UNIFORM COMPANION FOR LARGE LIE DIFFERENTIAL FIELDS OF CHARACTERISTIC ZERO

NICOLAS GUZY[1]

ABSTRACT. In this paper, we adapt the result of M. Tressl (see [3]) in order to show that there is a theory ($UC_{\mathrm{Lie},m}$) of Lie differential fields of characteristic zero, which serves as a model companion for every theory of large and Lie differential fields extending a model complete theory of pure fields. It allows us to introduce the Lie counterpart of classical theories of differential fields in several commuting derivations.

1. INTRODUCTION

The goal of this paper is to give the Lie counterpart of the theory (UC) of differential fields of characteristic zero in m commuting derivations established in [3].

In the next section we define all the Lie differential concepts necessary to this purpose. By using the work of [1], we establish the main ingredients use in [3] in order to write the Lie analogue ($UC_{\mathrm{Lie},m}$) of the theory (UC). These notions will allow us to prove the Lie version of the properties (I) and (II) of the theory (UC) developed in [3] which are in our case the following:

(I) Whenever the Lie differential fields L_1 and L_2 are models of ($UC_{\mathrm{Lie},m}$) and A is a common Lie subring of L_1 and L_2 such that L_1 and L_2 have the same universal theory over A as pure fields then they have the same universal theory over A as Lie differential fields.

(II) Every Lie differential fields F which is large can be extended to a model of ($UC_{\mathrm{Lie},m}$) and this extension is elementary in the language of rings.

In the last section we show that properties (I) and (II) of ($UC_{\mathrm{Lie},m}$) above imply that for every model complete theory T of large fields of characteristic zero in the language of rings, the theory $T_{\mathrm{Lie}} \cup (UC_{\mathrm{Lie},m})$ of Lie differential fields is model complete (where T_{Lie} is the corresponding theory of Lie differential fields). Moreover, if this is the case, $T_{\mathrm{Lie}} \cup (UC_{\mathrm{Lie},m})$ is complete if T is complete and $\widetilde{T}_{\mathrm{Lie}} \cup (UC_{\mathrm{Lie},m})$ has quantifier elimination if a definable expansion $\widetilde{T}$ has quantifier elimination (see Theorem 3.11).

Finally we apply our results in Proposition 3.12 to give the model completion of some theories of Lie differential fields; for example for Lie differential fields of characteristic zero we get the theory of Lie differentially closed fields obtained by Y. Yaffe in [5].

Date: May 8, 2006.

[1] Research Fellow at the "Fonds National de la Recherche Scientifique"; partially supported by grants of the "National Bank of Belgium", FRFC contract 2.4.559.06 F.

2. Lie differential algebra

Let us fix a field $\mathbb{F}$ of characteristic zero, a finite-dimensional $\mathbb{F}$-vector space $\mathbb{L}$ with a Lie multiplication making it a Lie algebra over a subfield of $\mathbb{F}$ and a vector space homomorphism $\phi_{\mathbb{F}} : \mathbb{L} \to Der(\mathbb{F})$, the Lie algebra of derivations on $\mathbb{F}$, preserving the Lie multiplication. We fix a basis $\{\delta_1, \dots, \delta_m\}$ of $\mathbb{L}$ and let $[\delta_i, \delta_j] := \sum_{k=1}^{m} c_{ij}^k \delta_k$ for some elements c_{ij}^k in $\mathbb{F}$. The elements (c_{ij}^k) are called the *structure constants* of $\mathbb{L}$.

Definition 2.1. A Lie differential field $\langle K, \delta_1, \dots, \delta_m \rangle$ is a differential field extension of $\mathbb{F}$ such that $K \models \forall x \ (\delta_i \delta_j x - \delta_j \delta_i x = \sum_{k=1}^{m} c_{ij}^k \cdot \delta_k x)$.

Let $\langle \mathbb{N}^m, + \rangle$ be the cartesian product of $\mathbb{N}$ considered as an additive abelian monoid with the addition defined component by component. If $\alpha := (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ then we denote by $|\alpha|$ the sum $\sum_{i=1}^{m} \alpha_i$. We let $\epsilon_k := (\underbrace{0, \dots, 0}_{k-1 \text{ times}}, 1, 0 \dots, 0) \in \mathbb{N}^m$.

Let $\mathbf{Y} := (Y_1, \dots, Y_N)$ be a set of differential indeterminates considered as an ordered set $Y_1 < \dots < Y_N$.

We consider the following set of indeterminates over K

$$\Theta\mathbf{Y} := \{\delta^\alpha y : y \in \mathbf{Y}, \alpha \in \mathbb{N}^m\}$$

and its elements are called *the derivatives* in $\mathbf{Y}$. We identify $\delta^0 y$ with $y \in \mathbf{Y}$.

Let us consider the polynomial algebra over K in the derivatives $(\delta^\alpha y)_{\alpha \in \mathbb{N}^m, y \in \mathbf{Y}}$ which is denoted by $K[\Theta\mathbf{Y}]$.

Now we define a *rank function* $rk : \Theta\mathbf{Y} \to \mathcal{O} := \mathbb{N} \times \mathbf{Y} \times \mathbb{N}^m$ which determines a total ordering on $\Theta\mathbf{Y}$ as follows

$$rk(y_\alpha) := (|\alpha|, y, \alpha)$$

where $\mathcal{O}$ is lexicographically ordered.

Definition 2.2. Let $f \in K[\Theta\mathbf{Y}] \setminus K$. The *leader* u_f of f is the variable $\delta^\alpha y \in \Theta\mathbf{Y}$ of highest rank rk which appears in f. Moreover $u_f^* := u_f^{\deg_{u_f} f}$ is the *highest power* of u_f in f.

Let $f = f_d u_f^d + \dots + f_1 u_f + f_0$ with polynomials $f_i \in K[\delta^\beta z : \delta^\beta z \neq u_f, \beta \in \mathbb{N}^m$, $z \in \mathbf{Y}]$ and $f_d \neq 0$. The *initial* $I(f)$ of f is defined as f_d and the *separant* $S(f)$ of f as $\frac{\partial f}{\partial u_f}$.

Now we extend the rank rk on $\Theta\mathbf{Y}$ to $K[\Theta\mathbf{Y}]$ as follows:

(1) if $f \in K$ then $rk(f) = -1 < \mathcal{O}$;
(2) if $f \in K[\Theta\mathbf{Y}] \setminus K$ then $rk(f) := rk(u_f)$.

Definition 2.3 (See also p. 169 in [1])**.** We can define a ranking $\prec$ on $K[\Theta\mathbf{Y}]$ from the total ordering on $\mathcal{O}$ which is a preorder as follows: $f \prec g$ if $rk(f) < rk(g)$.

