Dicots, and a taxonomic ranking for misère games

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Abstract

We study combinatorial games in misère version. In a general context, little can be said about misère games. For this reason, several universes were earlier considered for their study, which can be ranked according to their inclusion ordering. We study in particular a special universe of games called dicots, which turns out to be the known universe of lowest rank modulo which equivalence in misère version implies equivalence in normal version. We also prove that modulo the dicot universe, we can define a canonical form as the reduced form of a game that can be obtained by getting rid of dominated options and most reversible options. We finally count the number of dicot equivalence classes of dicot games born by day 3.

1 Introduction

In this paper, we study combinatorial games in misère version, and in particular a special family of games called dicots. We first recall basic definitions, following [1, 3, 4].

A combinatorial game is a finite two-player game with no chance and perfect information. The players, called Left and Right, alternate moves until one player has no available move. Under the normal convention, the last player to move wins the game while under the misère convention, that player loses the game. In this paper, we are mostly interested in games under the misère convention.

A game can be defined recursively by its sets of options $G = \{G^{\boldsymbol{L}}|G^{\boldsymbol{R}}\}$, where $G^{\boldsymbol{L}}$ is the set of games reachable in one move by Left (called Left options), and $G^{\boldsymbol{R}}$ the set of games reachable in one move by Right (called Right options). We note G^L for the typical option of $G^{\boldsymbol{L}}$, and G^R for the typical option of $G^{\boldsymbol{R}}$, and we use $\cdot$ to denote an empty set of options. The zero game $0 = \{\cdot|\cdot\}$, is the game with no options. The birthday of a game is defined recursively as one plus the maximum birthday of its options, with 0 being the only game with birthday 0. We say a game G is born *on day n* if its birthday is n and that it is born *by day n* if its birthday is at most n. The games born on day 1 are $\{0|\cdot\} = 1$, $\{\cdot|0\} = \overline{1}$ and $\{0|0\} = *$. A position reachable by any sequence of moves from a game is called a *follower*.

A game can also be depicted by its game tree, where the game trees of its options are linked to the root by downward edges, left-slanted for Left options and right-slanted for Right options. For instance, the game trees of games born by day 1 are depicted on Figure 1. More game trees appear on Figure 3 where s is the game $\{0, *|0\}$.

Given two games $G = \{G^{\boldsymbol{L}}|G^{\boldsymbol{R}}\}$ and $H = \{H^{\boldsymbol{L}}|H^{\boldsymbol{R}}\}$, the (disjunctive) *sum* of G and H is recursively defined as $G + H = \{G^{\boldsymbol{L}} + H, G + H^{\boldsymbol{L}}|G^{\boldsymbol{R}} + H, G + H^{\boldsymbol{R}}\}$ (where $G^{\boldsymbol{L}} + H$ is the set of

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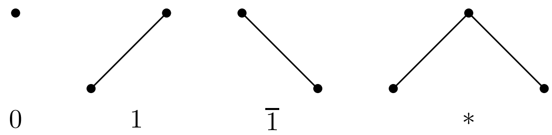


Figure 1: Game trees of games born by day 1.

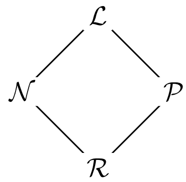


Figure 2: Partial ordering of outcomes

sums of H with an element of $G^{\boldsymbol{L}}$), i.e. the game where each player chooses on their turn which one of G or H to play on. The *conjugate* $\overline{G}$ of a game $G = \{G^{\boldsymbol{L}}|G^{\boldsymbol{R}}\}$ is recursively defined by $\overline{G} = \{\overline{G^{\boldsymbol{R}}}|\overline{G^{\boldsymbol{L}}}\}$ (where $\overline{G^{\boldsymbol{R}}}$ is the set of conjugates of elements of $G^{\boldsymbol{R}}$).

One of the main objectives of combinatorial game theory is to determine for a game G the outcome of its sum with any other game. Under both conventions, there are four possible outcomes for a game. Games for which Left has a winning strategy whatever Right does have outcome $\mathcal{L}$ (for *left*). Similarly, $\mathcal{N}$, $\mathcal{P}$ and $\mathcal{R}$ (for *next*, *previous* and *right*) denote respectively the outcomes of games for which the first player, the second player, and Right has a winning strategy. We denote by $o^+(G)$ the normal outcome of a game G, i.e. its outcome under the normal convention, and by $o^-(G)$ the misère outcome of G. Outcomes are partially ordered according to Figure 2, with greater games being more advantageous for Left. Note that there is no general relationship between the normal outcome and the misère outcome of a game.

Given two games G and H, we say that G is greater than or equal to H whenever Left prefers the game G rather than the game H, that is $G \geq^- H$ if for every game X, $o^-(G+X) \geq o^-(H+X)$. We say that G and H are equivalent in misère play, denoted $G \equiv^- H$, when for every game X, $o^-(G + X) = o^-(H + X)$ (i.e. $G \geq^- H$ and $H \geq^- G$). Inequality and equivalence are defined similarly in normal convention, using superscript $+$ instead of $-$.

General equivalence and comparison are very limited in misère play (see [5, 11]). In particular, while all games with outcome $\mathcal{P}$ are equivalent to 0 in normal play, 0 is the only member of its equivalence class in misère play. This is why Plambeck and Siegel defined in [8, 9] an equivalence relationship under restricted universes, leading to a breakthrough in the study of misère play games.

Definition 1.1 ([8, 9]) Let $\mathcal{U}$ be a universe of games, G and H two games (not necessarily in $\mathcal{U}$). We say G is greater than or equal to H modulo $\mathcal{U}$ in misère play and write $G \geq^- H \pmod{\mathcal{U}}$ if $o^-(G + X) \geq o^-(H + X)$ for every $X \in \mathcal{U}$. We say G is equivalent to H modulo $\mathcal{U}$ in misère play and write $G \equiv^- H \pmod{\mathcal{U}}$ if $G \geq^- H \pmod{\mathcal{U}}$ and $H \geq^- G \pmod{\mathcal{U}}$.

For instance, Plambeck and Siegel [8, 9] considered the universe of all positions of given games, especially octal games. Other universes have been considered, including the universes of impartial games $\mathcal{I}$ [3, 4], dicot games $\mathcal{D}$ [2, 6], dead-ending games $\mathcal{E}$ [7], and all games $\mathcal{G}$ [11]. These classes are ordered (ranked) by inclusion as follows:

$$\mathcal{I} \subset \mathcal{D} \subset \mathcal{E} \subset \mathcal{G}\,.$$

To simplify notation, we sometimes use $\geq^-_{\mathcal{U}}$ and $\equiv^-_{\mathcal{U}}$ to denote superiority and equivalence modulo the universe $\mathcal{U}$. Note that the symbol $=$ is reserved for equality between game trees. Observe also that if $\mathcal{U}$ and $\mathcal{U}'$ are two universes with $\mathcal{U} \subseteq \mathcal{U}'$, then for any two games G and H, $G \leq^-_{\mathcal{U}} H$ whenever $G \leq^-_{\mathcal{U}'} H$.

The canonical form of a game is the simplest game of its equivalence class (i.e. the earliest born game with minimum number of followers, and with options in canonical form). It is therefore natural to consider canonical forms modulo a given universe. In normal play, impartial games have the same canonical form when considered modulo the universe of impartial games or modulo the universe of all games. In misère play, the corresponding canonical forms are different.

In the following, we focus on the universe of dicots. A game is said to be dicot either if it is $\{\cdot|\cdot\}$ or if it has both Left and Right options and all these options are dicot. Note that the universe of dicots, denoted $\mathcal{D}$ is closed under conjugate, sum of games and taking options. Among dicots, there are some of the most famous combinatorial games, such as Hex or Clobber.

The paper is organized as follows. In Section 2, we give some useful definitions and state general results that are true for any universe, following the ideas of Siegel [11] who got similar results for the universe $\mathcal{G}$ of all games. In Section 3, we focus on dicots and describe their misère canonical form in the dicot universe $\mathcal{D}$. This result allows us to compute the exact number of dicot misère games born by day 3 in Section 4. Finally, in Section 5, we show that for any three outcomes o_1, o_2 and o_3, there exist dicots G_1 and G_2 with outcomes o_1 and o_2 whose sum has outcome o_3, as opposed to what happens under normal play.

2 Definitions and universal properties

In [11], Siegel introduced the notion of the adjoint of a game. Recall that a Left end is a game with no Left options, and a Right end is a game with no Right options.

