Dicots, and a taxonomic ranking for misère games

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Abstract

We study combinatorial games in misère version. In a general context, little can be said about misère games. For this reason, several universes were earlier considered for their study, which can be ranked according to their inclusion ordering. We study in particular a special universe of games called dicots, which turns out to be the known universe of lowest rank modulo which equivalence in misère version implies equivalence in normal version. We also prove that modulo the dicot universe, we can define a canonical form as the reduced form of a game that can be obtained by getting rid of dominated options and most reversible options. We finally count the number of dicot equivalence classes of dicot games born by day 3.

Keywords: Canonical form, Combinatorial game, Dicot, Misère play

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We study combinatorial games in misère version, and in particular a special universe (i.e. family) of games called dicots. We first recall basic definitions, following [1,3,4].

A combinatorial game is a finite two-player game with no chance and perfect information. The players, called Left and Right, alternate moves until one player has no available move. Under the normal convention, the last player

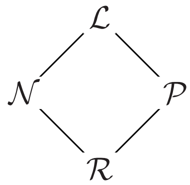

Fig. 1. Partial ordering of outcomes

to move wins the game while under the misère convention, that player loses the game.

A game can be defined recursively by its sets of options $G = \{G^{\boldsymbol{L}}|G^{\boldsymbol{R}}\}$, where $G^{\boldsymbol{L}}$ is the set of games reachable in one move by Left (called Left options), and $G^{\boldsymbol{R}}$ the set of games reachable in one move by Right (called Right options). The zero game $0 = \{\cdot|\cdot\}$, is the game with no options. The birthday of a game is defined recursively as $birthday(G) = 1+\max_{G' \in G^{\boldsymbol{L}} \cup G^{\boldsymbol{R}}} birthday(G')$, with 0 being the only game with birthday 0. We say a game G is born on day n if $birthday(G) = n$, and that it is born by day n if $birthday(G) \leq n$. The games born on day 1 are $\{0|\cdot\} = 1$, $\{\cdot|0\} = \overline{1}$ and $\{0|0\} = *$.

Given two games $G = \{G^{\boldsymbol{L}}|G^{\boldsymbol{R}}\}$ and $H = \{H^{\boldsymbol{L}}|H^{\boldsymbol{R}}\}$, we recursively define the (disjunctive) sum of G and H as $G+H = \{G^{\boldsymbol{L}}+H, G+H^{\boldsymbol{L}}|G^{\boldsymbol{R}}+H, G+H^{\boldsymbol{R}}\}$ (where $G^{\boldsymbol{L}}+H$ is the set of sums of H and an element of $G^{\boldsymbol{L}}$), i.e. the game where each player chooses on his turn which one of G and H to play on. One of the main objectives of combinatorial game theory is to determine for a game G the outcome of its sum with any other game.

For both conventions, there are four possible outcomes for a game. Games for which Left player has a winning strategy whatever Right does have outcome $\mathcal{L}$ (for *left*). Similarly, $\mathcal{N}$, $\mathcal{P}$ and $\mathcal{R}$ (for *next*, *previous* and *right*) denote respectively the outcomes of games for which the first player, the second player, and Right has a winning strategy. We note $o^+(G)$ the normal outcome of a game G i.e. its outcome under the normal convention and $o^-(G)$ the misère outcome of G. Outcomes are partially ordered according to Figure 1, with greater games being more advantageous for Left. Note that there is no general relationship between the normal outcome and the misère outcome of a game.

Given two games G and H, we say that G is greater than or equal to H in misère play whenever Left prefers the game G rather than the game H, that is $G \geq^- H$ if for every game X, $o^-(G+X) \geq o^-(H+X)$. We say that G and H are equivalent in misère play, denoted $G \equiv^- H$, when for every game X, $o^-(G+X) = o^-(H+X)$ (i.e. $G \geq^- H$ and $H \geq^- G$). Inequality and equivalence are defined similarly in normal convention, using superscript $+$ instead of $-$.

General equivalence and comparison are very limited in misère play (see [5,10]), this is why Plambeck and Siegel defined in [8,9] an equivalence relationship under restricted universes, leading to a breakthrough in the study of misère play games.

Definition 1 ([8,9]) Let $\mathcal{U}$ be a universe of games, G and H two games. We say G is greater than or equal to H modulo $\mathcal{U}$ in misère play and write $G \geq^- H \pmod{\mathcal{U}}$ if $o^-(G+X) \geq o^-(H+X)$ for every $X \in \mathcal{U}$. We say G is equivalent to H modulo $\mathcal{U}$ in misère play and write $G \equiv^- H \pmod{\mathcal{U}}$ if $G \geq^- H \pmod{\mathcal{U}}$ and $H \geq^- G \pmod{\mathcal{U}}$.

For instance, Plambeck and Siegel [8,9] considered the universe of all positions of given games, especially octal games. Other universes have been considered, including the universes of impartial games $\mathcal{I}$ [3,4], dicot games $\mathcal{D}$ [2,6], dead-ending games $\mathcal{E}$ [7], and all games $\mathcal{G}$ [10]. These classes are ordered (ranked) by inclusion as follows:

$$\mathcal{I} \subset \mathcal{D} \subset \mathcal{E} \subset \mathcal{G}.$$

The canonical form of a game is the simplest game of its equivalence class. It is therefore natural to consider canonical forms modulo a given universe. In normal play, impartial games have the same canonical form when considered modulo the universe of impartial games or modulo the universe of all games. In misère play, the corresponding canonical forms are different.

In the following, we focus on the universe of dicots. A game is said to be dicot either if it is $\{\cdot|\cdot\}$ or if it has both Left and Right options and all these options are dicot. Note that the universe of dicots, denoted $\mathcal{D}$ is closed under sum of games and taking option.

Theorem 2 *Let G and H be any games. If $G \geq^-_{\mathcal{D}} H$, then $G \geq^+ H$.*

The dicot universe is the universe of lowest rank known to have this property.

Modulo the dicot universe, we propose a reduced form of a game that can be obtained by getting rid of dominated options and most reversible options.

Theorem 3 *Consider two dicot games G and H. If $G \equiv^-_{\mathcal{D}} H$ and both are in reduced form, then either G and H are the games $0 = \{\cdot|\cdot\}$ and $\{*|*\}$, or there exists a bijection between the Left (resp. Right) options of G and of H such that an option and its image are equivalent modulo $\mathcal{D}$.*

As a consequence, we can define the canonical form of a game as its reduced form, except when the game reduces to $\{*|*\}$, in which case the canonical form is 0.

Thanks to that result, we are able to count the number of dicot equivalence classes (modulo $\mathcal{D}$) of games born by day 3, improving the bound of 5041 proposed by Milley in [6].

Theorem 4 *The* 1046530 *dicot games born by day* 3 *are distributed among* 1268 *equivalence classes modulo* $\mathcal{D}$*.*

By comparison, Milley proved in [6] that the number of misère dicot equivalence classes of dicot games born by day 2 is 9. In normal play, there are 50 non-equivalent dicot games born by day 3 (both modulo the universe of all games or the universe of dicots).

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