = 0$.

In the canonical example $\mathbf{k}((t^{\mathbf{G}}))$ of this class of D-henselian valued fields, t plays the role of a in Definition 4.1.

Now we apply Corollary 3.14 of [8] in order to prove a model completeness result for the theory of henselian residually p-adically closed D-fields which can be expressed in the first-order language $\mathcal{L}_{D,p,a} := \mathcal{L}_{p,a} \cup \{D\}$. We denote this $\mathcal{L}_{D,p,a}$-theory by $HRpCDF$. This model-theoretic result will be needed in the proof of Theorem 4.4 which is a differential Hilbert's Seventeenth problem for henselian residually p-adically closed D-fields.

Proposition 4.2. *The $\mathcal{L}_{D,p,a}$-theory $HRpCDF$ is model complete.*

Proof. It is well-known that the theory of $\mathbb{Z}$-groups admits quantifier elimination in the language $\mathcal{L}_V$ of totally ordered abelian groups with divisibility predicates and that the theory $pCDF$ admits quantifier elimination in the differential Macintyre's language $\mathcal{L}_R := \mathcal{L}^D_{\mathcal{D}_p, p_\omega}$. We have to show that any formula is equivalent to an existential formula. So we consider an $\mathcal{L}_{D,p,a}$-formula $\phi(\bar{x})$ where $\bar{x}$ are the free variables. By using [8, Appendix], we can translate this $\mathcal{L}_{D,p,a}$-formula to an $(\mathcal{L}_D, \mathcal{L}_V, \mathcal{L}_R)$-formula $\phi_*(\bar{x})$ where $\mathcal{L}_D := \mathcal{L}_{\text{rings}} \cup \{D, a; P_n, n \in \mathbb{N} \setminus \{0, 1\}\}$ such that D is a derivation and the P_n's are the nth powers predicates. Now we apply Corollary 4.2 in [8] to obtain an $(\mathcal{L}_D, \mathcal{L}_V, \mathcal{L}_R)$-quantifier-free formula $\psi_*(\bar{x})$ equivalent to $\phi_*(\bar{x})$. Since the divisibility predicates $n|.$ of the language of $\mathbb{Z}$-groups are existentially definable in the language $\{+, -, \leqslant, 0, 1\}$ and the p-valuation v_p, the predicates for the nth powers and their negations are existentially definable in the language of fields in $pCDF$, we get by using Lemma 2.26 and the reciprocal translation of [8, Appendix], an existential $\mathcal{L}_{D,p,a}$-formula $\psi(\bar{x})$ equivalent to $\phi(\bar{x})$ (we also used $v(a) = 1$). □

Lemma 4.3. *Let $\langle K, D, v_p, v, a\rangle$ be a valued D-field which is residually p-valued. Then we can extend $\langle K, D, v_p, v, a\rangle$ to a model $\langle L, D, w_p, w, a\rangle$ of $HRpCDF$.*

Proof. We know that if H is a discrete totally ordered abelian group and $\alpha = 1_H$ is the least positive element of H then there exists G an extension of H contained in $\widetilde{H}$, the divisible hull of H such that G is a $\mathbb{Z}$-group with least positive element α (see Lemma 4 in [11]). First we build an henselian unramified valued D-field extension K' of K such that its residue differential field is a model of $pCDF$. Since $pCDF$ is the model companion of the theory of differential p-valued fields, we can consider a p-valued extension k' of k_K which is a model of $pCDF$. By using the existence part of Lemma 7.12 in [16], we obtain our extension K'. Moreover, by Lemma 2.22, we can equip K' with a p-valuation which extends the one of K, is compatible with the valuation on K' and induces the p-valuation on k' (moreover, we can assume K' henselian). Then we build a p-valued totally ramified valued D-field extension K'' of K' such that its value group $v(K''^{\times})$ is equal to G. To this effect, it suffices to use Lemma 2.23 and to apply the calculations in Proposition 7.17 in [16]. Hence we obtain a totally ramified valued D-field extension. Now by using the same construction as in Proposition 3.12 of [8] and the first step of the proof, we obtain an unramified valued D-field extension K''' of K'' which has enough constants and its differential residue field is a model of $pCDF$.