Definition 2.1 (Siegel [11]) The adjoint of G, denoted G^o, is given by

$$G^o = \begin{cases} * & \text{if } G = 0\,, \\ \{(G^{\boldsymbol{R}})^o|0\} & \text{if } G \neq 0 \text{ and } G \text{ is a Left end,} \\ \{0|(G^{\boldsymbol{L}})^o\} & \text{if } G \neq 0 \text{ and } G \text{ is a Right end,} \\ \{(G^{\boldsymbol{R}})^o|(G^{\boldsymbol{L}})^o\} & \text{otherwise.} \end{cases}$$

where $(G^{\boldsymbol{R}})^o$ denotes the set of adjoints of elements of $G^{\boldsymbol{R}}$.

Observe that we can recursively verify that the adjoint of any game is a dicot. In normal play, the conjugate of a game is considered as its opposite and is thus denoted $-G$, since $G + \overline{G} \equiv^+ 0$. The interest of the adjoint of a game is that it plays a similar role in misère play as the opposite of a game in normal play, as the following proposition suggests:

Proposition 2.2 (Siegel [11]) *In misère play, $G + G^o$ is a $\mathcal{P}$-position.*

Note that $G+G^o$ is not necessarily a $\mathcal{P}$-position in normal play (in particular, $o^+(0+*) = \mathcal{N}$). In the remaining of this section and Section 3, we propose ways to reduce the number of Left and Right options of a game to reach a canonical form. With the following proposition, we observe that passing by conjugates in the universe of conjugates, any result on the Left options can be extended to the Right options.

Proposition 2.3 *Let G and H be any two games, and $\mathcal{U}$ a universe. Denote by $\overline{\mathcal{U}}$ the universe of the conjugates of the elements of $\mathcal{U}$. If $G \geq^-_{\mathcal{U}} H$, then $\overline{G} \leq^-_{\overline{\mathcal{U}}} \overline{H}$. As a consequence, $G \equiv^-_{\mathcal{U}} H \iff \overline{G} \equiv^-_{\overline{\mathcal{U}}} \overline{H}$.*

Proof. For a game $X \in \overline{\mathcal{U}}$, suppose Left can win $\overline{G} + X$ playing first (respectively second). We show that she also has a winning strategy on $\overline{H} + X$. Looking at conjugates, Right can win $\overline{\overline{G} + X} = G + \overline{X}$. As $\overline{X} \in \mathcal{U}$ and $G \geq^-_{\mathcal{U}} H$, Right can win $H + \overline{X}$. Thus Left can win $\overline{H + \overline{X}} = \overline{H} + X$ and $\overline{G} \leq^-_{\overline{\mathcal{U}}} \overline{H}$. □

Relying on this proposition, we give the results only on Left options in the following, keeping in mind that they naturally extend to the Right options provided the result holds on the universe of conjugate. This is the case in the following since we either prove our results on all universes, or on the universe $\mathcal{D}$ of dicots which is its own conjugate.

Considering a game, it is quite natural to observe that adding an option to a player who already has got some can only improve his position (hand-tying principle). It was already proved in [5] in the universe $\mathcal{G}$ of all games. As a consequence, this is true for any subuniverse $\mathcal{U}$ of $\mathcal{G}$.

Proposition 2.4 *Let G be a game with at least one Left option, S a set of games and $\mathcal{U}$ a universe of games (S need not be a subset of $\mathcal{U}$). Let H be the game defined by $H^{\boldsymbol{L}} = G^{\boldsymbol{L}} \cup S$ and $H^{\boldsymbol{R}} = G^{\boldsymbol{R}}$. Then $H \geq_{\mathcal{U}}^{-} G$.*

The following proposition was stated in [11] for the universe $\mathcal{G}$ of all games. Mimicking the proof, we extend it to any universe.

Proposition 2.5 *Let $\mathcal{U}$ be a universe of games, G and H two games (not necessarily in $\mathcal{U}$). We have $G \geq_{\mathcal{U}}^{-} H$ if and only if the following two conditions hold:*

(i) For all $X \in \mathcal{U}$ with $o^{-}(H+X) \geq \mathcal{P}$, we have $o^{-}(G+X) \geq \mathcal{P}$; and

(ii) For all $X \in \mathcal{U}$ with $o^{-}(H+X) \geq \mathcal{N}$, we have $o^{-}(G+X) \geq \mathcal{N}$.

Proof. The sufficiency follows from the definition of $\geq_{\mathcal{U}}^{-}$. For the converse, we must show that $o^{-}(G+X) \geq o^{-}(H+X)$ for all $X \in \mathcal{U}$. Since we always have $o^{-}(G+X) \geq \mathcal{R}$, if $o^{-}(H+X) = \mathcal{R}$, then there is nothing to prove. If $o^{-}(H+X) = \mathcal{P}$ or $\mathcal{N}$, the result directly follows from (i) or (ii), respectively. Finally, if $o^{-}(H+X) = \mathcal{L}$, then by (i) and (ii) we have both $o^{-}(G+X) \geq \mathcal{P}$ and $o^{-}(G+X) \geq \mathcal{N}$, hence $o^{-}(G+X) = \mathcal{L}$. $\square$

To obtain the canonical form of a game, we generally remove or bypass options that are not relevant. These options are of two types: dominated options can be removed because another option is always a better move for the player, and reversible options are bypassed since the answer of the opponent is 'predictible'. Under normal play, simply removing dominated options and bypassing reversible options is sufficient to obtain a canonical form. Under misère play, things are more complicated. Mesdal and Ottaway [5] proposed definitions of dominated and reversible options under misère play in the universe $\mathcal{G}$ of all games. Then Siegel [11] proved that deleting dominated options and bypassing reversible options actually define a canonical form in the universe $\mathcal{G}$. However, modulo smaller universes, games with different canonical forms may be equivalent. In the following, we adapt the definition of dominated and reversible options to restricted universes of games. We show in the next section that a canonical form modulo the universe of dicots can be obtained by removing dominated options and applying a slightly more complicated treatment to reversible options.

Definition 2.6 ($\mathcal{U}$-dominated and reversible options)
Let G be a game, $\mathcal{U}$ a universe of games.

(a) A Left option G^L is $\mathcal{U}$-dominated by some other Left option $G^{L'}$ if $G^{L'} \geq_{\mathcal{U}}^{-} G^L$.

(b) A Right option G^R is $\mathcal{U}$-dominated by some other Right option $G^{R'}$ if $G^{R'} \leq_{\mathcal{U}}^{-} G^R$.

(c) A Left option G^L is $\mathcal{U}$-reversible through some Right option G^{LR} if $G^{LR} \leq_{\mathcal{U}}^{-} G$.

(d) A Right option G^R is $\mathcal{U}$-reversible through some Left option G^{RL} if $G^{RL} \geq_{\mathcal{U}}^{-} G$.

To obtain the known canonical forms for the universe $\mathcal{G}$ of all games [11] but also for the universe $\mathcal{I}$ of impartial games [4], one may just remove dominated and bypass reversible options as defined. The natural question that arises is whether a similar process gives canonical forms in other universes. Indeed, it is remarkable that in all universes closed by taking options, dominated options can be ignored, as shown by the following lemma.

Lemma 2.7 *Let G be a game and let $\mathcal{U}$ be a universe of games closed by taking options of games. Suppose G^{L_1} is $\mathcal{U}$-dominated by G^{L_2}, and let G' be the game obtained by removing G^{L_1} from $G^{\boldsymbol{L}}$. Then $G \equiv^-_{\mathcal{U}} G'$.*

Proof. By Proposition 2.4, we have $G' \leq^-_{\mathcal{U}} G$. We thus only have to show that $G' \geq^-_{\mathcal{U}} G$. For a game $X \in \mathcal{U}$, suppose Left can win $G + X$ playing first (respectively second), we show that she also has a winning strategy in $G' + X$. Actually, she can simply follow the same strategy on $G' + X$, unless she is eventually supposed to make a move from some $G + Y$ to $G^{L_1} + Y$. In that case, she is supposed to move to the game $G^{L_1} + Y$ and then win, so $o^-(G^{L_1} + Y) \geq \mathcal{P}$. But $G^{L_2} \geq^-_{\mathcal{U}} G^{L_1}$ and $Y \in \mathcal{U}$, thus by Proposition 2.5, $o^-(G^{L_2} + Y) \geq \mathcal{P}$. Therefore, Left can win by moving from $G' + Y$ to $G^{L_2} + Y$, concluding the proof. □

Unfortunately, the case involving reversible options is more complex. Nevertheless, we show in the next section how we can deal with them in the specific universe of dicot games. Beforehand, we adapt the definition of downlinked or uplinked games from [11] to restricted universes. Recall that this definition is already a generalization of the definition of linked (impartial) games, for further reference we refer the reader to [12].

