Valued fields with K commuting derivations

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Abstract

Let T_1 be a theory of henselian valued fields of equicharacteristic zero in the language of rings equipped with a linear divisibility predicate (for the valuation). Assume that T_1 is the model companion (model completion) of a theory T_0 of valued fields. In this paper, we establish a uniform scheme of axioms (UC'_K) such that $T_1^* \cup (UC'_K)$ is the model companion (model completion) of the corresponding differential theory T_0^* (note that if T is a theory of valued fields then we denote by T^* the theory T with the axioms for K pairwise commuting derivations). For this purpose, we proceed in a similar way as M. Tressl who deals with this problem but in the case of field theories in an expansion by definition of the pure field language. In the valued case, we take advantage (as in [4] for one derivation) of the valuation topology in order to obtain an axiomatization similar to the one given in [13].

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1 Introduction

In [13], M. Tressl introduces a first-order theory of differential fields of characteristic zero, in K pairwise commuting derivations, denoted by (UC_K) (for Uniform Companion), with the following properties:

(I) Whenever L and M are models of (UC_K) and A is a common differential subring of L and M such that L and M have the same universal theory over A as pure fields then they have the same universal theory over A as differential fields.

(II) Every differential field F which is large can be extended to a model of (UC_K) and this extension is elementary in the language of rings.

The second property uses the important concept of large fields. A large field is a field F which is existentially closed in $F((t))$, the formal Laurent series field over F. In particular we know that henselian valued fields are large fields (see [8]).

More generally, properties (I) and (II) of (UC_K) above imply that for every model complete theory T of large fields in the language of rings, the theory $T \cup (UC_K)$ of differential fields is model complete. Moreover, if this is the case, $T \cup (UC_K)$ is complete if T is complete and $T^* \cup (UC_K)$ has quantifier elimination if a definable expansion T^* of T has quantifier elimination (see Theorem 7.2 in [13]).

So, in this paper, we consider the same problem for differential valued fields equipped with K commuting derivations (no interaction between the valuation and the derivations is demanded).

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First we introduce basic differential terminology. It allows us to define a notion of J-algebraically prepared system which is central in order to write down a scheme of axioms (UC'_K) which will play the same role as (UC_K) in the valued setting of the problems (I) and (II).

In Theorem 3.14 we prove a differential transfer result of model completeness. The proof of this result uses technical tools as the concept of *regular semigeneric point of a polynomial ideal* and the *Newton's Lemma* for henselian valued fields. Hence as a Corollary of Theorem 3.14, we can state the valued analogue of Theorem 7.2 in [13].

Finally we enclose the paper with some applications. The first one is a differential version of the classical Ax-Kochen-Ersov theorem for valued fields (see Theorem 4.2; this result was proved in [4] for one derivation). Then we apply our results to some particular theories of differential valued fields; namely algebraically closed valued fields, p-adically closed fields and real closed valued fields. Note that the theory of p-adically closed fields can be treated by using the results of M. Tressl since the p-valuation is definable in the language of rings and p-adically closed fields admit quantifier elimination in the language of Macintyre (which is an expansion by definition of the language of rings, more precisely by nth power predicates). But since our approach is topological, we can prove more easily a differential version of a Hilbert's Seventeenth problem for existentially closed differential p-adically closed fields.

2 Preliminaries

In this paper, we always use the classical notations and terminology of [5] for differential algebra.

We always consider unitary commutative domains of characteristic zero.

For any valued field $\langle F, v\rangle$, we denote the valuation ring, the residue field, the value group and the residue map by $\mathcal{O}_F$, k_F, $v(F^\times)$ and $\pi : \mathcal{O}_F \longmapsto k_F$, respectively.

Let $\mathcal{L}_\mathcal{D}$ be the language of rings equipped with a binary predicate of linear divisibility $\mathcal{D}$ (called a l.d. relation) for the valuation and a constant element $\underline{c}$. This predicate is interpreted as the set of tuples (x, y) such that $v(x) \leqslant v(y)$ for some x, y in a valued field $\langle F, v\rangle$ (see Section 4.2 in [7]) and the constant element $\underline{c}$ is interpreted as an element of non-zero value.

We will make use of henselian valued fields in the sequel. So we recall that a henselian valued field is a valued field $\langle F, v\rangle$ which satisfies the following property, called *the Newton's Lemma*:

for any polynomial f with coefficients in $\mathcal{O}_F$ and any element b in $\mathcal{O}_F$ which satisfies $v(f(b)) > 2v(f'(b))$ then there exists a unique element a in $\mathcal{O}_F$ such that $f(a) = 0$ such that $v(a-b) > v(f'(b))$.

We will see in Section 3 that the Newton's Lemma will play the role of large fields in [13].

All the valuations in the paper are assumed to be non-trivial unless something else is said.

Let $\langle F, \delta_1, \cdots, \delta_K\rangle$ be a differential field equipped with K pairwise commuting derivations. Let $\mathbb{D}$ be the free abelian monoid generated by $\{\delta_1, \cdots, \delta_K\}$ which we denote multiplicatively and let $\mathbb{D}Y$ be the set of ΘY_j such that $\Theta \in \mathbb{D}$ and $j \in \{1, \cdots, N\}$, where $Y := (Y_1, \cdots, Y_N)$ is a set of differential indeterminates. Let us note that $\mathbb{D}Y$ can be equipped with the following rank rk defined as follows:

$$rk(\delta_1^{i_1} \cdots \delta_K^{i_K} Y_n) = (i_1 + \ldots + i_K, n, i_K, \ldots, i_1).$$

It allows us to well-order elements of a subset of $\mathbb{D}Y$. The differential polynomial ring over F in K commuting derivations and n indeterminates will be denoted by $F\{Y\}$ together with its natural derivations such that it is a differential ring extension of F.

Let us first recall some definitions and notations.

Definition 2.1.
- We say that a variable $y \in \mathbb{D}Y$ *appears in* $f \in F\{Y\}$ if y appears in f considered as an ordinary polynomial.
- The *leader* u_f of $f \in F\{Y\} \setminus F$ is the variable $y \in \mathbb{D}Y$ of highest rank which appears in f.

- Let $f \in F\{Y\} \setminus F$, $f = f_d u_f^d + \ldots + f_1 u_f + f_0$ with polynomials $f_0, \ldots, f_d \in F[y \in \mathbb{D}Y | y \neq u_f]$ and $f_d \neq 0$. The *initial* $I(f)$ of f is defined as f_d and the *separant* $S(f)$ of f as $\frac{\partial}{\partial u_f} f$.

- For every subset $G = \{g_1, \ldots, g_l\}$ of $F\{Y\} \setminus F$, we define

$$H(G) = \prod_{i=1}^{l} I(g_i) \cdot S(g_i).$$

- For any set $I \subseteq F\{Y\} \setminus F$, we let $A(I) := F[\Theta Y_j | \Theta \in \mathbb{D}, j \in \{1, \cdots, N\}$ and ΘY_j appears in some $f \in I]$.

In the sequel we always assume that the derivations are pairwise commuting.

Let F be a field and let $F[X_1, \cdots, X_n]$ be the polynomial ring in n indeterminates over F (that we also denote by $F[\underline{X}]$ when n is understood). Let us consider a prime ideal I of $F[\underline{X}]$ with Krull dimension d.

Definition 2.2. A sequence $\{Q_1, \ldots, Q_{n-d}\}$ of polynomials in I is said to be *a set of semigenerators* of I if (possibly after renaming the variables X_j)

- each polynomial Q_i belongs to $F[X_1, \ldots, X_{d+i}] \setminus F[X_1, \ldots, X_{d+i-1}]$;

- the following equality holds:

$$I = (Q_1, \cdots, Q_{n-d}) : l(Q_1, \cdots, Q_{n-d})^\infty$$

  which is defined as follows

$$\big\{ f \in F[\underline{X}] \mid l(Q_1, \cdots, Q_{n-d})^m \cdot f \in (Q_1, \cdots, Q_{n-d}) \text{ for some positive integer } m \big\}$$

  where $l(Q_1, \cdots, Q_{n-d})$ is the product of leading coefficients of the Q_i's considered as polynomials in X_{d+i} over $F[X_1, \cdots, X_{d+i-1}]$.

Now we get the following

Lemma 2.3. *Any prime ideal I in $F[\underline{X}]$ has a set of semigenerators $\{Q_1, \ldots, Q_{n-d}\}$.*

Proof. Let $\bar{g} = (g_1, \cdots, g_n)$ be a generic point of I in a field extension of F. We may assume that $g_1, \cdots, g_d$ are algebraically independent over F if the Krull dimension of I is d. Then we let $q_i(X_{d+i})$ be the minimal polynomial of g_{d+i} over $F(g_1, \ldots, g_{d+i-1})$ (for $i \in \{1, \ldots, n-d\}$) with degree d_i. Let us consider the following polynomial $\widehat{Q}_i(X_{d+i})$ over the field $F(X_1, \ldots, X_{d+i-1})$:

$$\widehat{Q}_i(X_{d+i}) = \sum_{j=0}^{d_i} X_{d+i}^j \frac{G_i^j(X_1, \ldots, X_{d+i-1})}{H_i^j(X_1, \ldots, X_{d+i-1})}$$

such that for any $j \in \{0, \ldots, d_i\}$, $H_i^j(g_1, \ldots, g_{d+i-1}) \neq 0$ and $\widehat{Q}_i(g_1, \ldots, g_{d+i-1}, X_{d+i}) = q_i(X_{d+i})$.

Then we let

$$Q_i = \big(\prod_{j=0}^{d_i} H_i^j \big) \cdot \widehat{Q}_i := l(Q_i) \cdot \widehat{Q}_i \quad \text{where } l(Q_i) \in F[X_1, \ldots, X_{d+i-1}].$$

By the construction of the Q_i's, we have $(Q_1, \cdots, Q_{n-d}) : l(Q_1, \cdots, Q_{n-d})^\infty \subseteq I$.