We may refine this preorder by the following function on $K[\Theta\mathbf{Y}]$, denoted by rk^*

$rk^*(f) := (rk(u_f), \deg_{u_f} f) \in \mathcal{O} \times \mathbb{N}$, where $\mathcal{O} \times \mathbb{N}$ is lexicographically ordered.

We shall note $f \prec^* g$ if $rk^*(f) < rk^*(g)$.

Now we extend the derivations δ_k on K to $K[\Theta\mathbf{Y}]$ and then we show that it is a Lie differential field extending the Lie differential field $\langle K, \delta_1, \ldots, \delta_m\rangle$; it is called the Lie differential polynomial ring over K and is denoted by $K\{\mathbf{Y}\}$.

Definition 2.4 (See Definition 1.13 in [5])**.** Let $f \in K[\Theta\mathbf{Y}]$ and $k \in \{1, \ldots, m\}$.

(1) If $rk(f) = -1 < \mathcal{O}$ then by definition, $f \in K$ and so, $\delta_k(f)$ is defined.
(2) If f is of the form $\delta^\alpha y$ then $\delta_k(\delta^\alpha y)$ is defined as follows:
  (a) $\forall t < k$ $\alpha_t = 0$ then

$$\delta_k(\delta^\alpha y) = \delta^{\alpha+\epsilon_k} y.$$

  (b) otherwise we let $l = \min\{t : \alpha_t \neq 0\}$ (so $l < k$) and $\alpha' := \alpha - \epsilon_l$, then

$$\delta_k(\delta^\alpha y) = \delta_l\delta_k(\delta^{\alpha'} y) + \sum_{i=1}^{m} c^i_{kl} \cdot \delta_i(\delta^{\alpha'} y).$$

(3) If $f := \sum_{i=0}^{d} f_i \cdot u_f^i$ (see Definition 2.2) then

$$\delta_k(f) := \sum_{i=0}^{d} \delta_k(f_i) \cdot u_f^i + \underbrace{\sum_{i=0}^{d} f_i \cdot i \cdot u_f^{i-1}}_{S(f)} \cdot \delta_k(u_f).$$

Clearly, for any $i \in \{1, \ldots, m\}$, δ_k is a derivation on $K[\Theta\mathbf{Y}]$.

Proposition 2.5. *Let $f \in K[\Theta\mathbf{Y}]$, $1 \leq k \leq m$ and $u_f := \delta^\alpha y$.*
Then $\delta_k(f)$ is well-defined in $K[\Theta\mathbf{Y}]$. Moreover we have the following properties:

(1) $rk(\delta_k f) = (|\alpha| + 1, y, \alpha + \epsilon_k)$;
(2) $u_{\delta_k u_f} = u_{\delta_k f} = \delta^{\alpha+\epsilon_k} y$;
(3) $rk(\delta_k\delta^\alpha y - \delta^{\alpha+\epsilon_k} y) < rk(\delta^{\alpha+\epsilon_k} y)$ *which implies* $rk(\delta_k\delta^\alpha y) = rk(\delta^{\alpha+\epsilon_k} y)$.

Proof. We proceed by induction on $rk(f)$.

Suppose that the properties are checked for any polynomial g in $K[\Theta Y]$ such that $rk(g) < rk(f)$ and for any $1 \leq k \leq m$. We first show the proposition for $f := \delta^\alpha y$, i.e. $f = u_f := \delta^\alpha y$. By definition of the derivations on $K[\Theta\mathbf{Y}]$, we have to distinguish two cases.

First, $\forall t < k$, $\alpha_t = 0$ then all the properties are clear.

Otherwise we let $l = \min\{t : \alpha_t \neq 0\}$ (so $l < k$) and $\alpha' := \alpha - \epsilon_l$. By definition

$$\delta_k(\delta^\alpha y) = \delta_l\delta_k(\delta^{\alpha'} y) + \sum_{i=1}^{m} c^i_{kl}\delta_i(\delta^{\alpha'} y).$$

Since $\alpha' < \alpha$, we have $\delta^{\alpha'} y \prec \delta^\alpha y$.

By induction, we can apply (1) for $\delta_k(\delta^{\alpha'} y)$ and we obtain

$$rk(\delta_k\delta^{\alpha'} y) = (|\alpha|, y, \alpha' + \epsilon_k) < rk(\delta^\alpha y) \text{ since } \mathcal{O} \text{ is lexicographically ordered.}$$

By induction, $\delta_l\delta_k(\delta^{\alpha'} y)$ and $\delta_i\delta^{\alpha'} y$ $(1 \leq i \leq m)$ are well-defined; hence $\delta_k\delta^\alpha y$ is well-defined.

Let us prove (1). We have $rk(\delta_l(\delta_k\delta^{\alpha'}y)) = (|\alpha|+1, y, \alpha+\epsilon_k)$ and $rk(\delta_i(\delta^{\alpha'}y)) = (|\alpha|, y, \alpha'+\epsilon_i)$ for all $i \in \{1,\dots,m\}$. By definition of rk on $K[\Theta\mathbf{Y}]$, we get $rk(\delta_k\delta^\alpha y) = rk(\delta^{\alpha+\epsilon_k}y)$.

Let us prove (3), which will imply (2). By induction, we apply (3) and we get $\delta_k\delta^{\alpha'}y = \delta^{\alpha'+\epsilon_k}y + z$ where $rk(z) < rk(\delta^{\alpha'+\epsilon_k}y)$. Since $\delta^{\alpha'+\epsilon_k}y \prec \delta^\alpha y$, we can apply (3) by induction, and we get

$$\delta_l\delta_k\delta^{\alpha'}y = \delta_l\delta^{\alpha'+\epsilon_k}y + \delta_l z = \delta^{\alpha+\epsilon_k}y + w + \delta_l z$$

where $rk(w) < rk(\delta^{\alpha+\epsilon_k}y)$ and $rk(\delta_l z) < rk(\delta^{\alpha+\epsilon_k}y)$ by induction of (1), (2) and (3).

Since $rk(\delta_i\delta^\alpha y) = (|\alpha|, y, \alpha'+\epsilon_i) < rk(\delta^{\alpha+\epsilon_k}y)$, we have $rk(\delta_l\delta^\alpha y - \delta^{\alpha+\epsilon_k}y) < rk(\delta^{\alpha+\epsilon_k}y)$.

Moreover, it proves that $u_{\delta_k\delta^\alpha y} = u_{\delta_l\delta_k\delta^{\alpha'}y} = \delta^{\alpha+\epsilon_k}y$ and we get (2) for $f := \delta^\alpha y$.