Definition 2.8 (Siegel [11]) Let G and H be any two games. If there exists some $T \in \mathcal{U}$ such that $o^-(G + T) \leq \mathcal{P} \leq o^-(H + T)$, we say that G is *$\mathcal{U}$-downlinked* to H (by T). In that case, we also say that H is *$\mathcal{U}$-uplinked* to G by T.

Note that if two games are $\mathcal{U}$-downlinked and $\mathcal{U} \subseteq \mathcal{U}'$, then these two games are also $\mathcal{U}'$-downlinked. Therefore, the smaller the universe $\mathcal{U}$ is, the less 'likely' it is that two games are $\mathcal{U}$-downlinked.

Lemma 2.9 *Let G and H be any two games and $\mathcal{U}$ be a universe of games. If $G \geq^-_{\mathcal{U}} H$, then G is $\mathcal{U}$-downlinked to no H^L and no G^R is $\mathcal{U}$-downlinked to H.*

Proof. Let $T \in \mathcal{U}$ be any game such that $o^-(G + T) \leq \mathcal{P}$. Since $G \geq^-_{\mathcal{U}} H$ and $T \in \mathcal{U}$, $o^-(H + T) \leq \mathcal{P}$ as well. Hence for any $H^L \in H^{\boldsymbol{L}}$, $o^-(H^L + T) \leq \mathcal{N}$, and G is not $\mathcal{U}$-downlinked to H^L by T. Similarly, let $T' \in \mathcal{U}$ such that $o^-(H + T') \geq \mathcal{P}$. Then $o^-(G + T') \geq \mathcal{P}$ and therefore, for any $G^R \in G^{\boldsymbol{R}}$, $o^-(G^R + T') \geq \mathcal{N}$ and G^R is not $\mathcal{U}$-downlinked to H by T'. □

3 Canonical form of dicots

In this section, we consider games within the universe $\mathcal{D}$ of dicots, and show that we can define precisely a canonical form in that context. In order to do so, we first describe how to bypass the $\mathcal{D}$-reversible options in Lemmas 3.1 and 3.2.

Lemma 3.1 *Let G be a dicot game. Suppose G^{L_1} is $\mathcal{D}$-reversible through $G^{L_1R_1}$ and either $G^{L_1R_1} \neq 0$ or there exists another Left option G^{L_2} of G such that $o^-(G^{L_2}) \geq \mathcal{P}$. Let G' be the game obtained by bypassing G^{L_1}:*

$$G' = \{(G^{L_1R_1})^{\boldsymbol{L}}, G^{\boldsymbol{L}} \setminus \{G^{L_1}\} | G^{\boldsymbol{R}}\}.$$

Then G' is a dicot and $G \equiv^-_{\mathcal{D}} G'$.

Proof. First observe that since G is a dicot, all options of G' are dicots, and under our assumptions, G' has both Left and Right options. Thus G' is a dicot. We now prove that for any dicot X, the games $G + X$ and $G' + X$ have the same misère outcome.

Suppose Left can win playing first on $G + X$. Among all the winning strategies for Left, consider one that always recommends a move on X, unless the only winning move is on G. In the game $G' + X$, let Left follow the same strategy except if the strategy recommends precisely the move from G to G^{L_1}. If this move from G to G^{L_1} is the recommended move, then the game started on $G' + X$ has reached a position of the form $G' + Y$, where Y is a follower of X, with $o^-(G^{L_1} + Y) \geq \mathcal{P}$. Thus $o^-(G^{L_1R_1} + Y) \geq \mathcal{N}$.

Suppose Left has a winning move in Y from $G^{L_1R_1} + Y$, i.e. there exists some Y^L such that $o^-(G^{L_1R_1} + Y^L) \geq \mathcal{P}$. But then by reversibility, $o^-(G + Y^L) \geq \mathcal{P}$, contradicting our choice of Left's strategy. So either Left has a winning move of type $G^{L_1R_1L} + Y$, which she can play directly from $G' + Y$, or she wins because she has no possible moves, meaning that $G^{L_1R_1} = 0$ and $Y = 0$. In that case, she can also win in $G' + Y = G'$ by choosing the winning move to G^{L_2}. Now if Left can win playing second on $G + X$, following the same arguments we get a strategy for her to win playing second on $G' + X$.

Now suppose Right can win playing first on $G + X$. Consider any winning strategy for Right, and let him follow exactly the same strategy on $G' + X$ unless Left moves from some position $G' + Y$ to $G^{L_1R_1L} + Y$. First note that by our assumption, G' is not a Left end, thus if Right follows this strategy, Left can never run out of move prematurely.

Suppose now that Left made a move from some position $G' + Y$ to $G^{L_1R_1L} + Y$. Until that move, Right was following his winning strategy, so $o^-(G + Y) \leq \mathcal{P}$. Since $G^{L_1R_1} \leq^-_{\mathcal{D}} G$ and Y is a dicot, we have $o^-(G^{L_1R_1} + Y) \leq \mathcal{P}$. Thus $G^{L_1R_1L} + Y \leq \mathcal{N}$ and Right can adapt his strategy. Here also, the same arguments give a strategy for Right to win playing second in $G' + Y$ if he can win playing second in $G + Y$. □

With the previous lemma, we do not bypass reversible options through 0 when all other Left options have misère outcome at most $\mathcal{N}$. Such reversible options cannot be treated similarly, as shows the example of the game $\{0, *|*\}$. Note that as shown in [2], $\{*|*\} = * + * \equiv^-_{\mathcal{D}} 0$ and thus, by Proposition 2.4, $\{0, *|*\} \geq^-_{\mathcal{D}} 0$. Therefore, the Left option $*$ is $\mathcal{D}$-reversible through 0. However, $\{0, *|*\} \not\equiv^-_{\mathcal{D}} \{0|*\}$ since the first is an $\mathcal{N}$-position and the second an $\mathcal{R}$-position. Yet, we prove with the following lemma that all reversible options ignored by Lemma 3.1 can be replaced by $*$.

Lemma 3.2 *Let G be a dicot game. Suppose G^{L_1} is $\mathcal{D}$-reversible through $G^{L_1R_1} = 0$. Let G' be the game obtained by replacing G^{L_1} by star:*

$$G' = \{*, G^{\boldsymbol{L}} \setminus \{G^{L_1}\} | G^{\boldsymbol{R}}\}.$$

Then G' is a dicot and $G \equiv^-_{\mathcal{D}} G'$.

Proof. First observe that since G and $*$ are dicots, all options of G' are dicots, and G' has both Left and Right options. Thus G' is a dicot. We now prove that for any dicot X, the games $G + X$ and $G' + X$ have the same misère outcome.

Suppose Left can win playing first (respectively second) on $G + X$. Among all the winning strategies for Left, consider one that always recommends a move on X, unless the only winning move is on G. In the game $G' + X$, let Left follow the same strategy except if the strategy recommends precisely the move from G to G^{L_1}. In that case, the position is of the form $G' + Y$, with $o^-(G^{L_1} + Y) \geq \mathcal{P}$. Thus $o^-(G^{L_1R_1} + Y) \geq \mathcal{N}$.

Suppose Left has a winning move in $G^{L_1R_1} + Y = 0 + Y = Y$, i.e. there exists some Y^L such that $o^-(Y^L) \geq \mathcal{P}$. But then by reversibility, $o^-(G + Y^L) \geq \mathcal{P}$, contradicting our choice of Left's strategy. So Left has no winning move in Y, and she wins because she has no possible moves, i.e. $Y = 0$. In that case, she can also win in $G' + Y = G'$ by choosing the winning move to $*$.

Now suppose Right can win playing first (respectively second) on $G+X$. Consider any winning strategy for Right, and let him follow exactly the same strategy on $G' + X$ unless Left moves from some position $G' + Y$ to $* + Y$. First note that by our assumption, G' is not a Left end, thus if Right follows this strategy, Left can never run out of move prematurely.