Let H be in I. So we have $H(\bar{g}) = 0$ and by construction, $l(Q_1, \ldots, Q_{n-d})(\bar{g}) \neq 0$. By considering the polynomial $H(X_n, X_{n-1}, \ldots, X_1)$ in X_n over the field $F(X_1, \ldots, X_{n-1})$, we get

$$l(Q_{n-d})^{m_n} \cdot H = Q_{n-d} \cdot G_n + R_n$$

where G_n, $R_n \in F[\underline{X}]$ such that $\deg_{X_n} R_n < \deg_{X_n} Q_{n-d}$ and m_n is a natural number.

We can now express R_n under the following form

$$R_n(X_n) = \sum_{j=0}^{d_{n-d}-1} X_n^j \cdot R_{n,j}(X_1, \ldots, X_{n-1})$$

for some $R_{n,j} \in F[X_1, \ldots, X_{n-1}]$.

Since $R_n(\bar{g}) = 0$ and $\frac{Q_{n-d}}{l(Q_{n-d})}(X_n, g_{n-1}, \ldots, g_1)$ is the minimal polynomial of g_n over $F(g_1, \ldots, g_{n-1})$, we get $R_{n,j}(g_1, \ldots, g_{n-1}) = 0$ for all $j \in \{0, \ldots, d_{n-d} - 1\}$.

We continue this process by induction; it suffices to replace H, Q_{n-d} and $l(Q_{n-d})$ by R_n, Q_{n-d-1} and $l(Q_{n-d-1})$, respectively; and at the last step, we use the fact that $g_1, \ldots, g_d$ are algebraically independent over F. This finishes the proof. $\square$

Convention 2.4. If $\bar{a}$ is a tuple of elements then $\bar{a}_i$ denotes the ith element of the tuple $\bar{a}$. If Λ is an element of $\mathbb{D}Y$ and $\bar{a}$ is an N-tuple of elements of a differential field F then we denote by $\Lambda\bar{a}$ the element $\Theta(\bar{a}_i)$ such that $\Lambda := \Theta Y_i$ with $\Theta \in \mathbb{D}$.

Definition 2.5. 1. We say that a tuple $\bar{a}$ in F is a *regular semigeneric zero* of I with respect to a set of semigenerators $\{Q_i\}_i$ of I if $\bar{a}$ is a common zero of $\{Q_1, \cdots, Q_{n-d}\}$ such that

$$\bigwedge_{i=1}^{n-d} s_{Q_i}(\bar{a}) := \frac{\partial Q_i}{\partial X_{d+i}}(\bar{a}) \neq 0 \wedge l(Q_1, \ldots, Q_{n-d})(\bar{a}) \neq 0.$$

2. A *J-point* $\bar{a}$ is a tuple with the length equal to the cardinal of J and indexed by the elements of J, i.e. $\bar{a} := (b_j)_{j \in J}$ where J is a subset of $\mathbb{D}Y$ with $Y := (Y_1, \cdots, Y_N)$.

3. Let $\langle F, v, \delta_1, \cdots, \delta_K \rangle$ be a valued field equipped with K derivations. An N-tuple $\bar{a}$ is said to be *γ-close* (with respect to F) to a J-point $(b_\Lambda)_{\Lambda \in J}$ in F for some $J \subseteq \mathbb{D}Y$ and $\gamma \in v(F^\times)$ if $\{\Lambda\bar{a}\}_{\Lambda \in J}$ belongs to $B_{>\gamma}((b_\Lambda)_{\Lambda \in J}) := \{(\bar{x}_\Lambda)_{\Lambda \in J} | v(\bar{x}_\Lambda - b_\Lambda) > \gamma \ \forall \Lambda \in J\}$ where the valuation of a tuple is defined as the minimum of the value of each element of the tuple (and using Convention 2.4).

Now we define the notion of J-algebraically prepared system which will be essential for our uniform scheme of axioms (UC'_K).

Definition 2.6. A *J-algebraically prepared system* over F in K derivations (with respect to a finite subset J of $\mathbb{D}Y$) is a sequence $\{f_1, \cdots, f_l\}$ of differential polynomials in $F\{Y_1, \cdots, Y_N\} \setminus F$ with two tuples $\bar{a}$, $\bar{a}'$ in F (with their lengths determined by the f_i's and by the set J) and a sequence of polynomials $\{Q_1, \ldots, Q_{n-d}\}$ in $A(f_1, \cdots, f_l)$ such that the following conditions are satisfied:

- (AP1) $\{f_1, \cdots, f_l\}$ is a characteristic set of a prime differential ideal, so $\{f_1, \cdots, f_l\}$ is an autoreduced and coherent set of l polynomials and the ideal $(f_1, \cdots, f_l) : H(f_1, \cdots, f_l)^\infty$ of $A(f_1, \cdots, f_l)$ does not contain non-zero elements, reduced with respect to $f_1, \cdots, f_l$;
- (AP2) The ideal $(f_1, \cdots, f_l) : H(f_1, \cdots, f_l)^\infty$ of $A(f_1, \cdots, f_l)$ is prime, the Q_i's are semigenerators of $(f_1, \cdots, f_l) : H(f_1, \cdots, f_l)^\infty$ and the tuple $\bar{a}$ is a regular semigeneric F-rational point of $(f_1, \cdots, f_l) : H(f_1, \cdots, f_l)^\infty$ with respect to the Q_i's such that $H(f_1, \cdots, f_l)(\bar{a}) \neq 0$;
- (AP3) No element of J is a proper derivative of a leader of f_i, no element of J appears in f_i $(1 \leqslant i \leqslant l)$ and $\bar{a}'$ is a J-point.

Now the notion of J-algebraically prepared system allows us to state the scheme of axioms (UC'_K) which will give us the desired result (see Theorem 3.14).

Definition 2.7. The *scheme of axioms* (UC'_K) says that in a valued field equipped with K derivations $\langle F, v, \delta_1, \cdots, \delta_K\rangle$ we have:

for any J-algebraically prepared system $\{f_1, \cdots, f_l; Q_1, \ldots, Q_{n-d}\}$ over F with respect to two tuples $\bar{a}$, $\bar{a}'$ in F and any $\gamma \in v(F^\times)$, there is a differential solution $\bar{b}$ of $f_1, \cdots, f_l$ in F which is γ-close to the K-point $(\bar{a}, \bar{a}')$ with respect to F (where $K = J \cup \{\Lambda \in \mathbb{D}Y | \Lambda \text{ occurs in some } f_i\}$) such that $H(f_1, \cdots, f_l)(\bar{b}) \neq 0$.

Lemma 2.8 (See Lemma 3.2 in [13]). *Let A be a domain, let I be an ideal and let Z be an indeterminate over A. Then*

$$(I : h^\infty) := \{a \in A | h^n \cdot a \in I \text{ for some } n \in \mathbb{N}\} = (I, Z \cdot h - 1)_{A[Z]} \cap A.$$

Moreover, $h \notin \sqrt{I}$ if and only if $A \neq (I : h^\infty)$ and in this case, h is a non-zero divisor of $A/(I : h^\infty)$ and the induced map

$$(A/(I : h^\infty))_{h/(I:h^\infty)} \longmapsto A[Z]/(I, Z \cdot h - 1)$$

which sends $\frac{1}{h/(I:h^\infty)}$ to $Z/(I, Z \cdot h - 1)$ is an isomorphism. In particular, $(I : h^\infty)$ is prime if and only if $(I, Z \cdot h - 1)_{A[Z]}$ is prime, provided that $h \notin \sqrt{I}$.

Proposition 2.9. *The class of all valued differential fields equipped with K derivations which satisfies the scheme of axioms (UC'_K) is axiomatizable in the language $\mathcal{L}^*_{\mathcal{D}} := \mathcal{L}_{\mathcal{D}} \cup \{\delta_1, \cdots, \delta_K\}$.*

Proof. It suffices to use some results of first-order definability in Section 4 of [13] and we directly see that this scheme of axioms is expressible by first-order statements (namely in $\mathcal{L}^*_{\mathcal{D}} := \mathcal{L}_{\mathcal{D}} \cup \{\delta_1, \cdots, \delta_K\}$). The only difference is that the condition (AP2) in the definition of J-algebraically prepared system uses Lemma 2.8 and Theorem 4.2 (i) in [13] in order to show the definability in the language of rings about the coefficients of the sequence of polynomials $\{f_i; Q_j\}$. $\square$

First we recall an important theorem of differential algebra which is used in the Extension Theorem 3.4.

Theorem 2.10 (See Theorem 1 of [12]). *Let F be a differential field equipped with K derivations and let $I \subseteq F\{Y\}$ be a differential prime ideal with $Y := (Y_1, \cdots, Y_N)$. Let $\varphi : F\{Y\} \longmapsto F\{Y\}/I =: S$ the residue map, let G be a characteristic set of I and let $H(G)$ be the product of all initials and separants of polynomials in G as in Definition 2.1.*

Let $h := \varphi(H(G))$, $V := \{y \in \mathbb{D}Y | y$ is not a proper derivative of any leader of an element $g \in G\}$, $V_B := \{y \in V | y$ appears in some $g \in G\}$, $B := \varphi(F[V_B])$ and $P := \varphi(F[V \setminus V_B])$.

Then $h \in B$, $h \neq 0$ and

- *B is a finitely generated F-algebra and P is F-isomorphic to a polynomial ring over F in at most countably many indeterminates (the case $P = F$ is not excluded, for example in the case $K = 1, N = 1$);*
- *$S_h = (B.P)_h$ is a differentially finitely generated F-algebra;*
- *the homomorphism $B \otimes_F P \longmapsto B.P$ induced by the multiplication is an isomorphism of F-algebras;*
- *the restriction of φ to $F[V \setminus V_B]$ is injective.*

Definition 2.11. Let $\langle L, v\rangle$ be a valued field and let $\langle \widehat{L}, w\rangle$ be a valued field extension of $\langle L, v\rangle$.