Now we prove the required properties for $f := \sum_{i=0}^d f_i u_f^i$ where $u_f \in \Theta\mathbf{Y}$. By definition, we have

$$\delta_k f = \sum_{i=0}^d \delta_k(f_i)\cdot u_f^i + \sum_{i=0}^d i\cdot f_i\cdot u_f^{i-1}\cdot\delta_k(u_f) = \big[\sum_{i=0}^d \delta_k(f_i)\cdot u_f^i\big] + S(f)\cdot\delta_k(u_f).$$

We get from $f_i \prec u_f$ (i.e. $rk(f_i) < rk(u_f)$) and $\delta_k u_f$ is well-defined that $\delta_k f_i$ is well-defined and so, $\delta_k f$ is well-defined. By induction of (1), we have $rk(f_i) < rk(\delta_k u_f)$.

Therefore, it remains to show that $rk(\delta_k f_i) < rk(\delta_k u_f)$ and we deduce that $rk(\delta_k f) = rk(\delta_k u_f)$ and $u_{\delta_k f} = u_{\delta_k u_f} = \delta^{\alpha+\epsilon_k}y$. Since $rk(f_i) < rk(u_f) = rk(f)$, we have $u_{\delta_k f_i} = u_{\delta_k u_{f_i}} = \delta^{\beta+\epsilon_k}z$ if $u_{f_i} = \delta^\beta z$ and since $\delta^\beta z \prec u_f$, we have clearly $\delta^{\beta+\epsilon_k}z \prec \delta^{\alpha+\epsilon_k}y$, which proves the result. □

Notation 2.6. For any $f \in K[\Theta\mathbf{Y}]$, we denote $\delta_1^{\alpha_1}\dots\delta_m^{\alpha_m}f$ by $\delta^\alpha f$.

Remark 2.7. (1) We have also proved that

$$\delta^\beta(y_\alpha) - y_{\alpha+\beta} \prec y_{\alpha+\beta} \text{ where } \alpha,\beta \in \mathbb{N}^m.$$

Following the terminology in [1], it says that the commutation rules are non-trivial.

(2) Among the lines of the previous proof, we show that for any polynomial in $K[\Theta\mathbf{Y}] \setminus K$ and $\beta \in \mathbb{N}^m$ with $|\beta| > 0$ and $u_f := \delta^\alpha y$

$$u_{\delta^\beta f} = \delta^{\alpha+\beta}y \text{ and } I(\delta^\beta f) = S(f) \text{ (see also Proposition 6.1 in [1]).}$$

(3) The ranking $\prec$ has the following properties:
- $\delta^\alpha y \prec \delta_i\delta^\alpha y$;
- $\delta^\alpha y \prec \delta^\beta z$ implies $\delta_i\delta^\alpha y \prec \delta_i\delta^\beta z$;
- the compatibility with the commutation rules following the terminology in [1] $[\delta_i,\delta_j](\delta^\alpha y) \prec \delta^{\alpha+\epsilon_i+\epsilon_j}y$;
- the two following says that the ranking is admissible:
  - $|\alpha| < |\beta|$ implies $\delta^\alpha y \prec \delta^\beta y$;
  - $\delta^\alpha y \prec \delta^\beta z$ implies $\delta^{\alpha+\gamma}y \prec \delta^{\beta+\gamma}z$.

The previous proposition allows us to show that the differential field $K[\Theta\mathbf{Y}]$ is a Lie differential ring which is denoted by $K\{\mathbf{Y}\}$.

It suffices to use the previous results and the proof of Y. Yaffe in Proposition 1.19 of [5].

Proposition 2.8. *For any $f \in K\{\mathbf{Y}\}$ and $1 \leq k, l \leq m$, we have*

$$[\delta_k, \delta_l](f) = \sum_{i=1}^{m} c_{kl}^{i} \delta_i(f).$$

In the sequel, for any $\alpha \in \mathbb{N}^m$, we denote $\delta_1^{\alpha_1} \dots \delta_m^{\alpha_m}$ by δ^α; in particular, we will identify $\delta^{\epsilon_k} y$ with $\delta_k y$.

Notation 2.9. Let $\mathbb{N}_m := \{1, \dots, m\}$ and let us consider W_m the semi-group of words formed on $\mathbb{N}_m$: an element $I \in W_m$ can be represented by an empty tuple () or a tuple $(i_1, \dots, i_p)$ for some $p \in \mathbb{N} \setminus \{0\}$ and $i_k \in \mathbb{N}_m$. The length of a word $I = (i_1, \dots, i_p)$ is $|I| = p$. A word $I = (i_1, \dots, i_p)$ is monotone if $i_1 \leq \dots \leq i_p$. We shall denote by $\bar{I}$ the monotone word which is determined as follows: if $I = (i_1, \dots, i_p)$ is a word then $\bar{I}$ is the m-tuple $\alpha := (\alpha_1, \dots, \alpha_m)$ such that α_i is the cardinal of the set $\{j \in \mathbb{N} : \ i_j = i\}$.

We will use this terminology in the two next results of [1].

Proposition 2.10 (See Proposition 5.2 in [1]). *For any $I \in W_m$ with $|I| \geq 2$, for any $f \in K\{\mathbf{Y}\} \setminus K$, there is a family $\{a_\alpha\}_{\alpha \in \mathbb{N}^m}$ of polynomial functions in $\{\delta^K(c_{ij}^l) : |K| \leq |I| - 2\}$ with coefficients in $\mathbb{Z}$ such that*

$$\delta^I f = \delta^{\bar{I}} f + \sum_{\alpha \in \mathbb{N}^m, |\alpha| < |I|} a_\alpha \delta^\alpha f.$$

Corollary 2.11 (See Corollary 5.3 in [1]). *For all $\alpha, \beta \in \mathbb{N}^m$ and any $f \in K\{\mathbf{Y}\}$, there is a family $\{a_\gamma\}_{\gamma \in \mathbb{N}^m}$ of polynomial functions in $\{\delta^\mu(c_{ij}^l) : |\mu| < |\alpha + \beta| - 2\}$ with coefficients in $\mathbb{Z}$ such that*

$$\delta^\alpha \delta^\beta f = \delta^{\alpha+\beta} f + \sum_{|\gamma| < |\alpha+\beta|} a_\gamma \delta^\gamma f.$$

Definition 2.12. A *(Lie) differential ideal* in $K\{\mathbf{Y}\}$ is an ideal that is closed under the derivations $\delta_1, \dots, \delta_m$. We shall note $[G]$ the differential ideal generated by a non-empty set G of $K\{\mathbf{Y}\}$. We define $[G]$ as the intersection of all the differential ideals containing G.