Suppose now that Left made a move from some position $G' + Y$ to $* + Y$. Until that move, Right was following his winning strategy, so $o^-(G + Y) \leq \mathcal{P}$. Since $0 = G^{L_1R_1} \leq^-_{\mathcal{D}} G$ and Y is a dicot, we have $o^-(Y) = o^-(0 + Y) \leq o^-(G + Y) \leq \mathcal{P}$. So Right can move from $* + Y$ to Y and win. □

Note that some reversible options may be dealt with using both Lemmas 3.1 and 3.2. Yet, it is still possible to apply Lemma 3.1 and remove such an option after having applied Lemma 3.2.

At this point, we want to define a reduced form for each game obtained by applying the preceding lemmas as long as we can. In addition, it was proved by Allen in [2] that the game $\{*|*\}$ is equivalent to 0 modulo the universe of dicots, and we thus reduce this game to 0. Therefore, we define the reduced form of a dicot as follows:

Definition 3.3 (Reduced form) Let G be a dicot. We say G is in *reduced form* if:

(i) it is not $\{*|*\}$,

(ii) it contains no dominated option,

(iii) if Left has a reversible option, it is $*$ and no other Left option has outcome $\mathcal{P}$ or $\mathcal{L}$,

(iv) if Right has a reversible option, it is $*$ and no other Right option has outcome $\mathcal{P}$ or $\mathcal{R}$,

(v) all its options are in reduced form.

Observe first the following:

Theorem 3.4 *Every game G is equivalent modulo the universe of dicots to a game in reduced form H whose birthday is no larger than the birthday of G.*

Proof. To prove the theorem, we show that every game G not in reduced form is equivalent to a game with no larger birthday with fewer followers. Since 0 is in reduced form, the process converges by induction to a game in reduced form, whose birthday is no larger than the birthday of G.

Suppose G is not in reduced form. If G is $\{*|*\}$ (not satisfying property (i)), then it is equivalent to 0, in reduced form, born on day 0. If G does not satisfy property (ii), i.e. has a dominated option, then we can apply Lemma 2.7 and obtain a game of no larger birthday with fewer followers. If G does not satisfy property (iii), then it has a reversible Left option G^{L_1}. If it is not $*$, either it is reversible through $G^{L_1R_1} = 0$ and we can apply Lemma 3.2 and replace it by $*$, or it is reversible through $G^{L_1R_1} \neq 0$ and we can apply Lemma 3.1 to just bypass it. If it is $*$ but there is another Left option that has outcome $\mathcal{P}$ or $\mathcal{L}$, then by Lemma 3.1, we can bypass it (here, this is simply to remove it). In both cases, we obtain a game of no larger birthday with fewer followers. The case when it does not satisfy property (iv) is identical. Finally, if it does not satisfy (v), we can apply the same on the options. □

We now prove that the reduced form of a game can be seen as a canonical form. Before stating the main theorem, we need the two following lemmas.

Lemma 3.5 *Let G and H be any games. If $G \not\geq_{\mathcal{D}}^{-} H$, then:*

(a) There exists some $Y \in \mathcal{D}$ such that $o^-(G+Y) \leq \mathcal{P}$ and $o^-(H+Y) \geq \mathcal{N}$; and

(b) There exists some $Z \in \mathcal{D}$ such that $o^-(G+Z) \leq \mathcal{N}$ and $o^-(H+Z) \geq \mathcal{P}$.

Proof. Negating the condition of Proposition 2.5, we get that (a) or (b) must hold. To prove the lemma, we show that (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (a).

Consider some $Y \in \mathcal{D}$ such that $o^-(G+Y) \leq \mathcal{P}$ and $o^-(H+Y) \geq \mathcal{N}$, and set

$$Z = \{(H^{\boldsymbol{R}})^o, 0|Y\}\,.$$

First note that since Z has both a Left and a Right option, and all its options are dicots, Z is also a dicot. We now show that Z satisfies $o^-(G+Z) \leq \mathcal{N}$ and $o^-(H+Z) \geq \mathcal{P}$, as required in (b). From the game $G+Z$, Right has a winning move to $G+Y$, so $o^-(G+Z) \leq \mathcal{N}$. We now prove that Right has no winning move in the game $H+Z$. Observe first that $H+Z$ is not a Right end since Z is not. If Right moves to some H^R+Z, Left has a winning response to $H^R+(H^R)^o$. If instead

Right moves to $H+Y$ then, since $o^-(H+Y) \geq \mathcal{N}$, Left can win. Therefore $o^-(H+Z) \geq \mathcal{P}$, and (a) $\Rightarrow$ (b).

To prove (b) $\Rightarrow$ (a), for a given Z we set $Y = \{Z|0, (G^{\boldsymbol{L}})^o\}$ and prove similarly that Left wins if she plays first on $H+Y$ and loses if she plays first on $G+Y$. $\square$

Lemma 3.6 *Let G and H be any games. The game G is $\mathcal{D}$-downlinked to H if and only if no $G^L \geq_{\mathcal{D}}^- H$ and no $H^R \leq_{\mathcal{D}}^- G$.*

Proof. Consider two games G and H such that G is $\mathcal{D}$-downlinked to H by some third game T, i.e. $o^-(G+T) \leq \mathcal{P} \leq o^-(H+T)$. Then Left has no winning move from $G+T$, thus $o^-(G^L+T) \leq \mathcal{N}$ and similarly $o^-(H^R+T) \geq \mathcal{N}$. Therefore, T witnesses both $G^L \not\geq_{\mathcal{D}}^- H$ and $G \not\geq_{\mathcal{D}}^- H^R$.

Conversely, suppose that no $G^L \geq_{\mathcal{D}}^- H$ and no $H^R \leq_{\mathcal{D}}^- G$. Set $G^{\boldsymbol{L}} = \{G_1^L, \ldots, G_k^L\}$ and $H^{\boldsymbol{R}} = \{H_1^R, \ldots, H_\ell^R\}$. By Lemma 3.5, we can associate to each $G_i^L \in G^{\boldsymbol{L}}$ a game $X_i \in \mathcal{D}$ such that $o^-(G_i^L + X_i) \leq \mathcal{P}$ and $o^-(H+X_i) \geq \mathcal{N}$. Likewise, to each $H_j^R \in H^{\boldsymbol{R}}$, we associate a game $Y_j \in \mathcal{D}$ such that $o^-(G+Y_j) \leq \mathcal{N}$ and $o^-(H_j^R + Y_j) \geq \mathcal{P}$. Let T be the game defined by

$$T^{\boldsymbol{L}} = \begin{cases} \{0\} & \text{if both } G \text{ and } H \text{ are Right ends,} \\ (G^{\boldsymbol{R}})^o \cup \{Y_j \mid 1 \leq j \leq \ell\} & \text{otherwise.} \end{cases}$$

$$T^{\boldsymbol{R}} = \begin{cases} \{0\} & \text{if both } G \text{ and } H \text{ are Left ends,} \\ (H^{\boldsymbol{L}})^o \cup \{X_i \mid 1 \leq i \leq k\} & \text{otherwise.} \end{cases}$$

If $H^{\boldsymbol{R}}$ (respectively $G^{\boldsymbol{R}}$) is non-empty, then so is $\{Y_j \mid 1 \leq j \leq \ell\}$ (respectively $(G^{\boldsymbol{R}})^o$), and T has a left option. If both $G^{\boldsymbol{R}}$ and $H^{\boldsymbol{R}}$ are empty, then $T^{\boldsymbol{L}} = \{0\}$, so T always has a Left option. Similarly, T always have also a Right option. Moreover, all these options are dicots, so T is a dicot. We claim that G is $\mathcal{D}$-downlinked to H by T.

To show that $o^-(G+T) \leq \mathcal{P}$, we just prove that Left loses if she plays first in $G+T$. Since T has a Left option, $G+T$ is not a Left end. If Left moves to some $G_i^L + T$, then by our choice of X_i, Right has a winning response to $G_i^L + X_i$. If Left moves to some $G + (G^R)^o$, then Right can respond to $G^R + (G^R)^o$ and win (by Proposition 2.2). If Left moves to $G + Y_j$, then by our choice of Y_j, $o^-(G+Y_j) \leq \mathcal{N}$ and Right can win. The only remaining possibility is, when G and H are Right ends, that Left moves to $G+0$. But then Right cannot move and wins.