An element in $\widehat{L}$ is *infinitesimal with respect to* L if its value is bigger than γ for all $\gamma \in v(L^\times)$. For any two elements $l, \widehat{l} \in \widehat{L}$, we say that $\widehat{l}$ is *infinitesimally close to* l (with respect to L) if $\widehat{l} - l$ is infinitesimal with respect to L.

3 Model Companion

In the proof of Theorem 3.4 which allows us to extend a differential henselian valued field in a model of the scheme (UC'_K), we need three lemmas on pure henselian valued fields.

Lemma 3.1. *Let $\langle L, v\rangle$ be a henselian valued field and let $\langle \widehat{L}, w\rangle$ be an $\mathcal{L}_{\mathcal{D}}$-elementary $|L|^+$-saturated extension of L.*

Then we can find n elements in $\widehat{L}$, say $t_1, \ldots, t_n$, which are algebraically independent over L and are infinitesimal with respect to L.

Proof. By the $|L|^+$-saturation of $\widehat{L}$, it is sufficient to show that for any polynomial $f(X_1, \ldots, X_n)$ with coefficients in L and any $\gamma \in v(L^\times)$, there exist $x_1, \ldots, x_n$ in $\widehat{L}$ such that

$$f(x_1, \ldots, x_n) \neq 0 \wedge \bigwedge_{i=1}^{n} v(x_i) > \gamma.$$

The proof is by induction on n and the case $n = 1$ is trivial since the sets $\{x \in L | v(x) > \gamma\}$ are infinite and $f(X)$ has finitely many roots.

For the induction step, we apply the induction to the polynomials f_i in $L[X_1, \ldots, X_n]$ such that

$$f(X_1, \ldots, X_n, X_{n+1}) := \sum_{i=0}^{d} f_i(X_1, \ldots, X_n) \cdot X_{n+1}^i.$$

□

Lemma 3.2. *Let $\langle L, v\rangle$ be a field, let $f(X_0, \ldots, X_n)$ be a non-zero polynomial in $n+1$ indeterminates with coefficients in L and let $\bar{a} := (a_0, \ldots, a_n)$ be an $(n+1)$-tuple in L. Then there exists an element σ in $\mathcal{O}_L$ and a positive integer M such that:*

- $\sigma a_0, \ldots, \sigma a_d \in \mathcal{O}_L$;
- $\sigma^{M+1} \cdot f(\bar{a}) = \widetilde{f}(\sigma \bar{a})$ *with* $\widetilde{f} \in \mathcal{O}_L[X_0, \ldots, X_n]$ *and;*
- $\frac{\partial \widetilde{f}}{\partial X_k}(\sigma a_0, \ldots, \sigma a_n) = \sigma^M \cdot \frac{\partial f}{\partial X_k}(a_0, \ldots, a_n)$ *for all* $k \in \{0, \ldots, n\}$.

Proof. Let us consider the polynomial

$$f(X_0, \ldots, X_n) = \sum_{(i_0, \ldots, i_n)} c_{(i_0, \ldots, i_n)} . M_{(i_0, \ldots, i_n)}(X) \in L[X_0, \ldots, X_n]$$

such that the $c_{(i_0, \ldots, i_n)}$'s are non-zero elements in L and $M_{(i_0, \ldots, i_n)}(X) := X_0^{i_0} \cdots X_n^{i_n}$.

We let $\gamma = \min_{(i_0, \ldots, i_n)} v(c_{(i_0, \ldots, i_n)})$, $\delta = \min_{i \in \{0, \ldots, n\}} v(a_i)$ and $\Sigma = \max\{|\gamma|, |\delta|\} \in v(L^\times)$. Let us choose $\sigma \in \mathcal{O}_L$ such that $v(\sigma) = \Sigma$. We let $M = \max_{(i_0, \ldots, i_n)}(\sum_{j=0}^{n} i_j)$ and we obtain the required properties by easy calculations. □

Lemma 3.3 below is the most important point in the proof of the consistency of the axioms (UC'_K). A central tool in the proof of this lemma is the Newton's Lemma. As said above in the introduction, it plays the role of large fields in the work of M. Tressl.

Lemma 3.3. *Let $\langle L, v\rangle$ be an henselian valued field, let $\langle \widehat{L}, w\rangle$ be an elementary $|L|^+$-saturated extension of L, let I be a prime ideal of the polynomial ring $L[\underline{X}]$ with Krull dimension d and let $\{Q_i\}_{i=1}^{n-d}$ be a set of semigenerators of I. Let us suppose that I vanishes at a regular semigeneric L-rational point $\bar{b}$ of L^n (with respect to the Q_i's).*

Then there exists an n-tuple $\bar{f}$ in $\widehat{L}$ such that:

- *$L(\bar{f})$ is L-isomorphic to $L(I)$, the fraction field of $L[\underline{X}]/I$ and,*
- *the point $\bar{f}$ is infinitesimally close to $\bar{b}$ with respect to L.*

Proof. We want to find a common zero $\bar{f}$, in $\widehat{L}$, of the polynomials Q_i which is infinitesimally close to $\bar{b}$ with respect to L (i.e. $v(\bar{f}-\bar{b}) > v(L^\times)$) with $f_1, \ldots, f_d$ algebraically independent over L.

Claim: We may assume that the polynomials Q_i have their coefficients in $\mathcal{O}_L$ and $\bar{b}$ is an n-tuple of elements in $\mathcal{O}_L$.

By using Lemma 3.2, there exist a positive integer M and an element σ of $\mathcal{O}_L$ such that $\sigma\bar{b} \in \mathcal{O}_L^n$ and $\sigma^{M+1} \cdot Q_1(\bar{b}) = \widetilde{Q_1}(\sigma\bar{b}), \ldots, \sigma^{M+1} \cdot Q_{n-d}(\bar{b}) = \widetilde{Q_{n-d}}(\sigma\bar{b})$ where the polynomials $\widetilde{Q_i}$ are in $\mathcal{O}_L[X_0, \ldots, X_n]$.

Moreover we have $\sigma^M \cdot s_{Q_i}(\bar{b}) = s_{\widetilde{Q}_i}(\sigma\bar{b}) \neq 0$ for all $i \in \{1, \ldots n-d\}$. So $\sigma\bar{b}$ is a regular semigeneric $\mathcal{O}_L$-rational point of the prime ideal $(\widetilde{Q}_1, \ldots, \widetilde{Q}_{n-d}) : l(\widetilde{Q}_1, \ldots, \widetilde{Q}_{n-d})^\infty$ in $L[\underline{X}]$ (with respect to the $\widetilde{Q}_i$'s).

Now if we find a point $\bar{f}$ in $\widehat{L}$ with the required properties (i.e. $\bar{f}$ is infinitesimally close to $\sigma\bar{b}$ with respect to L with d coordinates algebraically independent over L) then the point $\sigma^{-1}\bar{f}$ proves the lemma since $\sigma \in \mathcal{O}_L$.

So the problem is reduced to the following case: the Q_i's are polynomials with coefficients in $\mathcal{O}_L$ and $\bar{b}$ is an n-tuple of elements in $\mathcal{O}_L$.

Since $\widehat{L}$ is an $\mathcal{L}_\mathcal{D}$-elementary $|L|^+$-saturated extension of L, Lemma 3.1 allows us to choose d elements in $\mathcal{O}_{\widehat{L}}$, say $t_1, \ldots, t_d$, which are algebraically independent over L and infinitesimal with respect to L.

We then define the elements $f_1, \cdots, f_d$ in $\mathcal{O}_{\widehat{L}}$ as $f_i := \bar{b}_i + t_i$ for any $i \in \{1, \ldots, d\}$. Hence the element $Q_1(f_1, \ldots, f_d, b_{d+1})$ is infinitesimal with respect to L and belongs to $\mathcal{O}_{\widehat{L}}$. Moreover, $v(s_{Q_1}(f_1, \ldots, f_d, b_{d+1})) \in v(\mathcal{O}_L \setminus \{0\})$.

By applying the Newton's Lemma to the polynomial $\widetilde{Q}_1(X) := Q_1(f_1, \ldots, f_d, X)$ and the element b_{d+1}, we obtain a unique element f_{d+1} in $\mathcal{O}_{\widehat{L}}$ such that $Q_1(f_1, \ldots, f_d, f_{d+1}) = 0$ and $v(f_{d+1} - b_{d+1}) > v(\widetilde{Q}_1'(b_{d+1})) \in v(\mathcal{O}_L \setminus \{0\})$. By using a Taylor expansion of $\widetilde{Q}_1$ at the point b_{d+1}, we get

$$0 = \widetilde{Q}_1(f_{d+1}) = \widetilde{Q}_1(b_{d+1}) + \sum_i \widetilde{Q}_1^{[i]}(b_{d+1}) \cdot (f_{d+1} - b_{d+1})^i$$

where $\widetilde{Q}_1^{[i]}$ is the ith formal derivative of $\widetilde{Q}_1$.

By applying v and using the facts that $\widetilde{Q}_1(b_{d+1})$ is infinitesimally close with respect to L and $v(s_{Q_1}(f_1, \ldots, f_d, b_{d+1})) \in v(\mathcal{O}_L \setminus \{0\})$, we deduce that f_{d+1} is infinitesimally close to b_{d+1} with respect to L (i.e. $v(f_{d+1} - b_{d+1}) > v(L^\times)$).

So by induction on $i \in \{1, \ldots, n-d-1\}$, we find an element $f_{d+i+1} \in \mathcal{O}_{\widehat{L}}$ such that f_{d+i+1} is infinitesimally close to b_{d+i+1} with respect to L and $Q_{i+1}(f_1, \ldots, f_{d+i+1}) = 0$, which proves the lemma with $\bar{f} := (f_1, \ldots, f_n)$. □

Now we prove the valued analogue of Theorem 6.2 (II) in [13] for differential henselian valued fields.