Proposition 2.13 (See Proposition 5.7 in [1]). *Let G be a non-empty subset of $K\{\mathbf{Y}\}$. The differential ideal $[G]$ is the ideal generated by $\Theta G := \{\delta^\alpha g : \ g \in G, \ \alpha \in \mathbb{N}^m\}$.*

That means that any element f of $[G]$ can be written as $f = \sum_{g \in G, \alpha \in \mathbb{N}^m} a_{\alpha,g} \delta^\alpha g$ where the $a_{\alpha,g}$ form a family in $K\{\mathbf{Y}\}$ with finite support.

Definition 2.14. A differential ideal is *radical* if whenever a positive power of an element belongs to the differential ideal, the element itself belongs to the differential ideal. The differential radical ideal of G, denoted by $\{G\}$, is defined as the intersection of all the differential radical ideals containing G.

As in the classical differential setting, we get that

$$\{G\} := \{f \in K\{\mathbf{Y}\} : \exists k \in \mathbb{N} \setminus \{0\},\ f^k \in [G]\}.$$

Definition 2.15. If H is a subset of $K\{\mathbf{Y}\}$, we denote by H^∞ the multiplicative set (containing 1) generated by the elements of H.

Let I be a differential ideal in $K\{\mathbf{Y}\}$. We define $I : H^\infty$ as follows:

$$\{f \in K\{\mathbf{Y}\} : \exists h \in H^\infty,\ hf \in I\}.$$

The following property shows that $I : H^\infty$ is a differential ideal.

Lemma 2.16. *(See Lemma 5.8 in [1]) Let $f, g \in K\{\mathbf{Y}\}$ and $\alpha \in \mathbb{N}^m$. We have the following*

$$f^{|\alpha|}\delta^\alpha g \equiv f^{|\alpha|}\delta^\alpha(fg) \ mod\ (\delta^\gamma(fg) || \gamma| < |\alpha|).$$

Definition 2.17.
(1) If $f, g \in K\{\mathbf{Y}\}$, $g \notin K$, $u_g := \delta^\alpha y$ then f is said to be *weakly reduced with respect to g* if f involves no derivatives of the type $\delta^{\alpha+\gamma} y$ where $|\gamma| > 0$; f is called *reduced with respect to g* if f is weakly reduced with respect to g and if $\deg_{u_g} f < \deg_{u_g} g$.
(2) The polynomial f is called *(weakly) reduced with respect to a non-empty subset* G of $K\{\mathbf{Y}\} \setminus K$ if f is (weakly) reduced with respect to every $g \in G$.
(3) A non-empty subset G of $K\{\mathbf{Y}\} \setminus K$ is called *autoreduced* if every $f \in G$ is reduced with respect to all $g \in G$, $g \neq f$. If G consists of a single element then G is called autoreduced as well.

It is easy to see that $u_f \neq u_g$, hence $rk^*(f) \neq rk^*(g)$ if f, g are different polynomials from an autoreduced set.

Moreover we have

Proposition 2.18. *Every autoreduced set of $K\{\mathbf{Y}\} \setminus K$ is finite.*

Proof. See Lemma 15 (a) in [2, Chapt. 0, Sect. 17]. □

Let ∞ be an element bigger than every element in $\mathcal{O}$ and let $(\mathcal{O} \cup \{\infty\})^{\mathbb{N}}$ be equipped with the lexicographic order. We define the rank rk^* of an autoreduced set G to be an element of $(\mathcal{O} \cup \{\infty\})^{\mathbb{N}}$ as follows.

Let $G := (g_1, \ldots, g_l)$ with $rk^*(g_1) < \ldots < rk^*(g_l)$. Then

$$rk^*(G) := (rk^*(g_1), \ldots, rk^*(g_l), \infty, \infty, \ldots).$$

Proposition 2.19. *There is no infinite sequence $G_1, G_2, \ldots$ of autoreduced sets with the property $rk^*(G_1) > rk^*(G_2) > \ldots$.*

Proof. See the classical proof in [2, Chap. I, Sect. 10, Prop. 3]. □

Definition 2.20. If $M \subseteq K\{\mathbf{Y}\}$ is a set not contained in K then by Proposition 2.19, the set $\{rk^*(G) : G \subseteq M$ is autoreduced $\}$ has a minimum. Every autoreduced subset G of M with this rank $rk^*(G)$ is called *a characteristic set of M*.

Proposition 2.21 (See Proposition 2.7 in [3]). *If G is a characteristic set of $M \subseteq K\{\mathbf{Y}\}$ and $f \in M \setminus K$, then f is not reduced with respect to G.*

Definition 2.22. Let $G := (g_1, \ldots, g_l)$ be a subset of $K\{\mathbf{Y}\} \setminus K$. We define

$$H(G) := \prod_{i=1}^{l} I(g_i)S(g_i) \text{ and } H_G := \{\prod_{i=1}^{l} I(g_i)^{n_i} S(g_i)^{m_i} : n_i, m_i \in \mathbb{N}\}.$$

We know that $S(g)$ and $I(g)$ are reduced with respect to G ($g \in G$) if G is an autoreduced set in $K\{\mathbf{Y}\}$. We will use this fact in the two next results which will give the Lie counterpart of Theorem 2.9 in [3].

Lemma 2.23 (See Lemma 6 in Chapter 1 of [2])**.** *Let G be an autoreduced set in $K\{\mathbf{Y}\}$ and $f \in K\{\mathbf{Y}\}$.*

Then there exists a polynomial $\widetilde{f} \in K\{\mathbf{Y}\}$ which is weakly reduced with respect to G and natural numbers s_g ($g \in G$) such that

$$\prod_{g \in G} S(g)^{s_g} f \equiv \widetilde{f} \textit{ mod } [G].$$

More precisely, $\prod_{g \in G} S(g)^{s_g} f - \widetilde{f}$ can be written as a linear combination over $K\{\mathbf{Y}\}$ of derivatives $\delta^\gamma g$ ($|\gamma| > 0$ and $g \in G$) with $rk(\delta^{\alpha+\gamma} y) \leq rk(f)$ whenever $\delta^\alpha y := u_g$.

Proof. If f is weakly reduced with respect to G, we define $\widetilde{f} := f$ and $s_g = 0$ ($g \in G$).

So we may assume that f is not weakly reduced with respect to G, that is, f involves a proper derivative $\delta^{\alpha+\gamma} y$ with $u_g := \delta^\alpha y$ for some $g \in G$, $|\gamma| > 0$.

We are going to define $s_{g'}$ ($g' \in G$) and $\widetilde{f}$ by induction on the highest such $\delta^{\alpha+\gamma} y$ (with respect to $\prec$).