Now, we show that $o^-(H+T) \geq \mathcal{P}$ by proving that Right loses playing first in $H+T$. If Right moves to some $H_j^R + T$, then Left has a winning response to $H_j^R + Y_j$. If Right moves to $H + (H^L)^o$, then Left wins by playing to $H^L + (H^L)^o$, and if Right moves to $H + X_i$, then by our choice of X_i, $o^-(H+X_i) \geq \mathcal{N}$ and Left can win. Finally, the only remaining possibility, when G and H are Left ends, is that Right moves to 0. But then Left cannot answer and wins. $\square$

As a consequence of this lemma together with Lemma 2.9, we have the following corollary:

Corollary 3.7 *Let G and H be any games. If $G \geq_{\mathcal{D}}^- H$, then for every Left option H^L, either there is a Left option G^L such that $G^L \geq_{\mathcal{D}}^- H^L$, or there is a Right option H^{LR} of H^L such that $G \geq_{\mathcal{D}}^- H^{LR}$.*

Proof. Let G and H be any games such that $G \geq_{\mathcal{D}}^- H$. By Lemma 2.9, for all Left options H^L, G is not $\mathcal{D}$-downlinked to H^L. Then, applying contrapositive of Lemma 3.6, we get the result. $\square$

We now prove the main theorem of the section.

Theorem 3.8 *Consider two dicots G and H. If $G \equiv_{\mathcal{D}}^- H$ and both are in reduced form, then $G = H$.*

Proof. If $G = H = \{\cdot|\cdot\}$, then the result is clear. We proceed by induction on the birthday of the games. Assume without loss of generality that G has an option. Since G is a dicot, it has both a Left and a Right option.

Consider a Left option G^L. Suppose first that G^L is not $\mathcal{D}$-reversible. Since $H \equiv^-_{\mathcal{D}} G$, $H \geq^-_{\mathcal{D}} G$ and Corollary 3.7 implies that there exists some $H^L \geq^-_{\mathcal{D}} G^L$, or there exists some Right option G^{LR} of G^L with $G^{LR} \leq^-_{\mathcal{D}} H$. The latter would imply that $G \geq^-_{\mathcal{D}} G^{LR}$ and thus that G^L is $\mathcal{D}$-reversible, contradicting our assumption. So we must have some option H^L such that $H^L \geq^-_{\mathcal{D}} G^L$. Now, since $G \geq^-_{\mathcal{D}} H$, by Corollary 3.7, there exists some $G^{L'} \geq^-_{\mathcal{D}} H^L$, or there exists some Right option H^{LR} of H^L with $H^{LR} \leq^-_{\mathcal{D}} G$.

Assume first we are in the latter case, i.e. $H^{LR} \leq^-_{\mathcal{D}} G \leq^-_{\mathcal{D}} H$. Then H^L is $\mathcal{D}$-reversible through H^{LR}, though since H is in reduced form, we must then have $H^L = *$ and $H \geq^-_{\mathcal{D}} 0$. Now, since $G^L \leq H^L = *$, we have $G^L + H^L \leq * + * = 0$. Again, by Corollary 3.7, considering the possibility for Left to move from $G^L + *$ to G^L, Right must have a good answer G^{LR} such that $G^{LR} \leq^-_{\mathcal{D}} 0$. But then we have $G^{LR} \leq^-_{\mathcal{D}} 0 \leq^-_{\mathcal{D}} G$, which contradicts our assumption that G^L is not $\mathcal{D}$-reversible. Hence we are in the former case, where there exists some $G^{L'}$ such that $G^{L'} \geq^-_{\mathcal{D}} H^L$.

So we have $G^{L'} \geq^-_{\mathcal{D}} H^L \geq^-_{\mathcal{D}} G^L$. If $G^{L'}$ and G^L are two different options, then G^L is dominated by $G^{L'}$, contradicting our assumption that G is in reduced form. Thus, $G^{L'}$ and G^L are the same option, and $G^L \equiv^-_{\mathcal{D}} H^L$. But G^L and H^L are in reduced form, so by induction hypothesis, $G^L = H^L$. The same argument applied to the Right options of G and to both Left and Right options of H shows that there is a pairwise correspondence of all non-$\mathcal{D}$-reversible options of G and H.

Assume now that G^L is a $\mathcal{D}$-reversible option. Then $G^L = *$ and for all other Left options $G^{L'}$, we have $o^-(G^{L'}) \leq \mathcal{N}$, and by reversibility, there exists some Right option G^{LR} of G^L such that $G^{LR} \leq^-_{\mathcal{D}} G$. Since the only Right option of $*$ is 0, $G \geq^-_{\mathcal{D}} 0$. Thus $H \geq^-_{\mathcal{D}} 0$, so either $H = 0$ or Left has a winning move in H, namely a Left option H^L such that $o^-(H^L) \geq \mathcal{P}$. First assume $H = 0$. Then by the pairwise correspondence proved in the first part of this proof, G has no non-$\mathcal{D}$-reversible options. Yet it is a dicot and must have both a Left and a Right option, and since it is in reduced form, both are $*$. Then $G = \{*|*\}$, a contradiction. Now assume H has a Left option H^L such that $o^-(H^L) \geq \mathcal{P}$. If H^L is not $\mathcal{D}$-reversible, then it is in correspondence with a non-$\mathcal{D}$-reversible option $G^{L'}$, but then we should have $o^-(H^L) = o^-(G^{L'}) \leq \mathcal{N}$, a contradiction. So H^L is $\mathcal{D}$-reversible, and $H^L = G^L = *$. The same argument applied to possible Right $\mathcal{D}$-reversible options of G and to $\mathcal{D}$-reversible options of H concludes the proof that $G = H$. $\square$

This proves that the reduced form of a game is unique, and that any two $\mathcal{D}$-equivalent games have the same reduced form. Therefore, the reduced form as described in Definition 3.3 can be considered as the canonical form of the game modulo the universe of dicots.

Siegel showed in [11] that for any games G and H, if $G \geq^-_{\mathcal{G}} H$, then $G \geq^+ H$ also in normal play. This result can be strengthened as follows :

Theorem 3.9 *Let G and H be any games. If $G \geq^-_{\mathcal{D}} H$, then $G \geq^+ H$.*

Proof. Consider any two games G and H such that $G \geq^-_{\mathcal{D}} H$. We show that $G - H \geq^+ 0$, i.e. that Left can win $G - H$ in normal play when Right moves first [3], by induction on the birthday of G and H. Suppose Right plays to some $G - H^L$. Since $G \geq^-_{\mathcal{D}} H$, by Corollary 3.7, either there exists some Left option G^L of G with $G^L \geq^-_{\mathcal{D}} H^L$, or there exists some Right option H^{LR} of H^L with $H^{LR} \leq^-_{\mathcal{D}} G$. In the first case, we get by induction that $G^L \geq^+ H^L$ and Left can win by moving to $G^L - H^L$. In the second case, by induction we have $H^{LR} \leq^+ G$ and Left can win by moving to $G - H^{LR}$. The argument when Right plays to some $G^R - H$ is similar, using Corollary 3.7 applied to Right options. $\square$

Theorem 3.9 implies in particular that if two games are equivalent in misère play modulo $\mathcal{D}$, then they are also equivalent in normal play. It allows us to use any normal play tools to prove incomparability or distinguishability (i.e. non equivalence) to deduce it modulo the universe of dicots. Moreover, a corollary of Theorem 3.9 is that its statement is also true for any universe containing $\mathcal{D}$, in particular for the universe $\mathcal{G}$ of all games (implying the result of [11]) and for the universe $\mathcal{E}$ of dead-ending games (studied in [7]). However, concerning the universe of impartial

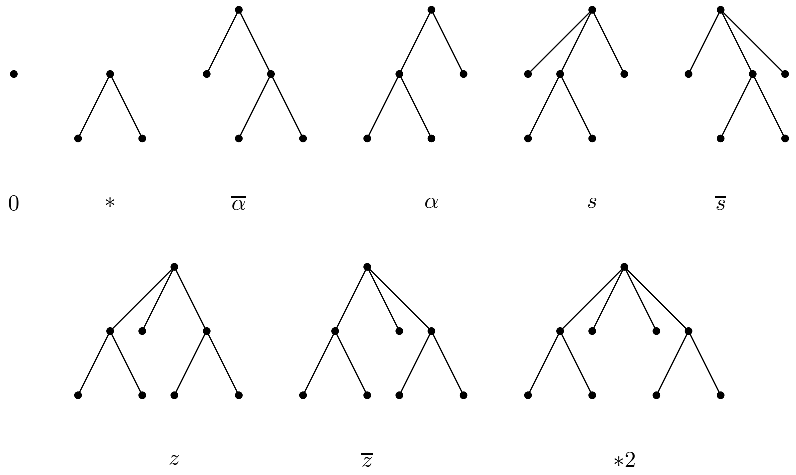


Figure 3: Game trees of the 9 dicots born by day 2

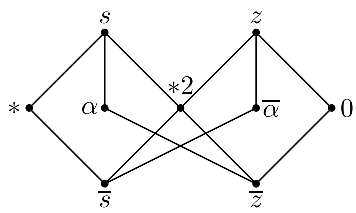


Figure 4: Partial ordering of dicots born by day 2

games, it was recently proved in [10] that there exist dicots equivalent in misère play modulo the universe $\mathcal{I}$ of impartial games but non equivalent in normal play.