To finish the proof, we proceed as in [16], more precisely we use Lemma 7.25 of [16] to produce the necessary pseudo-convergent sequence in K''' and then use Proposition 7.32 of [16] to actually find a solution in an immediate valued D-field extension. So we obtain the required valued D-field extension L. Since the extension is immediate, the valuation v is henselian on L and $k_{L,v} \models pCDF$ with $v(L^{\times})$ a $\mathbb{Z}$-group. By using Lemma 2.24, we can define a p-valuation on L and then, v is convex for this p-valuation on L; so L is also a p-valued extension of $K\langle \underline{X} \rangle$. □

Now we can prove an analogue of the Hilbert's Seventeenth problem for the theory of henselian residually p-adically closed D-fields as in Theorem 2.28. We will use the following notation for the logarithmic derivative operator: †, i.e. $x^{\dagger} = \frac{Dx}{x}$. We denote by $K\{\underline{X}\}$ the differential ring of differential polynomials in n indeterminates over K and its fraction field by $K\langle \underline{X} \rangle$.

Theorem 4.4. *Let $\langle K, D, v_p, v, a\rangle$ be a henselian residually p-adically closed valued D-field and let f be in $K\langle \underline{X} \rangle$. If $v_p(f(\bar{x})) \geqslant 0$ for every $\bar{x} \in K^n$ such that $f(\bar{x})$ is defined (*).*

Then f belongs to $R^M_{\gamma_p}(K\langle \underline{X} \rangle)$ where M is equal to $A \cdot \mathcal{M}_{K,v}$ such that A is the subring of $K\langle \underline{X} \rangle$ generated by $(K\langle \underline{X} \rangle^{\bullet})^{\dagger}$ and $\gamma(K\langle \underline{X} \rangle)$.

Proof. We proceed as in Theorem 2.28. Suppose that f does not belong to $R^M_{\gamma_p}(K\langle \underline{X} \rangle)$. Since there exists a p-valuation v_p on $K\langle \underline{X} \rangle$ which extends the one of K such that $v_p(M) \geqslant 0$ (see Remark 4.5), we can extend the p-valuation v_p of K to a p-valuation w_p on $K\langle \underline{X} \rangle$ such that $w_p(M) \geqslant 0$ and $w_p(f) < 0$ by applying Lemma 2.21.

We consider $B = pcH(A, K\langle \underline{X} \rangle)$. We get the same properties for B as the ones in Theorem 2.28; furthermore, since $(K\langle \underline{X} \rangle^{\bullet})^{\dagger} \subseteq B$, B is a differential ring in the following sense: if $x \in B$ then $x^{\dagger}$ belongs to B and so $D(x)$ is in B (**). We use Proposition 4.3 instead Proposition 2.23 in Theorem 2.28 in order to obtain an extension $\langle L, D, \overline{w}_p, \overline{w}, a\rangle$ of $\langle K\langle \underline{X} \rangle, D, w_p, w, a\rangle$ with $\langle L, D, \overline{w}\rangle$ D-henselian, $\langle k_{L,\overline{w}}, D, \widetilde{\overline{w}_p}\rangle$ is a p-adically closed differential field, $\langle L, D, \overline{w}_p\rangle$ is a p-valued differential field extension of $\langle K\langle \underline{X} \rangle, D, w_p\rangle$ and $\overline{w}(L^{\times})$ a $\mathbb{Z}$-group such that $\overline{w}(a) = 1_{\overline{w}(L^{\times})}$.

Now it suffices to conclude as in Theorem 2.28 by using the model completeness result of Proposition 4.2 in order to deduce that $\langle K, D, \mathcal{D}_v, a\rangle \prec_{\mathcal{L}_{\mathcal{D},a}\cup\{D\}} \langle L, D, \mathcal{D}_{\overline{w}}\rangle$. □

Remark 4.5. As in Remark 2.29, we have to find, in the previous theorem, a p-valuation v_p on $K\langle \underline{X}\rangle$ which extends the one of K such that $v_p(M) \geqslant 0$, i.e. $v_p(A \cdot \mathcal{M}_{K,v}) \geqslant 0$. We take a $|K|^+$-saturated $\mathcal{L}_{D,p,a}$-elementary extension L of K and so, we satisfy in L the n-type required for $X_1, \cdots, X_n$. This n-type is consistent since in L, we have that $(L^\bullet)^\dagger \subseteq \mathcal{O}_{L,v}$ and so $(L^\bullet)^\dagger \cdot \mathcal{M}_{L,v} \subseteq \mathcal{M}_{L,v} \subseteq \mathcal{O}_{L,v_p}$.

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