Let $\delta^{\alpha+\gamma} y$ be the highest derivative which appears in f such that $|\gamma| > 0$ and $\delta^\alpha y := u_g$ for some $g \in G$. By Remark 2.7 (2), we may write $S(g)\delta^{\alpha+\gamma} y = \delta^\gamma g + t$ where $t \in K\{\mathbf{Y}\}$ and $rk(t) < rk(\delta^{\alpha+\gamma} y)$.

Letting $e = \deg_{\delta^{\alpha+\gamma} y} f$, we may write $f = \sum_{i=0}^{e} f_i \cdot (\delta^{\alpha+\gamma} y)^i$ where $f_i \in K\{\mathbf{Y}\}$ are of rank rk less than $rk(\delta^{\alpha+\gamma} y)$.

Then

$$S(g)^e f = \sum_{i=0}^{e} S(g)^{e-i} \cdot (f_i \cdot \delta^{\alpha+\gamma} y)^i \equiv \sum_{i=0}^{e} S(g)^{e-i} \cdot f_i \cdot t^i \ (\text{mod } \delta^\gamma g).$$

Obviously $h := \sum_{i=0}^{e} S(g)^{e-i} \cdot f_i \cdot t^i$ cannot involve a proper derivative as high as $\delta^{\alpha+\gamma} y$ (with respect to $\prec$) and $rk(h) \leq rk(f)$.

Therefore by induction, $\widetilde{h}$ and the corresponding natural numbers $\tilde{s}_g$ are defined and have the required properties. We now define $\widetilde{f} := \widetilde{h}$, $s_g := \tilde{s}_g + e$ and $s_{g'} := \tilde{s}_{g'}$ ($g' \in G$ and $g' \neq g$), which gives the result. □

Let us remark that in order to write this lemma under its actual form with $[G]$, we use Proposition 2.13.

Proposition 2.24 (See Proposition 1 in Chapter 1 of [2])**.** *Let G be an autoreduced set in $K\{\mathbf{Y}\}$ and $f \in K\{\mathbf{Y}\} \setminus K$.*

Then there exists a polynomial $\widehat{f} \in K\{\mathbf{Y}\}$ *which is reduced with respect to* G *(so* $rk^*(\widehat{f}) < rk^*(f)$*) and natural numbers* i_g, s_g *(*$g \in G$*) such that*

$$\prod_{g \in G} I(g)^{i_g} S(g)^{s_g} f \equiv \widehat{f} \ mod \ [G].$$

More precisely, $\prod_{g\in G} I(g)^{i_g} S(g)^{s_g} f - \widehat{f}$ *can be written as a linear combination over* $K\{\mathbf{Y}\}$ *of derivatives* $\delta^\gamma g$ *(*$|\gamma| > 0$ *and* $g \in G$*) with* $rk(\delta^{\alpha+\gamma} y) \leq rk(f)$ *whenever* $\delta^\alpha y := u_g$.

Proof. Let $G := \{g_1, \ldots, g_r\}$. By applying Lemma 2.23, we get a polynomial $\widetilde{f}$ that is weakly reduced with respect to G. Let $e_r := \deg_{u_{g_r}} \widetilde{f}$ and $d_r := \deg_{u_{g_r}} g_r$. We define $i_r := e_r - d_r + 1$ or $i_r = 0$ according $e_r \geq d_r$ or $e_r < d_r$. In either case, by using the pseudo euclidian division, we may write

$$I(g_r)^{i_r} \widetilde{f} \equiv f_{(r)} \text{ mod } (g_r)$$

where $f_{(r)} \in K\{\mathbf{Y}\}$ is weakly reduced with respect to G and is reduced with respect to g_r. We proceed in the same way with $\widetilde{f}$, g_r and r replaced by $f_{(r)}$, g_{r-1} and $r-1$; and so on for each polynomial of G until g_1.

Finally we get $f_{(1)} \in K\{\mathbf{Y}\}$ which is reduced with respect to G and we let $\widehat{f} := f_{(1)}$. □

Now we introduce the notion of *coherence* for autoreduced sets in $K\{\mathbf{Y}\} \setminus K$. It will allow us to state a Lie analogue of *Rosenfeld's Lemma* (see Theorem 2.14 in [3]).

Notation 2.25. For $\delta^\alpha y \in \Theta\mathbf{Y}$, we note $\Theta\mathbf{Y}_{<\delta^\alpha y}$ the set of derivatives of rank rk lower than $rk(\delta^\alpha y)$.

Let G be an autoreduced set in $K\{\mathbf{Y}\} \setminus K$. We will denote by ΘG the set of derivatives of elements of G

$$\{\delta^\alpha g : \ g \in G, \alpha \in \mathbb{N}^m\}.$$

We denote by $\Theta G_{<\delta^\alpha y}$ the set $\Theta G \cap K[\Theta\mathbf{Y}_{<\delta^\alpha y}]$.

Definition 2.26. Let G be an autoreduced set in $K\{\mathbf{Y}\} \setminus K$. We say that G is *coherent* if whenever $g, g' \in G$ are such that $u_g := \delta^\alpha y$ and $u_{g'} := \delta^\beta y$ for some $y \in \mathbf{Y}$ and $\alpha, \beta \in \mathbb{N}^m$ then for any $\gamma \in (\alpha + \mathbb{N}^m) \cap (\beta + \mathbb{N}^m)$ we have

$$S(g')\delta^{\gamma-\alpha} g - S(g)\delta^{\gamma-\beta} g' \in (\Theta G_{<\delta^\gamma y}) : H(G)^\infty.$$

Theorem 2.27 (See Theorem 6.3 in [1]). *Let* G *be an autoreduced set in* $K\{\mathbf{Y}\} \setminus K$ *then any differential polynomial of the differential ideal* $[G] : H(G)^\infty$ *that is weakly reduced with respect to* G *belongs to* $(G) : H(G)^\infty$.

The following proposition shows the finiteness of the test for coherence.

For $\alpha, \beta \in \mathbb{N}^m$, we denote by $\alpha \diamond \beta$ the element of $\mathbb{N}^m$ having $\max(\alpha_i, \beta_i)$ for ith element. Any element of $(\alpha + \mathbb{N}^m) \cap (\beta + \mathbb{N}^m)$ can be written $\alpha \diamond \beta + \mu$ for some $\mu \in \mathbb{N}^m$.