4 Dicot misère games born by day 3

We now use Theorem 3.8 to count the dicot misère games born by day 3. Recall that the number of impartial misère games distinguishable modulo the universe $\mathcal{I}$ of impartial games that are born by day 0, 1, 2, 3 and 4 are respectively 1, 2, 3, 5 and 22 (see [4]). Siegel [11] proved that the number of misère games distinguishable modulo the universe $\mathcal{G}$ of all games that are born by day 0, 1 and 2 is respectively 1, 4 and 256, while the number of distinguishable misère games born by day 3 is at most 2^{183}. Notice that since impartial games form a subset of dicot games, the number of dicots born by day 3 lies between 5 and 2^{183}. Before showing that this number is exactly 1268, we state properties of the dicots born by day 2.

Proposition 4.1 *There are* 9 *dicots born by day 2 distinguishable modulo the universe* $\mathcal{D}$ *of dicots, namely* 0, $*$, $\overline{\alpha} = \{0|*\}$, $\alpha = \{*|0\}$, $s = \{0,*|0\}$, $z = \{0,*|*\}$, $\overline{s} = \{0|0,*\}$, $\overline{z} = \{*|0,*\}$, *and* $*2 = \{0,*|0,*\}$ *(see Figure 3). They are partially ordered according to Figure 4. Moreover, the outcomes of their sums are given in Table 1.*

Proof. There are 10 dicots born by day 2, of which 0 and $\{*|*\}$ are equivalent. We now prove that these nine games are pairwise distinguishable modulo the universe $\mathcal{D}$ of dicots[1]. First note that these games are all in reduced form. Indeed, since all options are either 0 or $*$ which are not comparable modulo $\mathcal{D}$, there are no dominated options. Moreover, $*$ might be reversible through

[1] Milley gave an alternate proof of this fact in [6].

	0	$*$	$\overline{\alpha}$	α	s	z	$\overline{s}$	$\overline{z}$	$*2$
0	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{N}$	$\mathcal{R}$	$\mathcal{N}$	$\mathcal{N}$
$*$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{L}$	$\mathcal{N}$	$\mathcal{R}$	$\mathcal{N}$
$\overline{\alpha}$	$\mathcal{R}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$
α	$\mathcal{L}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{L}$
s	$\mathcal{L}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{L}$
z	$\mathcal{N}$	$\mathcal{L}$	$\mathcal{P}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{L}$
$\overline{s}$	$\mathcal{R}$	$\mathcal{N}$	$\mathcal{R}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$
$\overline{z}$	$\mathcal{N}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$
$*2$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{P}$

Table 1: Outcomes of sums of dicots born by day 2

X	Y	Z_1 such that $o^-(X+Z_1) \not\leq o^-(Y+Z_1)$	Z_2 such that $o^-(X+Z_2) \not\geq o^-(Y+Z_2)$
s	z	$\overline{s}$	$\overline{s}$
s	$\overline{\alpha}$	$\overline{\alpha}$	$\overline{\alpha}$
s	0	$\overline{z}$	$\overline{z}$
z	$*$	0	0
z	α	$\overline{\alpha}$	$\overline{\alpha}$
$*$	α	α	α
$*$	0	0	0
$*$	$*2$	0	0
α	$*2$	0	α
α	0	$*$	$*$
α	$\overline{\alpha}$	α	α
$*2$	0	$*$	$*$

Table 2: Incomparability of dicots born by day 2

0, but since there are no other options $\mathcal{P}$ or $\mathcal{L}$, it cannot be reduced. Thus, by Theorem 3.8, these games are pairwise non-equivalent.

The proof of the outcomes of sums of these games (given in Table 1) is tedious but not difficult, and omitted here.

We now show that these games are partially ordered according to Figure 4. Using the fact that $\{*|*\} \equiv^-_{\mathcal{D}} 0$ and Proposition 2.4, we easily infer the relations corresponding to edges in Figure 4. All other pairs are incomparable : for each pair (X, Y), there exist $Z_1, Z_2 \in \{0, *, \alpha, \overline{\alpha}, \overline{s}, \overline{z}\}$ such that $o^-(X+Z_1) \not\leq o^-(Y+Z_1)$ and $o^-(X+Z_2) \not\geq o^-(Y+Z_2)$ (see Table 2 for explicit such Z_1 and Z_2). $\square$

We now start counting the dicots born by day 3. Their Left and Right options are necessarily dicots born by day 2. We can consider only games in their canonical form, so with no dominated options.

Using Figure 4, we find the following 50 antichains:

$$\begin{cases} \text{all 32 subsets of } \{0, *, \overline{\alpha}, \alpha, *2\}, \\ \{s, z\} \text{ and } \{\overline{s}, \overline{z}\}, \\ \text{4 containing } s \text{ and any subset of } \{0, \overline{\alpha}\} \\ \text{4 containing } z \text{ and any subset of } \{*, \alpha\} \\ \text{4 containing } \overline{s} \text{ and any subset of } \{0, \alpha\} \\ \text{4 containing } \overline{z} \text{ and any subset of } \{*, \overline{\alpha}\} \end{cases}$$

Therefore, choosing $G^{\boldsymbol{L}}$ and $G^{\boldsymbol{R}}$ among these antichains, together with the fact that G is a dicot, we get $49^2 + 1 = 2402$ dicots born by day 3 with no dominated options.

To get only games in canonical form, we still have to remove games with reversible options. Note that an option from a dicot born by day 3 can only be reversible through 0 or $*$ since these are the only dicots born by day 1. To deal with the reversible options, we consider separately the games with different outcomes. If Left has a winning move from a game G, namely a move to $*$, α or s, then $o^-(G) \geq \mathcal{N}$. Otherwise, $o^-(G) \leq \mathcal{P}$. Likewise, if Right has a winning move from G, namely a move to $*$, $\overline{\alpha}$ or $\overline{s}$, then $o^-(G) \leq \mathcal{N}$. Otherwise, $o^-(G) \geq \mathcal{P}$. From this observation, we infer the outcome of any dicot born by day 3.

Consider first the games G with outcome $\mathcal{P}$, i.e. $G^{\boldsymbol{L}} \cap \{*, \alpha, s\} = \emptyset$ and $G^{\boldsymbol{R}} \cap \{*, \overline{\alpha}, \overline{s}\} = \emptyset$. Since $o^-(0) = \mathcal{N}$, G and 0 are $\mathcal{D}$-incomparable, so no option of G is $\mathcal{D}$-reversible through 0. The following lemma allows to characterize dicots born by day 3 whose outcome is $\mathcal{P}$ and that contain $\mathcal{D}$-reversible options through $*$.

Lemma 4.2 *Let G be a dicot born by day 3 with misère outcome $\mathcal{P}$. We have $G \geq^-_{\mathcal{D}} *$ if and only if $G^{\boldsymbol{L}} \cap \{0, z\} \neq \emptyset$.*

Proof. First suppose that $G^{\boldsymbol{L}} \cap \{0, z\} \neq \emptyset$. Let X be a dicot such that Left has a winning strategy on $* + X$ when playing first (respectively second). Left can follow the same strategy on $G + X$, unless the strategy recommends that she plays from some $* + Y$ to $0 + Y$, or Right eventually plays from some $G + Z$ to some $G^R + Z$. In the first case, we must have $o^-(0 + Y) \geq \mathcal{P}$. Left can move from $G + Y$ either to $0 + Y$ or to $z + Y$, which are both winning moves. Indeed, since $z \geq^-_{\mathcal{D}} 0$, we have $o^-(z + Y) \geq o^-(0 + Y) \geq \mathcal{P}$. Suppose now that Right just moved from $G + Z$ to some $G^R + Z$. By our choice of strategy, we have $o^-(* + Z) \geq \mathcal{P}$. If $G^R = 0$, then Left can continue her strategy since $0 + Z$ is also a Right option of $* + Z$. Otherwise, since $G^{\boldsymbol{R}} \cap \{*, \overline{\alpha}, \overline{s}\} = \emptyset$, G^R is one of $\alpha, s, z, \overline{z}, *2$ and $*$ is a Left option of G^R. Then Left can play from $G^R + Z$ to $* + Z$ and win. Thus, if Left wins $* + X$, she wins $G + X$ as well and thus $G \geq^-_{\mathcal{D}} *$.