Theorem 3.4. *Any differential henselian valued field can be extended to a differential henselian valued field which is a model of (UC'_K) and this extension is elementary in the language $\mathcal{L}_\mathcal{D}$.*

Proof. Since the theory of henselian valued field is inductive, it suffices to prove, by a classical argument of chains and transfinite induction, the following Proposition: □

Proposition 3.5. *Let $\langle F, v, \delta_1, \cdots, \delta_K\rangle$ be a differential henselian valued field equipped with K derivations and let $\{f_1, \cdots, f_l; Q_1, \ldots, Q_{n-d}\}$ be a J-algebraically prepared system over F with respect to a finite subset of variables J and two tuples $\bar{a}$, $\bar{a}'$ in F.*

Then there is a differential valued field extension, L of F, which is $\mathcal{L}_{\mathcal{D}}$-elementary and which has a differential solution of $f_1 = 0, \cdots, f_l = 0$, infinitesimally close to the K-point $(\bar{a}, \bar{a}')$ with respect to F (in particular this solution satisfies $H(f_1, \ldots, f_l) \neq 0$).

Proof. First we consider the differential ideal $I = [f_1, \cdots, f_l] : H(f_1, \cdots, f_l)^\infty$ where $[f_1, \ldots, f_l]$ is the differential ideal of $F\{Y\}$ generated by $\{f_1, \ldots, f_l\}$. Since $\{f_1, \cdots, f_l\}$ is a characteristic set of a prime differential ideal, this prime ideal is equal to I. By Theorem 2.10, the differential F-algebra $A := F\{Y\}/I$ localized in h is F-isomorphic to $B_h \otimes_F P$ where $B = A(f_1, \cdots, f_n)/I \cap A(f_1, \cdots, f_n)$ and $P := \varphi(V \setminus V_B)$ is a ring of polynomials over F in at most countably many variables (following the terminology of Theorem 2.10). By condition (AP3) in the Definition of J-algebraically prepared system, J is a finite subset of $V \setminus V_B$; and the subset $\varphi(J)$ of P is called the subset of J-indeterminates of P.

Now $I \cap A(f_1, \cdots, f_n)$ is the ideal $(f_1, \cdots, f_n) : H(f_1, \cdots, f_n)^\infty$ of $A(f_1, \cdots, f_n)$.

Since $\bar{a}$ is a regular semigeneric point of $(f_1, \cdots, f_n) : H(f_1, \cdots, f_n)^\infty$ (which is of Krull dimension d) with respect to the Q_i's, we apply Lemma 3.3 in order to get a point $\bar{d}$ in an $\mathcal{L}_{\mathcal{D}}$-elementary sufficiently saturated extenion L_0 of F which annihilates $(f_1, \cdots, f_n) : H(f_1, \cdots, f_n)^\infty$ such that $H(f_1, \cdots, f_n)(\bar{d}) \neq 0$, $F(\bar{d})$ is isomorphic to the fraction field of $A(f_1, \cdots, f_n)/I \cap A(f_1, \cdots, f_n)$ over F, and which is infinitesimally close to $\bar{a}$ with respect to F. Moreover, the transcendence degree of $F(\bar{d})$ over F is equal to d.

We now consider an $\mathcal{L}_{\mathcal{D}}$-elementary sufficiently saturated extension L of L_0. Then there is an F-embedding of P in L such that the J-indeterminates of P are infinitesimally close to $\bar{a}'$ with respect to L_0 and are sent to $\bar{a}' + \bar{t}'$ where $\bar{t}'$ are algebraically independent elements over L_0 and are infinitesimal with respect to L_0 (see Lemma 3.1).

Since $A_h = B_h \otimes_F P$, A_h can be embedded into L over F. Finally, we can extend the derivations on A_h to commuting derivations on L and we find an $\mathcal{L}_{\mathcal{D}}$-elementary differential henselian valued field extension as desired. $\square$

Remark 3.6. With the previous Lemma, we find the desired extension by transfinite induction. It is possible because any J-algebraically prepared system over F is also a J-algebraically prepared system over L for any differential valued field $L \supseteq F$ with the property that F is algebraically closed in L- in particular $F \prec L$ in the language of rings.

Lemma 3.7. *Let $\langle F, v, \delta_1, \ldots, \delta_K \rangle$ be a differential valued field, let P be a prime differential ideal in $F\{Y\}$ with a characteristic set $G := \{f_1, \ldots, f_l\}$. Let $\phi(\bar{x})$ be a quantifier-free $\mathcal{L}^*_{\mathcal{D}}$-formula of the following form:*

$$q(\bar{x}) \neq 0 \wedge \bigwedge_{j=1}^{v} \mathcal{D}(g_j(\bar{x}), h_j(\bar{x})) \wedge \bigwedge_{k=1}^{w} \neg \mathcal{D}(l_k(\bar{x}), m_k(\bar{x}))$$

with differential polynomials $q(Y), g_j(Y), h_j(Y), l_k(Y), m_k(Y)$ with coefficients in F where $Y = (Y_1, \cdots, Y_N)$.

Then there exists a quantifier-free $\mathcal{L}^*_{\mathcal{D}}$-formula $\widetilde{\phi}$ with parameters in F such that:

- *for any differential valued field extension L of F, we get*

$$L \models \forall \bar{x} \big[\bigwedge_{i=1}^{l} f_i(\bar{x}) = 0 \wedge H(f_1, \ldots, f_l)(\bar{x}) \neq 0 \Rightarrow (\phi(\bar{x}) \iff \widetilde{\phi}(\bar{x})) \big];$$

- *all the differential polynomials occuring in $\widetilde{\phi}$ are weakly reduced with respect to $\{f_1, \ldots, f_l\}$.*

Proof. Since $\{f_1, \ldots, f_l\}$ is a characteristic set of the prime differential ideal P in $F\{Y\}$, we use the Rosenfeld's Lemma (see Theorem 2.14 in [13]) to get, for any $f \in F\{Y\}$,

$$H(f_1, \cdots, f_l)^d f \equiv \widetilde{f} \, mod[f_1, \cdots, f_n] \qquad (1)$$

where $\widetilde{f} \in F\{Y\}$ is weakly reduced with respect to $\{f_1, \cdots, f_l\}$ and d is a natural number. So we can deduce similar equations for differential polynomials $q, g_j, h_j, l_k, m_k \in F\{Y\}$ (in particular we can consider the same d if d is assumed big enough since $H(f_1, \cdots, f_l)$ is reduced with respect to $\{f_1, \cdots, f_l\}$).

Then we get the required quantifier-free $\mathcal{L}^*_{\mathcal{D}}$-formula $\widetilde{\phi}$:

$$\widetilde{q}(\bar{x}) \neq 0 \wedge \bigwedge_{j=1}^{v} \mathcal{D}(\widetilde{g}_j(\bar{x}), \widetilde{h}_j(\bar{x})) \wedge \bigwedge_{k=1}^{w} \neg\mathcal{D}(\widetilde{l}_k(\bar{x}), \widetilde{m}_k(\bar{x})).$$

$\square$

Definition 3.8. The $\mathcal{L}^*_{\mathcal{D}}$-formula $\widetilde{\phi}$ in Lemma 3.7 is called *an* $(f_1, \ldots, f_l)$*-reduced variant* of ϕ over F.

Notation 3.9. Given an $(f_1, \ldots, f_l)$-reduced variant $\widetilde{\phi}$ of ϕ over F as above, we associate to $\widetilde{\phi}$ a quantifier-free $\mathcal{L}_{\mathcal{D}}$-formula $\widetilde{\phi}_{\mathrm{alg}}$ with parameters in F as follows.

Let $\bar{u} := (u_1, \cdots, u_r)$ be an enumeration of all variables ΘY_j occuring in some of the f_i's and let $\bar{u}' := (u'_1, \cdots, u'_s)$ be an enumeration of the other variables $\Theta' Y_j$ occuring in $\widetilde{\phi}$. Writing $f(\bar{u}, \bar{u}')$ for the polynomials $f(Y) \in F\{Y\}$ occuring in $\widetilde{\phi}$, we get the $\mathcal{L}_{\mathcal{D}}$-formula $\widetilde{\phi}_{\mathrm{alg}}(\bar{u}, \bar{u}')$.

Definition 3.10. Let $\langle F, v, \delta_1, \ldots, \delta_K\rangle$ be a differential valued field which is algebraically closed in a differential valued field extension L of F, let $\bar{a}$ be an N-tuple in L and let ϕ be a quantifier-free $\mathcal{L}^*_{\mathcal{D}}$-formula with parameters in F as in Lemma 3.7.

A J-algebraically prepared system $\{f_1, \ldots, f_l; Q_1, \ldots, Q_{n-d}\}$ over L with respect to two tuples $\bar{c}$, $\bar{c}'$ is said to be *realized by* $\bar{a} \in L$ *with respect to* ϕ *and* F if:

- $\{f_1, \ldots, f_l\}$ is a characteristic set of the prime differential ideal $I(a/F) := \{f \in F\{Y\} | f(a) = 0\}$ with Krull dimension d (the f_i have their coefficients in F) and,
- the polynomials Q_i have their coefficients in F such that $(Q_1, \ldots, Q_{n-d}) : l(Q_1, \ldots, Q_{n-d})^\infty = I(a/F) \cap F[V_B]$ (see the notations of Theorem 2.10);
- J is the set of variables which occur in the $(f_1, \ldots, f_l)$-reduced variant of ϕ but not in any of the f_i's.