Proposition 2.28 (See Proposition 6.6 in [1]). *Let G be an autoreduced set in $K\{\mathbf{Y}\}\setminus K$. If for all $g, g' \in G$ such that $u_g := \delta^\alpha y$ and $u_{g'} := \delta^\beta y$ for some $y \in \mathbf{Y}$ and $\alpha, \beta \in \mathbb{N}^m$ we have*

$$\Delta(g, g') := S(g')\delta^{\alpha\diamond\beta-\alpha} g - S(g)\delta^{\alpha\diamond\beta-\beta} g' \in (\Theta G_{<\delta^{\alpha\diamond\beta} y}) : H(G)^\infty$$

then G is coherent.

Lemma 2.29 (See Lemma 6 in Chapter 3 of [2]). *Let G be an autoreduced coherent set in $K\{\mathbf{Y}\} \setminus K$. Then $[G] : H(G)^\infty$ is a prime differential ideal if $(G) : H(G)^\infty$ is prime.*

Proof. By Lemma 2.16, it is clearly differential. Let us prove that $[G] : H(G)^\infty$ is prime.

Let $f, h \in K\{\mathbf{Y}\}$ such that $f \cdot h \in [G] : H(G)^\infty$. By Lemma 2.23, we get $\prod_{g\in G} S(g)^{s_g} f \equiv \widetilde{f} \bmod [G]$ and $\prod_{g\in G} S(g)^{s'_g} h \equiv \widetilde{h} \bmod [G]$ such that $\widetilde{f}$ and $\widetilde{h}$ are weakly reduced with respect to G. So $\widetilde{f} \cdot \widetilde{h}$ is weakly reduced with respect to G and belongs to $[G] : H(G)^\infty$. By Theorem 2.27, $\widetilde{f} \cdot \widetilde{h} \in (G) : H(G)^\infty$ and so, since $(G) : H(G)^\infty$ is prime, $\widetilde{f}$ or $\widetilde{h}$ belongs to $(G) : H(G)^\infty$. Therefore we conclude that f or h belongs to $[G] : H(G)^\infty$. □

Now we can prove the Lie analogue of the Rosenfeld's Lemma.

Theorem 2.30.
1. *If G is a characteristic set of a prime differential ideal P in $K\{\mathbf{Y}\}$ such that $P \cap K = \{0\}$ then $P = [G] : H(G)^\infty$, G is coherent and $(G) : H(G)^\infty$ is a prime ideal not containing a non-zero element reduced with respect to G.*
2. *Conversely, if G is a coherent autoreduced set in $K\{\mathbf{Y}\} \setminus K$ such that $(G) : H(G)^\infty$ is a prime ideal and does not contain a non-zero element reduced with respect to G then G is a characteristic set of a prime differential ideal in $K\{\mathbf{Y}\}$.*

Proof. Let G be a characteristic set of a prime differential ideal P in $K\{\mathbf{Y}\}$. By Proposition 2.21, P does not contain a non-zero element reduced with respect to G; therefore $S(g), I(g) \notin P$. Let f be in P. By Proposition 2.24, there is a h in H_G (see Definition 2.22)

$$hf \equiv \widehat{f} \bmod [G]$$

with $\widehat{f}$ reduced with respect to G. So $\widehat{f} = 0$ and $f \in [G] : H(G)^\infty$; which implies $P = [G] : H(G)^\infty$.

Let g, g' be in G such that $u_g := \delta^\alpha y$ and $u_{g'} := \delta^\beta y$ for some $y \in \mathbf{Y}$ and $\alpha, \beta \in \mathbb{N}^m$. By Proposition 2.28, in order to show that G is coherent, it suffices to verify that

$$\Delta(g, g') := S(g')\delta^{\alpha\diamond\beta-\alpha} g - S(g)\delta^{\alpha\diamond\beta-\beta} g' \in (\Theta G_{<\delta^{\alpha\diamond\beta} y}) : H(G)^\infty.$$

But we know that

$$\delta^{\alpha\diamond\beta-\alpha} g = S(g)\delta^{\alpha\diamond\beta} y + t \text{ and } \delta^{\alpha\diamond\beta-\alpha} g' = S(g')\delta^{\alpha\diamond\beta} y + t'$$

with t, t' of rank lower than $rk(\delta^{\alpha\diamond\beta} y)$. So $\Delta(g, g') = S(g')t - S(g)t' \in P$ belongs to the ideal $(\Theta G_{<\delta^{\alpha\diamond\beta} y}) : H(G)^\infty$ by Proposition 2.24; and G is coherent.

Let V be the set of derivatives $(\delta^\alpha y)$ such that $G \subseteq K[V]$. It is easy to see that the ideal $(G) : H(G)^\infty$ in $K\{\mathbf{Y}\}$ is the ideal generated in $K\{\mathbf{Y}\}$ by the ideal $(G) : H(G)^\infty$ in $K[V]$ and that the ideal $(G) : H(G)^\infty$ in $K[V]$ is the intersection with $K[V]$ of the ideal $(G) : H(G)^\infty$ in $K\{\mathbf{Y}\}$. It follows that the condition that the ideal $(G) : H(G)^\infty$ be prime is independent of the polynomial algebra in which the ideal is taken.

In particular, we may take V to be the set of all derivatives $\delta^\alpha y$ that appear in at least one element of G. A similar remark holds for the condition that $(G) : H(G)^\infty$ does not contain a non-zero element reduced with respect to G. This concludes (1).

Conversely, let G be an autoreduced and coherent set in $K\{\mathbf{Y}\} \setminus K$ such that $(G) : H(G)^\infty$ is prime and does not contain a non-zero element reduced with respect to G. By Lemma 2.29, $[G] : H(G)^\infty$ is a prime differential ideal and, by Theorem 2.27, an element of $[G] : H(G)^\infty$ reduced with respect to G is contained in $(G) : H(G)^\infty$ and therefore must be equal to 0. From this, it easily follows that $[G] : H(G)^\infty$ does not contain an autoreduced set of rank rk^* lower than $rk^*(G)$; that is G is a characteristic set of $[G] : H(G)^\infty$. □

Thanks to all the previous results, we easily see that the proof of the following result in [4] follows the same lines.

Theorem 2.31. *Let P be a prime differential ideal in $K\{\mathbf{Y}\}$ such that $P \cap K = 0$, let G be a characteristic set of P and let $\varphi : K\{\mathbf{Y}\} \to S := K\{\mathbf{Y}\}/P$ be the canonical Lie differential homomorphism.*

We take $h := \varphi(H(G))$, $V := \{\delta^\gamma y \in \Theta\mathbf{Y} :\ \delta^\gamma y$ is not of the form $\delta^{\alpha+\mu} y$ with $|\mu| > 0$ and $\delta^\alpha y = u_g$ for some $g \in G\}$, $V_B := \{\delta^\gamma y \in V :\ \delta^\gamma y$ appears in some $g \in G\}$, $B := \varphi(K[V_B])$ and $P := \varphi(K[V \setminus V_B])$.