Suppose now that $G^{\boldsymbol{L}} \cap \{0, z\} = \emptyset$, that is $G^{\boldsymbol{L}} \subseteq \{\overline{\alpha}, \overline{s}, \overline{z}, *2\}$. Let $X = \{\overline{s}|0\}$. In $* + X$, Left wins playing to $0 + X$ and Right wins playing to $* + 0$, hence $o^-(* + X) = \mathcal{N}$. On the other hand, in $G + X$, Right wins by playing to $G + 0$, but Left has no other options than $\overline{\alpha} + X, \overline{s} + X, \overline{z} + X, *2 + X, G + \overline{s}$. In the last four, Right wins by playing to $0 + X$ or $G + 0$, both with outcome $\mathcal{P}$. In $\overline{\alpha} + X$, Right wins by playing to $\overline{\alpha} + 0$ which has outcome $\mathcal{R}$. So $o^-(G + X) \leq \mathcal{P}$, and since $o^-(* + X) = \mathcal{N}$, we have $G \not\geq^-_{\mathcal{D}} *$. □

We deduce the following theorem:

Theorem 4.3 *A dicot G born by day 3 with outcome $\mathcal{P}$ is in canonical form if and only if*

$$\begin{cases} G^{\boldsymbol{L}} \in \big\{\{\overline{\alpha}\}, \{\overline{\alpha}, *2\}, \{*2\}, \{\overline{s}\}, \{\overline{s}, \overline{z}\}, \{\overline{z}\}, \{\overline{\alpha}, \overline{z}\}, \{0\}\big\}, \text{ and} \\ G^{\boldsymbol{R}} \in \big\{\{\alpha\}, \{\alpha, *2\}, \{*2\}, \{s\}, \{s, z\}, \{z\}, \{\alpha, z\}, \{0\}\big\}. \end{cases}$$

This yields $8 \cdot 8 = 64$ dicots non equivalent modulo $\mathcal{D}$.

Proof. Let G be a dicot born by day 3 with misère outcome $\mathcal{P}$, in canonical form. By our earlier statement, $G^{\boldsymbol{L}} \subseteq \{0, \overline{\alpha}, \overline{s}, z, \overline{z}, *2\}$. By Lemma 4.2, options $\overline{\alpha}, \overline{s}, z, \overline{z}, *2$ are reversible through $*$ whenever $G^{\boldsymbol{L}} \cap \{0, z\} \neq \emptyset$. So z is not a Left option of G, and if 0 is, there are no other Left options. Thus the only antichains left for $G^{\boldsymbol{L}}$ are $\big\{\{\overline{\alpha}\}, \{\overline{\alpha}, *2\}, \{*2\}, \{\overline{s}\}, \{\overline{s}, \overline{z}\}, \{\overline{z}\}, \{\overline{\alpha}, \overline{z}\}, \{0\}\big\}$. A similar argument with conjugates gives all possibilities for $G^{\boldsymbol{R}}$. □

Now we consider games G with outcome $\mathcal{L}$, i.e. $G^{\boldsymbol{L}} \cap \{*, \alpha, s\} \neq \emptyset$ and $G^{\boldsymbol{R}} \cap \{*, \overline{\alpha}, \overline{s}\} = \emptyset$. Since $G \not\leq 0$ and $G \not\leq *$, no Right option of G is $\mathcal{D}$-reversible. The two following lemmas allow us to characterize dicots born by day 3 whose outcome is $\mathcal{L}$ and that contain $\mathcal{D}$-reversible Left options. First, we characterize positions that may contain $\mathcal{D}$-reversible Left options through $*$.

Lemma 4.4 *Let G be a dicot born by day 3 with misère outcome $\mathcal{L}$. We have $G \geq^-_{\mathcal{D}} *$ if and only if $G^{\boldsymbol{L}} \cap \{0, z\} \neq \emptyset$.*

Proof. The proof that if $G^{\boldsymbol{L}} \cap \{0, z\} \neq \emptyset$, then Left wins $G + X$ whenever she wins $* + X$ is the same as for Lemma 4.2.

Consider now the case when $G^{\boldsymbol{L}} \cap \{0, z\} = \emptyset$, that is $G^{\boldsymbol{L}} \subseteq \{*, \alpha, s, \overline{\alpha}, \overline{s}, \overline{z}, *2\}$. Assume first that $\{0, z\} \cap G^{\boldsymbol{R}} \neq \emptyset$ and let $X = \{\overline{s}|0\}$. Recall that in $* + X$, Left wins playing to $0 + X$ and Right wins playing to $* + 0$, hence $o^-(* + X) = \mathcal{N}$. On the other hand, in $G + X$, Left has no other options than $\overline{\alpha} + X, * + X, \alpha + X, s + X, \overline{s} + X, \overline{z} + X, *2 + X, G + \overline{s}$. In $\overline{\alpha} + X$, Right wins by playing to $\overline{\alpha} + 0$, whose outcome is $\mathcal{R}$. In $G + \overline{s}$, by our assumption, Right can play either to $0 + \overline{s}$ or to $z + \overline{s}$, with outcome $\mathcal{R}$ and $\mathcal{P}$ respectively, and thus wins. In all other cases, Right wins by playing to $0 + X$, whose outcome is $\mathcal{P}$. Thus $o^-(G + X) \leq \mathcal{P}$, and since $o^-(* + X) = \mathcal{N}$, we have $G \not\geq^-_{\mathcal{D}} *$.

Now assume $\{0, z\} \cap G^{\boldsymbol{R}} = \emptyset$, that is $G^{\boldsymbol{R}} \subseteq \{\alpha, s, \overline{z}, *2\}$. Let $X' = \{\overline{z}|0\}$. In $* + X'$, Left wins playing to $0 + X'$ and Right wins playing to $* + 0$, hence $o^-(* + X') = \mathcal{N}$. On the other hand, in $G + X'$, Left has no other options than $G + \overline{z}, \overline{\alpha} + X', * + X', \alpha + X', s + X', \overline{s} + X', \overline{z} + X', *2 + X'$. In $\overline{\alpha} + X'$, Right wins by playing to $\overline{\alpha} + 0$ whose outcome is $\mathcal{R}$. In $G + \overline{z}$, Right wins by playing either to $\alpha + \overline{z}$ or $s + \overline{z}$, both with outcome $\mathcal{P}$, or to $\overline{z} + \overline{z}$ or $*2 + \overline{z}$, both with outcome $\mathcal{R}$. In the remaining cases, Right wins by playing to $0 + X'$ whose outcome is $\mathcal{P}$. Thus $o^-(G + X') \leq \mathcal{P}$, and since $o^-(* + X') = \mathcal{N}$, we have $G \not\geq^-_{\mathcal{D}} *$. $\square$

Now, we characterize games that may contain $\mathcal{D}$-reversible Left options through 0. The following lemma actually can be proven for both games with outcome $\mathcal{L}$ or $\mathcal{N}$, and we also use it for proof of Theorem 4.7.

Lemma 4.5 *Let G be a dicot born by day 3 with misère outcome $\mathcal{L}$ or $\mathcal{N}$. We have $G \geq^-_{\mathcal{D}} 0$ if and only if $G^{\boldsymbol{R}} \cap \{0, \alpha, \overline{z}\} = \emptyset$.*

Proof. Suppose first that $G^{\boldsymbol{R}} \cap \{0, \alpha, \overline{z}\} = \emptyset$. Then every Right option of G has 0 as a Left option. Let X be a dicot such that Left has a winning strategy on $0 + X$ when playing first (respectively second). Left can follow the same strategy on $G + X$ until either Right plays on G or she has to move from $G + 0$. In the first case, she can answer in $G^R + Y$ to $0 + Y$ and continue her winning strategy. In the second case, she wins in $G + 0$ since $o^-(G) \geq \mathcal{N}$. Therefore, $G \geq^-_{\mathcal{D}} 0$.