Remark 3.11. Since $\{f_1, \ldots, f_l\}$ is a characteristic set of $I(a/F)$, we get that $I(a/F) = [f_1, \ldots, f_l] : H(f_1, \ldots, f_l)^\infty \subseteq F\{Y\}$ and $I := (f_1, \ldots, f_l) : H(f_1, \ldots, f_l)^\infty$ is a prime ideal of $F[V_B]$. As F is algebraically closed in L, we have that the ideal generated by I in $L[V_B]$, denoted by $I_{L[V_B]}$, is prime and so, by Rosenfeld's Lemma (see Theorem 2.14 in [13])), $\{f_1, \ldots, f_l\}$ is a characteristic set of the prime differential ideal $[f_1, \ldots, f_l] : H(f_1, \ldots, f_l)^\infty$ in $L\{Y\}$.

Moreover, since $(Q_1, \ldots, Q_{n-d}) : l(Q_1, \ldots, Q_{n-d})^\infty = (f_1, \ldots, f_l) : H(f_1, \ldots, f_l)^\infty$ in $F[V_B]$ then the same holds in $L[V_B]$ by Proposition 2.13 in [13]. So the Q_i's form a set of semigenerators of the prime ideal $(f_1, \ldots, f_l) : H(f_1, \ldots, f_l)^\infty$ in $L[V_B]$.

The next proposition establishes a new input which is needed to extend the results of M. Tressl for fields to valued fields.

Proposition 3.12. *Let* $\langle L_1, w, \delta_1, \ldots, \delta_K\rangle$ *be a differential valued field extension of* $\langle F_1, v, \delta_1, \ldots, \delta_K\rangle$ *and let* $\bar{a}$ *be an* N*-tuple in* L_1*. Let* F *be the algebraic closure of* F_1 *in* L_1*. Let us consider a quantifier-free* $\mathcal{L}^*_{\mathcal{D}}$*-formula* $\phi(\bar{x})$ *with parameters in* F_1 *as in Lemma 3.7 such that* $L_1 \models \phi(\bar{a})$*.*

Then there is a J*-algebraically prepared system* $\{f_1, \ldots, f_l; Q_1, \ldots, Q_{n-d}\}$ *over* L_1 *with respect to two tuples* $\bar{c}$*,* $\bar{c}'$ *which is realized by* $\bar{a}$ *with respect to* ϕ *and* F *such that* $L_1 \models \widetilde{\phi}_{alg}(\bar{c}, \bar{c}')$ *(see Notation 3.9).*

Proof. Let us consider a characteristic set $\{f_1,\dots,f_l\}$ of the prime differential ideal $P_1 := \{f \in F\{Y\} | f(\bar{a}) = 0\}$ in $F\{Y\}$. By Lemma 2.3, we find a set of semigenerators $\{Q_i\}_{i=1}^d$ of the prime ideal $I := (f_1,\cdots,f_n) : H(f_1,\cdots,f_n)^\infty$ of $A(f_1,\cdots,f_n)$ (with Krull dimension d).

Since $\bar{a}$ is a generic point of I (and by Remark 3.11), we can define in a unique way $\bar{c}$, $\bar{c}'$ and J in order to obtain the required J-algebraically prepared system over L_1 which is realized by $\bar{a}$ with respect to ϕ and F.

Indeed, the tuple $\bar{c} := (\Lambda\bar{a})_{\Lambda\in V_B}$ is a regular semigeneric point of I (see the notations of Theorem 2.10), J is the set of all variables which occur in the $(f_1,\dots,f_l)$-reduced variant of ϕ but not in any f_i and the tuple $\bar{c}' := (\Lambda\bar{a})_{\Lambda\in J}$ is the J-point such that $L_1 \models \widetilde{\phi}_{alg}(\bar{c},\bar{c}')$. Let us note that $\widetilde{\phi}_{\mathrm{alg}}$ is a quantifier-free $\mathcal{L}_{\mathcal{D}}$-formula with parameters in F. $\square$

Notation 3.13. If M, N are $\mathcal{L}$-structures in an arbitrary language $\mathcal{L}$ and A is a common subset of M and N then we write

$$M \equiv >_{\exists,A} N$$

if every existential $\mathcal{L}$-formula with parameters in A, that holds in M, also holds in N. We write $M \equiv_{\exists,A} N$ if $M \equiv>_{\exists,A} N$ and $N \equiv>_{\exists,A} M$. Hence $M \equiv_{\exists,A} N$ if and only if M and N have the same universal theory over A.

Now we give the proof of the analogue of Theorem 3.3 in [13].

Theorem 3.14. *Let $\langle F_0, v, \delta_1,\cdots,\delta_K\rangle$ be a differential valued subfield of the differential valued fields $\langle L_1, w_1, \delta_1,\cdots,\delta_K\rangle$ and $\langle L_2, w_2, \delta_1,\cdots,\delta_K\rangle$ equipped with K derivations. Let F_i be the algebraic closure F_0 in L_i. Assume that*

1. *$L_1 \equiv_{\exists,F_0} L_2$ in the language $\mathcal{L}_{\mathcal{D}}$;*
2. *L_2 satisfies the scheme of axioms (UC'_K).*

Then $L_1 \equiv>_{\exists,F_0} L_2$ in the language $\mathcal{L}^*_{\mathcal{D}} := \mathcal{L}_{\mathcal{D}} \cup \{\delta_1,\cdots,\delta_K\}$.

Proof. First the condition 1 implies that the valued fields $\langle F_i, w_i\rangle$ are isomorphic. Moreover, this isomorphism respects the K commuting derivations (observe that F_i is a differential subfield of L_i); so we may assume that

$$\langle F, w, \delta_1,\cdots,\delta_K\rangle := \langle F_1, w_1, \delta_1,\cdots,\delta_K\rangle = \langle F_2, w_2, \delta_1,\cdots,\delta_K\rangle$$

is the algebraic closure of F_0 in L_i.

Let $\varphi(\bar{x})$ be a quantifier-free $\mathcal{L}^*_{\mathcal{D}}$-formula with parameters in F_0, where $\bar{x}$ is an N-tuple of variables. Clearly we may assume that the formula φ has the following form:

$$p_1(\bar{x}) = \cdots = p_h(\bar{x}) = 0 \wedge \phi(\bar{x}) \text{ such that}$$

$$\phi(\bar{x}) := \left[q(\bar{x}) \neq 0 \wedge \bigwedge_{j=1}^{v} \mathcal{D}(g_j(\bar{x}), h_j(\bar{x})) \wedge \bigwedge_{k=1}^{w} \neg\mathcal{D}(l_k(\bar{x}), m_k(\bar{x}))\right]$$

and the differential polynomials $p_i(Y), q(Y), g_j(Y), h_j(Y), l_k(Y), m_k(Y)$ have their coefficients in F_0 where $Y = (Y_1,\cdots,Y_N)$.

We assume that there is an N-tuple $\bar{a}$ in L_1^N such that $L_1 \models \varphi(\bar{a})$ and we have to find an N-tuple $\bar{b}$ in L_2^N such that $L_2 \models \varphi(\bar{b})$.

Moreover, by modifying φ, we may assume that

$$L_1 \models \bigwedge_{j=1}^{v} 0 \neq g_j(\bar{a}) \neq h_j(\bar{a}) \neq 0 \wedge \bigwedge_{k=1}^{w} 0 \neq l_k(\bar{a}) \neq m_k(\bar{a}) \neq 0.$$

By Proposition 3.12 (applied to ϕ, F and $\bar{a}$), there is a J-algebraically prepared system $\{f_1,\dots,f_l; Q_1,\dots,Q_{n-d}\}$ over L_1 with two tuples $\bar{c}$, $\bar{c}'$ which is realized by $\bar{a}$ with respect to ϕ and F; i.e.

- $L_1 \models \forall \bar{x}\big[\bigwedge_{i=1}^{l} f_i(\bar{x}) = 0 \wedge H(f_1, \ldots, f_l)(\bar{x}) \neq 0 \Rightarrow (\phi(\bar{x}) \iff \widetilde{\phi}(\bar{x}))\big]$;
- $L_1 \models \widetilde{\phi}_{\mathrm{alg}}(\bar{c}, \bar{c}')$ and,
- $\widetilde{\phi}_{\mathrm{alg}}(\bar{u}, \bar{u}')$ is a quantifier free $\mathcal{L}_{\mathcal{D}}(F)$-formula of the following form

$$\widetilde{q}(\bar{u}, \bar{u}') \neq 0 \wedge \bigwedge_{j=1}^{v} \mathcal{D}(\widetilde{g}_j(\bar{u}, \bar{u}'), \widetilde{h}_j(\bar{u}, \bar{u}')) \wedge \bigwedge_{k=1}^{w} \neg\mathcal{D}(\widetilde{l}_k(\bar{u}, \bar{u}'), \widetilde{m}_k(\bar{u}, \bar{u}'))$$

$$\wedge \bigwedge_{j=1}^{v} 0 \neq g_j(\bar{u}, \bar{u}') \neq h_j(\bar{u}, \bar{u}') \neq 0 \wedge \bigwedge_{k=1}^{w} 0 \neq l_k(\bar{u}, \bar{u}') \neq m_k(\bar{u}, \bar{u'}) \neq 0$$

where $\bar{u} := (u_1, \cdots, u_r)$ is an enumeration of all variables ΘY_j occuring in some of the f_i's and $\bar{u}' := (u'_1, \cdots, u'_s)$ is an enumeration of the other variables $\Theta' Y_j$ occuring in some of the $\widetilde{q}$, $\widetilde{g}_j$, $\widetilde{h}_j$, $\widetilde{l}_k$, $\widetilde{m}_k$'s (see Lemma 3.7).