Then $h \in B$, $h \neq 0$ and

1. *B is a finitely generated K-algebra and P is isomorphic to a polynomial ring over K in at most countably many variables (the case $P = F$ is not excluded);*
2. *$S_h = (B \cdot P)_h$ is a differentially finitely generated K-algebra;*
3. *The homomorphism $B \otimes_K P \to B \cdot P$ induced from multiplication is an isomorphism of K-algebras;*
4. *The restriction of φ to $K[V \setminus V_B]$ is injective.*

An other important tool in Lie differential algebra is the basis theorem for radical differential ideals in $K\{\mathbf{Y}\}$.

Theorem 2.32 (See Appendix in [1])**.** *For any radical differential ideal J in $K\{\mathbf{Y}\}$ there exists a finite subset Φ in $K\{\mathbf{Y}\}$ such that $\{\Phi\} = J$.*

Any radical differential ideal in $K\{\mathbf{Y}\}$ is the intersection of a finite number of prime differential ideals. The set of prime differential ideals coming into such a decomposition with no superfluous component is unique.

3. Uniform model companion for Lie differential valued fields

Let us fix a field $\mathbb{F}$ of characteristic zero, a finite-dimensional $\mathbb{F}$-vector space $\mathbb{L}$ with a Lie multiplication making it a Lie algebra over a subfield of $\mathbb{F}$ and a vector space homomorphism $\phi_{\mathbb{F}} : \mathbb{L} \to Der(\mathbb{F})$, the Lie algebra of derivations on $\mathbb{F}$, preserving the

Lie multiplication. We fix a basis $\{\delta_1, \ldots, \delta_m\}$ of $\mathbb{L}$ and let $[\delta_i, \delta_j] := \sum_{k=1}^{m} c_{ij}^k \delta_k$ for some c_{ij}^k in $\mathbb{F}$.

In the sequel, we will consider the following language $\mathcal{L}_{\text{Lie}}$ for Lie differential fields:

- the language of rings together with unary function symbols δ_i, $i \in \{1, \ldots, m\}$;
- and constants symbols for each element of $\mathbb{F}$.

Definition 3.1. Let $\langle K, \delta_1, \ldots, \delta_m \rangle$ be a Lie differential field of characteristic 0. For every set $I \subseteq K\{\mathbf{Y}\} \setminus K$ of differential polynomials in $\mathbf{Y} := (Y_1, \ldots, Y_N)$, we write

$$A(I) := K[\delta^\alpha y : \alpha \in \mathbb{N}^m, y \in \mathbf{Y} \text{ and } \delta^\alpha y \text{ appears in some } f \in I].$$

Now we define as in Definition 3.1 of [3] the notion of algebraically prepared system of K.

Definition 3.2. An *algebraically prepared system* of K in m derivatives ($[\mathbb{L} : \mathbb{F}] = m$) is a sequence $(f_1, \ldots, f_l)$ of differential polynomials in $K\{\mathbf{Y}\} \setminus K$ such that the following two conditions hold:

(AP1) $\{f_1, \ldots, f_l\}$ is a characteristic set of a differential prime ideal; thus by Theorem 2.30, $\{f_1, \ldots, f_l\}$ is an autoreduced and coherent set of l polynomials and the ideal $(f_1, \ldots, f_l) : H(f_1, \ldots, f_l)^\infty$ of $A(f_1, \ldots, f_l)$ does not contain a non-zero element, reduced with respect to $f_1, \ldots, f_l$.

(AP2) the ideal $(f_1, \ldots, f_l) : H(f_1, \ldots, f_l)^\infty$ of $A(f_1, \ldots, f_l)$ is prime and there is a regular K-rational point of this ideal, where $H(f_1, \ldots, f_l)$ does not vanish.

Definition 3.3. We say that K *solves* an algebraically prepared system $(f_1, \ldots, f_l)$ of K if there is a differential solution $a \in K^n$ of $f_1 = \ldots = f_l = 0$. We say that an algebraically prepared system $(f_1, \ldots, f_l)$ of K *is defined over* a subring R of K if each polynomial f_i is over R.

Notation 3.4. If M, N are $\mathcal{L}$-structures in an arbitrary language $\mathcal{L}$ and A is a common subset of M, N then we write $M \equiv>_{\exists,A} N$ if every existential $\mathcal{L}$-formula with parameters in A that holds in M, also holds in N. We write $M \equiv_{\exists,A} N$ if $M \equiv>_{\exists,A} N$ and $N \equiv>_{\exists,A} M$. Hence $M \equiv_{\exists,A} N$ if and only if M and N have the same universal theory over A.

Now, by applying the results of Section 2, we can transpose the proof of Theorem 3.3 in [3] in order to establish the following theorem

Theorem 3.5. *Let A be a common Lie differential subring of Lie differential fields L_1, L_2 of characteristic zero. Let F_i be the algebraic closure of the quotient field F_0 of A in L_i. Suppose*

- *$L_1 \equiv_{\exists,A} L_2$ as pure fields in the language of rings and*
- *L_2 solves all algebraically prepared systems of L_2 defined over F_2.*

Then $L_1 \equiv>_{\exists,F_0} L_2$ as Lie differential fields in the language $\mathcal{L}_{Lie}$.

Proof. It suffices to follow the proof of Theorem 3.3 in [3] by using the corresponding results in the Lie differential case. For example, we can extend the Lie structure to the algebraic closure of a field since $\delta_i\delta_j - \delta_j\delta_i$ and $\sum_{i=1}^{m} \delta_i$ are derivations and in characteristic zero, the derivations extend uniquely to the algebraic closure of a differential field. Moreover, Proposition 2.12 in [3] has its analogue in Theorem 2.32. □

Now we are able to write a Lie version of the scheme of axioms (UC) in [3].

Definition 3.6. A Lie differential field $\langle K, \delta_1, \ldots, \delta_m \rangle$ satisfies $(UC)_{\mathrm{Lie},m}$ if every algebraically prepared system of K has a differential solution in K.

Moreover, Section 4 in [3] shows that the scheme of axioms $(UC)_{\mathrm{Lie},m}$ is expressible by first-order statements in the language of Lie differential fields. Indeed the class of all Lie differential fields of characteristic zero which solve all their algebraically prepared systems is axiomatizable in the language $\mathcal{L}_{\mathrm{Lie}}$.

Since we want to prove the same results as in [3] in the case of Lie differential fields, we need to use the notion of large fields.

Definition 3.7. A field K is called *large* if every smooth integral curve defined over K that has a K-rational point has infinitely many K-rational points.

A characterization of large fields is given in Proposition 5.3 of [3]. This asserts that K is large if and only if F is existentially closed in the formal Laurent series field $K((t_1, \ldots, t_n))$ for all $n \in \mathbb{N}$, for example.

In Section 2, Theorem 2.31 is a Lie analogue of Theorem 6.1 in [3] which yields an interesting representation of $S := K\{\mathbf{Y}\}/P$ for some prime differential ideal P in $K\{\mathbf{Y}\}$. It enables us to prove the following theorem (which is a Lie version of Main Theorem 6.2 in [3]).

Theorem 3.8. *Let $\langle K, \delta_1, \ldots, \delta_m \rangle$ be a Lie differential field of characteristic zero. Then the two following holds:*

(I) *If K is large as pure field then K can be extended to a Lie differential field which satisfies the scheme of axioms $(UC_{Lie,m})$.*

(II) *If L is a Lie differential field containing a Lie differential subring A of K such that L and K have the same universal theory over A as pure fields, then they have the same universal theory over A as Lie differential fields.*

Proof. (I) follows from the proof in [3] by using Theorem 2.31 and the notion of large fields like in the classical differential case. Moreover we know as in the classical differential case that the Lie structure of a Lie differential field can be extended to any field extension since $\delta_i\delta_j - \delta_j\delta_i$ and $\sum_{i=1}^m \delta_i$ are also derivations.

The item (II) holds by Theorem 3.5. □

Now we give the model-theoretic results which were the principal motivation of this section. It corresponds to the results in Section 7 of [3]. For this purpose, we use the notations and the terminology in [3].

Definition 3.9. Let T be an $\mathcal{L}$-theory of fields of characteristic zero. Then we denote by T_{Lie} the following theory in the corresponding Lie language $\mathcal{L}^*_{\mathrm{Lie}} := \mathcal{L} \cup \mathcal{L}_{\mathrm{Lie}}$:

- the $\mathcal{L}$-theory T;
- the diagram of $\mathbb{F}$ including the action of the δ_i $(i \in \{1, \ldots, m\})$;
- axioms saying that the δ_i are derivations;

- an axiom of the form $\forall x\ (\delta_k\delta_l c - \delta_l\delta_k x = \sum_{i=1}^{m} c_{kl}^i \cdot \delta_i x)$ for each $k, l \in \mathbb{N}_m$.

This is the content of the Lie version of Theorems 7.1 and 7.2 in [3].

Theorem 3.10. *Let $\mathcal{L}$ be the language of rings and let C be a set of new constants. Let T be a model complete $\mathcal{L}(C)$-theory such that every model of T is a large field.*

Let $\widetilde{T}$ be a theory in a language $\widetilde{\mathcal{L}} \supseteq \mathcal{L}(C)$ such that $\widetilde{T}$ contains T and $\widetilde{T}$ is an expansion by definition of T. Let F be an $\widetilde{\mathcal{L}}^*_{Lie}$-structure.

If $\widetilde{T} \cup diag(F \restriction \widetilde{\mathcal{L}})$ is complete then $\widetilde{T}_{Lie} \cup diag(F) \cup (UC_{Lie,m})$ is complete.

Theorem 3.11. *Under the same assumptions as in Theorem 3.10, assume moreover that $\widetilde{T}$ is a model companion of an $\widetilde{\mathcal{L}}$-theory $\widetilde{T}_0$ extending the theory of fields. Then*

(1) *$\widetilde{T}_{Lie} \cup (UC_{Lie,m})$ is a model companion of the $\widetilde{\mathcal{L}}^*_{Lie}$-theory $(\widetilde{T}_0)_{Lie}$.*
(2) *If $\widetilde{T}$ is a model completion of $\widetilde{T}_0$ then $\widetilde{T}_{Lie} \cup (UC_{Lie,m})$ is a model completion of the $\widetilde{\mathcal{L}}^*_{Lie}$-theory $(\widetilde{T}_0)_{Lie}$.*
(3) *If $\widetilde{T}$ has quantifier elimination then $\widetilde{T}_{Lie} \cup (UC_{Lie,m})$ has quantifier elimination.*
(4) *If T is complete and L is a Lie differential field and a model of T then $\widetilde{T}_{Lie} \cup (UC_{Lie,m}) \cup diag(F)$ is complete where F is the $\widetilde{\mathcal{L}}^*_{Lie}$-substructure generated by $\emptyset$ in L.*

Now as in Section 8 of [3], we use Proposition 8.1 which gives a class of examples of large fields in order to apply our results to classical theories of fields; in particular we can reformulate with the scheme $(UC_{\mathrm{Lie},m})$ the result of model completion for Lie differentially closed fields in Theorem 5.2 of [5].

Proposition 3.12 (See Proposition 8.2 in [3])**.** *The following hold:*

- *$(ACF_0)_{Lie} \cup (UC)_{Lie,m}$ is the theory $LDCF_0$ of Lie differentially closed fields of characteristic zero as introduced in Definition 5.1 of* [5].
- *$(RCF)_{Lie} \cup (UC_{Lie,m})$ is the complete and model complete theory of real closed ordered Lie differential fields in the language $\mathcal{L}_{Lie}$. Moreover, $RCF_{Lie} \cup (UC_{Lie,m})$ has quantifier elimination in the language of Lie differential fields equipped with a total ordering $\leq$.*
- *$(pCF_d)_{Lie} \cup (UC_{Lie,m})$ is the complete and model complete theory of p-adically closed Lie differential fields of p-rank d in the language $\mathcal{L}_{Lie}$.*

  Moreover, $(pCF_d)_{Lie} \cup (UC_{Lie,m})$ has quantifier elimination in the language of Lie differential fields equipped with unary predicates P_n for the nth powers of the field.

References

[1] E. Hubert, Differential Algebra for derivations with nontrivial commutation rules, Journ. Pure Appl. Algebra 200 (2005), pp. 163–190.

[2] E.R. Kolchin, Differential Algebra and Algebraic groups, Pure and Applied Mathematics, vol. 54, Academic Press 1973.

[3] M. Tressl, A Uniform Companion for Large Differential Fields of characteristic zero, Transactions of the American Mathematical Society 357 (10) (2005), pp. 3933–3951.

[4] M. Tressl, A structure theorem for differential algebras, Differential Galois Theory Banach Center Publications, Vol 58 (2002), pp. 201–206.

[5] Y. Yaffe, Model completion of Lie differential fields, Ann. Pure Appl. Logic 107 (1–3) (2001), pp. 49–86.

Nicolas Guzy, Institut de Mathématique, Université de Mons-Hainaut, Le Pentagone, 6, avenue du Champ de Mars, B-7000 Mons, Belgium

E-mail address: Nicolas.Guzy@umh.ac.be