Moreover $L_1 \models \psi(\bar{c})$ where $\psi(\bar{z})$ is the following quantifier-free formula in the language of rings with parameters in F:

$$\psi(\bar{z}) := \bigwedge_{i=1}^{n-d} [Q_i(\bar{z}) = 0 \wedge s_{Q_i}(\bar{z}) \neq 0] \wedge l(Q_1, \ldots, Q_{n-d})(\bar{z}) \neq 0.$$

Since F is algebraic over F_0, we can choose some $\alpha \in F$ such that the parameters of the formula ψ and $\widetilde{\phi}_{\mathrm{alg}}$ and the differential polynomials f_i have their coefficients in $F_0(\alpha)$. Since $F\{Y\}$ is differentially noetherian and F is algebraic over F_0, there are only finitely many prime ideals of $F\{Y\}$, lying over $P_0 := \{f \in F_0\{Y\} | f(\bar{a}) = 0\}$, say there are exactly t of them. Since F is algebraic over F_0, we may choose α so that in addition, there are t prime differential prime ideals of $F_0(\alpha)\{Y\}$, lying over P_0.

Let Z be a new differential indeterminate of rank smaller than $Y_1, \ldots, Y_N$, let $\mu(Z) \in F_0[Z]$ be the minimal polynomial of α over F_0. For $\epsilon \in \mathbb{N}_0^{r+s}$, let $h_\epsilon \in F_0[Z]$ be the uniquely determined polynomials of degree $< [F_0(\alpha) : F_0]$ such that $h(\bar{u}, \bar{u}') = h'(\bar{u}, \bar{u}', \alpha)$ where

$$h'(\bar{u}, \bar{u}', Z) := \sum_{\epsilon \in \mathbb{N}_0^{r+s}} h_\epsilon(Z) \cdot (\bar{u}\bar{u}')^\epsilon \text{ for any } h \in F_0(\alpha)[\bar{u}, \bar{u}'].$$

Again we write $h'(Y, Z)$ if we consider $h'(\bar{u}, \bar{u}', Z)$ as a differential polynomial in $Z, Y_1, \ldots, Y_N$. The same holds for any polynomial h in $F_0(\alpha)[\bar{z}]$.

Then for all zeroes γ of μ, we have

$$H(f'_1(Y, Z), \ldots, f'_l(Y, Z))(Y, \gamma) = H(f'_1(Y, \gamma), \ldots, f'_l(Y, \gamma)).$$

In L_1, there is a solution $(\bar{c}, \bar{c}', \alpha)$ of the following quantifier-free $\mathcal{L}_{\mathcal{D}}(F_0)$-formula $\Sigma(\bar{u}, \bar{u}', Z)$:

$$\mu(Z) = 0 \wedge \widetilde{\phi'}_{\mathrm{alg}}(\bar{u}, \bar{u}', Z) \wedge \psi'(\bar{u}, Z) \wedge \bigwedge_{i=1}^{l} f'_i(\bar{u}, Z) = 0 \wedge H(f'_1, \ldots, f'_l)(\bar{u}, Z) \neq 0$$

such that $\widetilde{\phi}'_{\mathrm{alg}}(\bar{u}, \bar{u}', \gamma)$ is a quantifier-free $\mathcal{L}_{\mathcal{D}}(F_0)$-formula obtained from $\widetilde{\phi}_{\mathrm{alg}}(\bar{u}, \bar{u}')$ by replacing polynomials h by h' for any γ in F with $\mu(\gamma) = 0$ and similarly for ψ' in the language of rings.

Since $L_1 \equiv\rangle_{\exists, F_0} L_2$ as valued fields, Σ is satisfied by $(\bar{e}, \bar{f}, \beta)$ in L_2.

Now we can define an isomorphism σ between $F_0(\alpha)$ and $F_0(\beta)$ where β is a solution of $\mu(Z)$ in L_2. We can extend this isomorphism in a natural way between $F_0(\alpha)\{Y\}$ and $F_0(\beta)\{Y\}$ and the same holds for $F_0(\alpha)[\bar{z}]$.

Hence, as in the Claim of Theorem 3.3 in [13], we get that $\{\sigma(f_1), \ldots, \sigma(f_l)\}$ is a characteristic set in L_2 and $(\sigma(f_1), \ldots, \sigma(f_l)) : H(\sigma(f_1), \ldots, \sigma(f_l))^\infty$ is a prime ideal in $L_2[\bar{u}]$. Moreover since $\{\sigma(Q_i)\}_{i=1}^{n-d}$ is a set of semigenerators of $(\sigma(f_1), \cdots, \sigma(f_l)) : H(\sigma(f_1), \cdots, \sigma(f_l))^\infty$ and $L_2 \models \psi'(\bar{e}, \beta)$, we get that $\bar{e}$ is a regular semigeneric point of $(\sigma(f_1), \cdots, \sigma(f_l)) : H(\sigma(f_1), \cdots, \sigma(f_l))^\infty$.

The following sets in L_2 are open for the valuation topology in L_2

$$\big\{(x, y) \in L_2^2 : L_2 \models \mathcal{D}(x, y) \wedge 0 \neq x \neq y \neq 0\big\} \text{ and } \big\{(x, y) \in L_2^2 : L_2 \models \neg\mathcal{D}(x, y) \wedge 0 \neq x \neq y \neq 0\big\},$$

and by the continuity of polynomial functions with respect to the valuation topology in L_2, we get that the following set D is open and contains the tuple $(\bar{e}, \bar{f})$

$$\bigcap_{j=1}^{v} \big\{(\bar{u}, \bar{u}') \subseteq L_2 : \mathcal{D}(\widetilde{g}'_j(\bar{u}, \bar{u}', \beta), \widetilde{h}'_j(\bar{u}, \bar{u}', \beta)) \wedge 0 \neq \widetilde{g}'_j(\bar{u}, \bar{u}', \beta) \neq \widetilde{h}'_j(\bar{u}, \bar{u}', \beta) \neq 0\big\}$$

$$\cap \bigcap_{k=1}^{w} \big\{(\bar{u}, \bar{u}') \subseteq L_2 : \neg\mathcal{D}(\widetilde{l}'_k(\bar{u}, \bar{u}', \beta), \widetilde{m}'_k(\bar{u}, \bar{u}', \beta)) \wedge 0 \neq \widetilde{l}'_k(\bar{u}, \bar{u}', \beta) \neq \widetilde{m}'_k(\bar{u}, \bar{u}', \beta) \neq 0\big\}$$

$$\cap \big\{(\bar{u}, \bar{u}') \subseteq L_2 : \widetilde{q}'(\bar{u}, \bar{u}', \beta) \neq 0\big\}.$$

So there exists a ball $B := B_{>\gamma}(\bar{e}, \bar{f}) \subseteq D$.

So we apply the scheme of axioms (UC'_K) in order to find a differential solution $\bar{b}$ of $\sigma(f_1) = 0, \ldots, \sigma(f_l) = 0$ in L_2^N which is γ-close to the tuple $(\bar{e}, \bar{f})$ such that $H(\sigma(f_1), \ldots, \sigma(f_l))(\bar{b}) \neq 0$. Since $p_1, \ldots, p_r \in P_0$, and $\sigma(P_1 \cap F_0(\alpha)\{Y\}) = [\sigma(f_1), \ldots, \sigma(f_n)] : H(\sigma(f_1), \ldots, \sigma(f_n))^\infty$ ly over P_0, $\bar{b}$ is a differential solution of $p_1 = \ldots = p_r = 0$.

Moreover the topological conditions about the point $\bar{b}$ with respect to $(\bar{e}, \bar{f})$ give us

$$L_2 \models \widetilde{q}'(\bar{b}, \beta) \neq 0 \wedge \bigwedge_{j=1}^{v} \mathcal{D}(\widetilde{g}'_j(\bar{b}, \beta), \widetilde{h}'_j(\bar{b}), \beta) \wedge \bigwedge_{k=1}^{w} \neg\mathcal{D}(\widetilde{l}'_k(\bar{b}, \beta), \widetilde{m}'_k(\bar{b}), \beta).$$

Since $\sigma(f_1)(\bar{b}) = \ldots = \sigma(f_l)(\bar{b}) = 0$ and $H(\sigma(f_1), \ldots, \sigma(f_l))(\bar{b}) \neq 0$, it suffices to apply the differential isomorphism $\sigma : F_0(\alpha)\{Y\} \to F_0(\beta)\{Y\}$ to Equation 1 in Lemma 3.7 (more precisely to the polynomials q, g_j, h_j, l_k, m_k in $F_0\{Y\}$ with respect to $\sigma(f_1), \ldots, \sigma(f_l)$) in order to get that $L_2 \models \phi(\bar{b})$; and so $L_2 \models \varphi(\bar{b})$. □

Notation 3.15. If T is a theory of valued fields then we denote by T^* the corresponding theory of differential valued fields equipped with K commuting derivations.

Now we are stating our valued analogues of the results of model companion and model completion of theories of differential fields in [13].

Corollary 3.16. *Let $\mathcal{L}_\mathcal{D}$ be the language of non-trivially valued fields and let C be a set of new constants. Let T be a model complete theory in the language $\mathcal{L}_\mathcal{D}(C)$ such that every model of T is henselian.*

Let $\widetilde{T}$ be a theory in a language $\widetilde{\mathcal{L}} \supseteq \mathcal{L}_\mathcal{D}(C)$ such that $\widetilde{T}$ contains T and $\widetilde{T}$ is an expansion by definition of T. Let F be an $\widetilde{\mathcal{L}}^*$-structure.

If $\widetilde{T} \cup diag(F{\restriction}\widetilde{\mathcal{L}})$ is complete then $\widetilde{T}^* \cup (UC'_K) \cup diag(F)$ is complete.

Proof. See the proof of Theorem 7.1 in [13]. □

As in [13], the next result is a consequence of Corollary 3.16.

Corollary 3.17. *Under the same assumptions as in Corollary 3.16, assume moreover that $\widetilde{T}$ is a model companion of an $\widetilde{\mathcal{L}}$-theory $\widetilde{T}_0$ extending the theory of valued fields. Then*

1. *$\widetilde{T}^* \cup (UC'_K)$ is a model companion of the $\widetilde{\mathcal{L}}^*$-theory $\widetilde{T}_0^*$.*

2. *If $\widetilde{T}$ is a model completion of $\widetilde{T}_0$ then $\widetilde{T}^* \cup (UC'_K)$ is a model completion of the $\widetilde{\mathcal{L}}^*$-theory $\widetilde{T}_0^*$.*

3. *If $\widetilde{T}$ has quantifier elimination then $\widetilde{T}^* \cup (UC'_K)$ has quantifier elimination.*

4. *If T is complete and L is a differential valued field and a model of T then $\widetilde{T}^* \cup (UC'_K) \cup diag(F)$ is complete where F is the $\widetilde{\mathcal{L}}^*$-substructure generated by $\emptyset$ in L.*

Proof. See Proof of Theorem 7.2 in [13]. □

Proposition 3.18. *Any differential henselian valued field $\langle F, v, \delta_1, \ldots, \delta_K \rangle$ which is a model of the scheme (UC'_K) is a model of the scheme (UC_K) (see Section 3 in [13]).*

Proof. It suffices to prove that if any prime ideal I of a polynomial ring over F has a regular F-rational point then it has a regular semigeneric F-rational point.

Let I be an ideal in $F[\underline{X}]$ of Krull dimension d with a set of semigenerators $\{Q_1, \ldots, Q_{n-d}\}$. Since F is a henselian valued field, F is a large field and; since I has a regular F-rational point, we have that F is existentially closed in $L := F[\underline{X}]/I$. So $\bar{f} := \underline{X} + I$ is a generic point of I in $F[\underline{X}]/I$. In particular, $\bar{f}$ is a regular semigeneric point of I. Since F is existentially closed in L in the language of rings and the fact that $\underline{X} + I$ is a regular semigeneric point of I with respect to the Q_i's is expressible by a first-order quantifier-free formula with parameters in F in the language of rings, we get a regular semigeneric F-rational point of I. □

4 Applications

4.1 Differential Ax-Kochen-Ersov Theorem

Now we are going to establish an Ax-Kochen-Ersov differential result for differential henselian valued fields which satisfy the scheme of axioms (UC'_K).

First we recall the classical "Ax-Kochen-Ersov" theorem in its existentially closed form. A proof of this result can be found in [6].

Theorem 4.1. *Let $\langle F_1, v_1 \rangle$ be a henselian valued field and let $\langle F_2, v_2 \rangle$ be a valued field extension of $\langle F_1, v_1 \rangle$ such that*

- $k_{F_1} \subseteq_{e.c} k_{F_2}$,
- $v_1(F_1^\times) \subseteq_{e.c} v_2(F_2^\times)$.

Then $\langle F_1, v_1 \rangle \subseteq_{e.c} \langle F_2, v_2 \rangle$ in the language $\mathcal{L}_{\mathcal{D}}$.

Now we can easily prove the following differential analogue.

Theorem 4.2. *Let $\langle F_1, \delta_1, \cdots, \delta_K, v_1 \rangle$ be a differential henselian valued field such that $F_1 \models (UC'_K)$ and let $\langle F_2, \delta_1, \cdots, \delta_K, v_2 \rangle$ be a differential valued field extension of $\langle F_1, \delta_1, \cdots, \delta_K, v_1 \rangle$ such that*

- $k_{F_1} \subseteq_{e.c} k_{F_2}$,
- $v_1(F_1^\times) \subseteq_{e.c} v_2(F_2^\times)$.

Then $\langle F_1, v_1, \delta_1, \cdots, \delta_K \rangle \subseteq_{e.c} \langle F_2, v_2, \delta_1, \cdots, \delta_K \rangle$ in $\mathcal{L}^*_{\mathcal{D}}$.

Proof. By Theorem 4.1, we deduce that $\langle F_2, v_2 \rangle \equiv_{\exists, F_1} \langle F_1, v_1 \rangle$ in the language $\mathcal{L}_{\mathcal{D}}$. Now it suffices to apply Theorem 3.14 to obtain the result. □

Now we prove a topological lemma which will be also useful in Theorem 4.10. In particular, it shows that the field of constants of a differential valued field satisfying the scheme (UC'_K) is dense in its underlying field with respect to the valuation topology.

Lemma 4.3. *Let $\langle F, \delta_1, \cdots, \delta_K\rangle$ be a differential henselian valued field which is a model of (UC'_K).*

Then for any natural numbers $N_1, \cdots, N_K$, the following set is dense in $F^{\sum_{i=1}^{K}(N_i+1)}$ with respect to the valuation topology

$$\{(x, \delta_1(x), \ldots, \delta_1^{(N_1)}(x), \ldots, x, \ldots, \delta_K^{(N_K)}(x))|x \in F\}.$$

Proof. Let us consider a tuple $\bar{a} := (a_{10}, \ldots, a_{1N_1}, \ldots, a_{K0}, \ldots, a_{KN_K})$ in F and $\gamma \in v(F^\times)$. We want to find an element $x \in F$ such that $\bigwedge_{i=1}^{K} \bigwedge_{j=0}^{N_i} v(\delta_i^{(j)}x - a_{ij}) > \gamma$ and $\bigwedge_{i=1}^{K} \delta_i^{(N_i+1)}(x) = 0$. To this effect, we consider the following differential ideal $[\delta_1^{(N_1+1)}(X), \ldots, \delta_K^{(N_K+1)}(X)]$ in $F\{X\}$.

It is a prime differential ideal and moreover, $\{\delta_1^{(N_1+1)}(X), \ldots, \delta_K^{(N_K+1)}(X)\}$ is a characteristic set of this ideal. Indeed, $\{X_{\delta_1^{(N_1+1)}}, \ldots, X_{\delta_K^{(N_K+1)}}\}$ is a set of autoreduced and coherent differential polynomials and $(X_{\delta_1^{(N_1+1)}}, \ldots, X_{\delta_K^{(N_K+1)}}) = (X_{\delta_1^{(N_1+1)}}, \ldots, X_{\delta_K^{(N_K+1)}}) : H(X_{\delta_1^{(N_1+1)}}, \ldots, X_{\delta_K^{(N_K+1)}})^\infty$ is a prime ideal of the corresponding polynomial ring $A(X_{\delta_1^{(N_1+1)}}, \ldots, X_{\delta_K^{(N_K+1)}})$ and does not contain non-zero element reduced with respect to $\{X_{\delta_1^{(N_1+1)}}, \ldots, X_{\delta_K^{(N_K+1)}}\}$. Hence by Rosenfeld's Lemma (see [5, p167]), $\{X_{\delta_1^{(N_1+1)}}, \ldots, X_{\delta_K^{(N_K+1)}}\}$ is a characteristic set of our considered differential ideal. Moreover $\bar{0}$ is a regular semigeneric point of $(X_{\delta_1^{(N_1+1)}}, \ldots, X_{\delta_K^{(N_K+1)}})$ with respect to the semigenerators $\{Q_i := X_{\delta_i^{(N_i+1)}}\}$.

So we can find an element x having the required properties by applying the scheme of axioms (UC'_K) at the regular semigeneric F-rational point $\bar{0}$ in F^K of the ideal $(X_{\delta_1^{(N_1+1)}}, \ldots, X_{\delta_K^{(N_K+1)}})$ for the neighbourhood $\bigwedge_{i=1}^{K} \bigwedge_{j=0}^{N_i} v(x_{ij} - a_{ij}) > \gamma$ and the J-point $\bar{a}$ where J is the set of differential indeterminates $\{\delta_i^{(j)}X \big| i \in \{1, \ldots, K\}, j \in \{0, \ldots, N_i\}\}$. □

4.2 Examples

Let VF_0 be the $\mathcal{L}_\mathcal{D}$-theory of non-trivially valued fields of equicharacteristic 0 (i.e. $\neg\mathcal{D}(c, 1)$ belongs to VF_0 and so, VF_0 is a universal $\mathcal{L}_\mathcal{D}$-theory). Then $ACVF_0$ is the $\mathcal{L}_\mathcal{D}$-theory of algebraically closed non-trivially valued fields and it is model complete (see [10]), moreover using prime extensions, it is easy to see that it admits quantifier elimination in $\mathcal{L}_\mathcal{D}$ (see [7, p. 83]). So we get the following

Corollary 4.4. *The $\mathcal{L}_\mathcal{D}^*$-theory $ACVF_0^* \cup (UC'_K)$ is the model completion of the $\mathcal{L}_\mathcal{D}^*$-theory VF_0^*.*

Proof. Since any algebraically closed valued field is clearly henselian, it suffices to apply Corollary 3.17. □

Remark 4.5. Since any model of $ACVF_0$ is a henselian valued field, Proposition 3.18 implies that any model of $ACVF_0^* \cup (UC'_K)$ is a model of the model completion of the theory of differential fields equipped with K commuting derivations, denoted by $ACF_0 \cup (UC_K)$ (see Definition 8.2 (i) in [13]).

Let us recall that a p-valued field K of p-rank d ($d \in \mathbb{N} \setminus \{0\}$), with p a prime number, is a valued field of characteristic 0, residue field of characteristic p and the dimension of $\mathcal{O}_K/(p)$ over the prime field $\mathbb{F}_p$ is equal to d.

Let $\widetilde{\mathcal{L}} := \mathcal{L}_\mathcal{D} \cup \{P_n; n \in \mathbb{N} \setminus \{0, 1\}\} \cup \{c_2, \cdots, c_d\}$. This language is an expansion by definition of the language $\mathcal{L}_\mathcal{D}$ since the predicates P_n are interpreted as the nth powers. The theory pCF_d of p-adically closed fields of p-rank d admits quantifier elimination in $\widetilde{\mathcal{L}}$ and so is the model completion of its universal part $(pCF_d)_\forall$. This last theory has been axiomatized by L. Bélair who denoted the set of axioms by T_d (see Theorem 2.4 in [2]).

Since p-adically closed fields are henselian by definition, we get the following result from Corollary 3.17

Corollary 4.6. *The $\widetilde{\mathcal{L}}^*$-theory $(pCF_d)^* \cup (UC'_K)$ is the model completion of $(T_d)^*$.*

Let p be a prime number, let d, f be positive natural numbers. Let $\langle F, v\rangle$ be a differential p-valued field of p-rank d where $v : F \longmapsto v(F^{\times}) \cup \{\infty\}$ is a p-valuation of p-rank d and $[k_F : \mathbb{F}_p] = f$. Let π belonging to F be such that $v(\pi)$ is the least positive element of the value group.

Set $q = p^f$ and let $\gamma(X) := \frac{1}{\pi}[\frac{X^q - X}{(X^q - X)^2 - 1}]$ be the π-adic Kochen operator.

We have that $\gamma(F) \subseteq \mathcal{O}_F$ (see Lemma 6.1 in [9]); if F is p-adically closed, then $\mathcal{O}_F = \gamma(F)$ (see Theorem 6.15 in [9]).

Let us denote by $F\langle \underline{X}\rangle := F\langle X_1, \ldots, X_n\rangle$ the field of differential rational functions in n indeterminates. Assume now that $\langle F, v\rangle$ is a differential p-valued field of p-rank d with valuation v and K commmuting derivations $\delta_1, \cdots, \delta_K$. Then we can extend the valuation and the derivation on $F\langle \underline{X}\rangle$ in such a way it becomes a differential p-valued field extension of F of p-rank d (see Section 5 in [4]).

Now assume that $\langle F, v\rangle$ is a differential p-adically closed field of p-rank d. Before recalling the analogue of the Hilbert's Seventeenth Problem for p-adically closed fields of p-rank d, we need to introduce the following notation.

Let $\langle L, v\rangle$ be a valued extension of $\langle F, v\rangle$.

Definition 4.7 (See Section 6.2 in [9]). The γ-Kochen ring R_L of L over F is the subring defined by:

$$R_L = \{\frac{t}{1 + \pi \cdot s} : t, s \in \mathcal{O}_F[\gamma(L)] \text{ and } 1 + \pi \cdot s \neq 0\}.$$

The quotient field of R_L is the field generated by F and $\gamma(L) \setminus \{\infty\}$ and by Merckel's Lemma, $K(\gamma(L)) = L$ (see Lemma 6.6 in [9]).

Theorem 4.8 (See Theorem 7.12 in [9]). *Let F be a model of pCF_d. If f, $g \in K[\underline{X}]$ and f/g is integral definite (i.e. $g(\bar{a}) \neq 0$ implies $\frac{f(\bar{a})}{g(\bar{a})} \in \mathcal{O}_K$ for all $\bar{a} \subseteq F^n$), then there are $t, t' \in \mathcal{O}_F[\gamma(F(\underline{X}))]$ such that*

$$\frac{f}{g} = \frac{t'}{1 + \pi \cdot t}.$$

Now, let us state and prove the differential case using the technology of Section 3 and the following result on holomorphy rings.

Theorem 4.9 (See Theorem 6.14 in [9]). *Let L be a field extension of F which admits a p-valuation of p-rank d. The γ-Kochen ring R_L of L is the holomorphy ring $\bigcap_{v \in \Gamma} \mathcal{O}_v$, where Γ is the set of all p-valuations of p-rank d of L which extends the p-valuation of p-rank d of F.*

Theorem 4.10. *Let $\langle F, v, \delta_1, \cdots, \delta_K\rangle$ be a model of $(pCF_d)^* \cup (UC'_K)$. Let us consider the ring $F\{\underline{X}\}$ of differential polynomials in n differential indeterminates over F. Let f, g be two differential polynomials in $F\{\underline{X}\}$ such that $\frac{f}{g}$ is integral definite (i.e $g(\bar{a}) \neq 0$ implies $\frac{f(\bar{a})}{g(\bar{a})} \in \mathcal{O}_F$ for all $\bar{a} \subseteq F^n$). Then $\frac{f}{g}$ belongs to the γ-Kochen ring $R_{F\langle \underline{X}\rangle}$ of $F\langle \underline{X}\rangle$ over K.*

Proof. Let us assume that $\frac{f}{g} \notin R_{F\langle \underline{X}\rangle}$. Then, by Theorem 4.9, there exists one p-valuation w of p-rank d of $F\langle \underline{X}\rangle$ extending v over K such that $w(\frac{f}{g}) < 0$.

We have:

$$\langle F\langle \underline{X}\rangle, w\rangle \models \phi := \exists \bar{y}\, [w(\frac{f^*(\bar{y})}{g^*(\bar{y})}) < 0 \wedge g^*(\bar{y}) \neq 0]$$

where f^*, g^* are the usual ordinary polynomials corresponding to f and g. Then, using the fact that pCF_d is the model completion of T_d, we embed $F\langle \underline{X}\rangle$ in a differential p-adically closed field of p-rank d. So, since F satisfies the scheme (UC'_K), we use the model completeness of pCF_d and apply Lemma 4.3. So, we get a contradiction with $\frac{f}{g}$ integral definite. $\square$

Let $\widetilde{\mathcal{L}} := \mathcal{L}_{\mathcal{D}} \cup \{\leqslant\}$ and let OVF be the theory of non-trivially valued ordered fields, namely the $\mathcal{L}_{fields} \cup \{\leqslant\}$-theory of ordered fields together with the $\mathcal{L}_{\mathcal{D}}$-theory of valued fields, and the following compatibility condition between the valuation topology and the order topology:

$$\forall a \, \forall b \quad \big(0 < a \leqslant b \Rightarrow \mathcal{D}(b, a)\big).$$

Let RVF be the $\widetilde{\mathcal{L}}$-theory of real closed valued fields, namely the theory OVF together with axioms of real closed fields. Note that an $\widetilde{\mathcal{L}}$-substructure of a model of RVF is a model of OVF.

The theory RVF is model complete. Indeed, a real closed valued field is a henselian valued ordered field with a real closed residue field and divisible ordered group (see Theorem 3 in [3]). Since the theory of real closed fields RCF and the theory of divisible ordered groups are model complete and complete, the $\widetilde{\mathcal{L}}$-theory RVF is model complete and complete by Ax-Kochen-Ersov Theorems (see Theorems A and B in [3]) (note that the order in a real closed field is existentially definable).

Then, we show that any $\widetilde{\mathcal{L}}$-substructure of a model of RVF has a prime extension and so RVF is the model completion of OVF (see [11]). Let F be a model of OVF and let $\mathcal{O}$ be its valuation ring and $\mathcal{M}$ its maximal ideal. Let $\widetilde{F}$ be the real closure of F and let $\widetilde{\mathcal{O}}$ be the convex hull of $\mathcal{O}$ in $\widetilde{F}$. Then, $\widetilde{\mathcal{O}}$ is a valuation ring of $\widetilde{F}$ and its maximal ideal $\mathcal{M}_{\widetilde{\mathcal{O}}}$ is such that $\mathcal{M}_{\widetilde{\mathcal{O}}} \cap \mathcal{O} = \mathcal{M}$ (see Lemma 1.1, Lemma 1.8 and its proof in [1]) and so $\widetilde{F}$ is an $\widetilde{\mathcal{L}}$-extension of F, satisfying RVF (see Theorems 1 and 2 in [3]).

Therefore, we get the following

Corollary 4.11. *The $\widetilde{\mathcal{L}}^*$-theory $(RVF)^* \cup (UC'_K)$ is the model completion of the $\widetilde{\mathcal{L}}^*$-theory $(OVF)^*$.*

Remark 4.12. Since any real closed valued field is henselian, Proposition 3.18 implies that any model of $(RVF)^* \cup (UC'_K)$ is a model of the model completion of the $\mathcal{L}_{\text{rings}} \cup \{\leqslant\}$-theory of differential ordered fields, denoted by $RCF \cup (UC_K)$ (see Definition 8.2 (ii) in [13]).

By using our previous results, we can state a more topological axiomatization of $RCF \cup (UC_K)$ as follows.

Let $\langle F, \leqslant, \delta_1, \cdots, \delta_K \rangle$ be a differential ordered field. We say that F is a model of $RCF \cup (UC'_K)$ if the following scheme of axioms, denoted by $(UC'_K)_{\leqslant}$, holds:

for any J-algebraically prepared system $\{f_1, \ldots, f_l; Q_1, \ldots, Q_{n-d}\}$ over F with respect to the two tuples $\bar{a}$, $\bar{a}'$ and any $\epsilon > 0$ in F, there is a differential solution $\bar{b}$ of $f_1, \ldots, f_l$ in F which is ϵ-close to $(\bar{a}, \bar{a}')$ with respect to the order topology on F.

Here the notion of ϵ-closeness of two tuples $\bar{c}$ and $\bar{d}$ means that for any i, $|a_i - b_i| > \epsilon$.

Let us prove that this scheme of axioms for differential real closed fields proves that these are models of $RCF \cup (UC_K)$. To this effect, it suffices to consider a differential real closed field F which satisfies the scheme of axioms $(UC'_K)_{\leqslant}$. We define on the ordered field F a non-trivial valuation with respect to the following valuation subring $\mathcal{O}_F$ of F

$$\{x \in F | \, 0 \leqslant |x| \leqslant q \text{ for some } q \in \mathbb{Q}^{>0}\}.$$

We easily see that this valuation v is compatible with the order on F and that $\langle F, v, \leqslant \rangle$ is then a model of (UC'_K). So by Remark 4.12, $\langle F, \leqslant \rangle$ is a model of $RCF \cup (UC_K)$.

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