Consider now the case when $G^{\boldsymbol{R}} \cap \{0, \alpha, \overline{z}\} \neq \emptyset$. Let $X = \{\overline{\alpha}|0\}$, note that $o^-(X) = \mathcal{P}$. When playing first on $G + X$, Right wins by playing either to $0 + X$ with outcome $\mathcal{P}$, or to $\alpha + X$ or $\overline{z} + X$, both with outcome $\mathcal{R}$. Hence $o^-(G + X) \leq \mathcal{N}$ so $G \not\geq^-_{\mathcal{D}} 0$. $\square$

We now are in position to state the set of dicots born by day 3 with outcome $\mathcal{L}$ in canonical form. Given two sets of sets A and B, we use the notation $A \uplus B$ to denote the set $\{a \cup b | a \in A, b \in B\}$.

Theorem 4.6 *A dicot G born by day 3 with outcome $\mathcal{L}$ is in canonical form if and only if either*

$$\begin{cases} G^{\boldsymbol{L}} \in \big(\big\{\{*\}, \{\alpha\}, \{*, \alpha\}\big\} \uplus \big\{\emptyset, \{0\}, \{\overline{\alpha}\}, \{*2\}, \{\overline{\alpha}, *2\}\big\}\big) \\ \qquad\quad \cup \; \big\{\{s\}, \{\overline{\alpha}, s\}, \{\alpha, \overline{s}\}, \{*, \overline{z}\}, \{s, 0\}, \{*, \overline{\alpha}, \overline{z}\}\big\}, \text{ and} \\ G^{\boldsymbol{R}} \in \big\{\{0\}, \{\alpha\}, \{0, \alpha\}, \{0, *2\}, \{\alpha, *2\}, \{0, \alpha, *2\}, \{\overline{z}\}, \{\alpha, z\}, \{0, s\}\big\}, \end{cases}$$

or

$$\begin{cases} G^{\boldsymbol{L}} \in \big\{\{*\}, \{*, 0\}, \{*, \overline{\alpha}\}\big\}, \text{ and} \\ G^{\boldsymbol{R}} \in \big\{\{*2\}, \{s\}, \{z\}, \{s, z\}\big\}. \end{cases}$$

This yields $21 \cdot 9 + 3 \cdot 4 = 201$ dicots non equivalent modulo $\mathcal{D}$.

Proof. Let G be a dicot game born by day 3 with outcome $\mathcal{L}$, in canonical form. By our earlier statement, $G^{\boldsymbol{L}} \cap \{*, \alpha, s\} \neq \emptyset$. By Lemma 4.4, options $\overline{\alpha}, \overline{s}, z, \overline{z}, *2$ are reversible Left options through $*$ whenever $G^{\boldsymbol{L}} \cap \{0, z\} \neq \emptyset$. Thus, we have 21 of the 50 antichains remaining for G^L, namely:

$$\begin{cases} \text{15 containing } \{*\}, \{\alpha\} \text{ or } \{*, \alpha\} \text{ together with } \{0\} \text{ or any subset of } \{\overline{\alpha}, *2\} \\ \{s\}, \{s, 0\} \text{ and } \{s, \overline{\alpha}\}, \\ \{\overline{s}, \alpha\} \\ \{\overline{z}, *\} \text{ and } \{\overline{z}, *, \overline{\alpha}\} \end{cases}$$

Now, by Lemma 4.5, options $*, \alpha, s, \overline{s}, \overline{z}$, and $*2$ are reversible through 0 whenever $G^{\boldsymbol{R}} \cap \{0, \alpha, \overline{z}\} = \emptyset$. By Lemma 3.2, these options should then be replaced by $*$. Thus the only antichains remaining for $G^{\boldsymbol{L}}$ when $G^{\boldsymbol{R}} \cap \{0, \alpha, \overline{z}\} = \emptyset$ are $\{*\}$, $\{*, 0\}$ and $\{*, \overline{\alpha}\}$.

Consider now Right options. By our earlier statement, $G^{\boldsymbol{R}} \subseteq \{0, \alpha, s, z, \overline{z}, *2\}$, and no Right option is reversible. Intersecting $\{0, \alpha, \overline{z}\}$, we have the antichains: $\{0\}$, $\{\alpha\}$, $\{0, \alpha\}$, $\{0, *2\}$, $\{\alpha, *2\}$, $\{0, \alpha, *2\}$, $\{\overline{z}\}$, $\{\alpha, z\}$ and $\{0, s\}$. Not intersecting $\{0, \alpha, \overline{z}\}$, we have $\{*2\}$, $\{s\}$, $\{z\}$ and $\{s, z\}$. Combining these sets, we get the theorem. □

The dicots born by day 3 with outcome $\mathcal{R}$ in canonical form are exactly the conjugates of those with outcome $\mathcal{L}$.

Now consider dicots with outcome $\mathcal{N}$. By our earlier statement, we have $G^{\boldsymbol{L}} \cap \{*, \alpha, s\} \neq \emptyset$ and $G^{\boldsymbol{R}} \cap \{*, \overline{\alpha}, \overline{s}\} \neq \emptyset$. Note that G and $*$ are $\mathcal{D}$-incomparable since $o^-(*) = \mathcal{P}$. Therefore no option of G is $\mathcal{D}$-reversible through $*$. Recall also that by Lemma 4.5, we can recognize dicots born by day 3 whose outcome is $\mathcal{N}$ and that may contain $\mathcal{D}$-reversible options through 0.

Theorem 4.7 *A dicot G born by day 3 with outcome $\mathcal{N}$ is in canonical form if and only if either $G = 0$ or*

$$\text{or} \begin{cases} G^{\boldsymbol{L}} \in \big\{\{*\}, \{\alpha\}, \{*, \alpha\}, \{*, *2\}, \{\alpha, *2\}, \{*, \alpha, *2\}, \{s\}, \{\alpha, \overline{s}\}, \{*, \overline{z}\}\big\}, \text{ and} \\ G^{\boldsymbol{R}} \in \big\{\{0, *\}, \{*, \alpha\}, \{0, *, \alpha\}, \{*, \overline{z}\}\big\}, \end{cases}$$

$$\text{or} \begin{cases} G^{\boldsymbol{L}} \in \big\{\{0, *\}, \{*, \overline{\alpha}\}, \{0, *, \overline{\alpha}\}, \{*, z\}\big\}, \text{ and} \\ G^{\boldsymbol{R}} \in \big\{\{*\}, \{\overline{z}\}, \{*, \overline{z}\}, \{*, *2\}, \{\overline{z}, *2\}, \{*, \overline{z}, *2\}, \{\overline{s}\}, \{\overline{\alpha}, s\}, \{*, z\}\big\}, \end{cases}$$

$$\text{or} \begin{cases} G^{\boldsymbol{L}} \in \big\{\{*\}, \{\alpha\}, \{*, \alpha\}\big\} \uplus \big\{\{0\}, \{\overline{\alpha}\}, \{0, \overline{\alpha}\}\big\} \uplus \big\{\emptyset, \{*2\}\big\} \\ \qquad \cup \ \big\{\{s, z\}, \{s, 0\}, \{s, \overline{\alpha}\}, \{s, \overline{\alpha}, 0\}, \{z, *\}, \{z, \alpha\}, \{z, \alpha, *\}, \{\alpha, \overline{s}, 0\}, \{*, \overline{z}, \overline{\alpha}\}\big\}, \text{ and} \\ G^{\boldsymbol{R}} \in \big\{\{*\}, \{\overline{\alpha}\}, \{*, \overline{\alpha}\}\big\} \uplus \big\{\{0\}, \{\alpha\}, \{0, \alpha\}\big\} \uplus \big\{\emptyset, \{*2\}\big\} \\ \qquad \cup \ \big\{\{\overline{s}, \overline{z}\}, \{\overline{s}, 0\}, \{\overline{s}, \alpha\}, \{\overline{s}, \alpha, 0\}, \{\overline{z}, *\}, \{\overline{z}, \overline{\alpha}\}, \{\overline{z}, \overline{\alpha}, *\}, \{\overline{\alpha}, s, 0\}, \{*, z, \alpha\}\big\}. \end{cases}$$

This yields $1 + 9 \cdot 4 + 4 \cdot 9 + 27 \cdot 27 = 802$ dicots non equivalent modulo $\mathcal{D}$.

Proof. Recall that by Lemma 4.5, if $G^{\boldsymbol{R}} \cap \{0, \alpha, \overline{z}\} \neq \emptyset$, then Left options $*, \alpha, s, \overline{s}, \overline{z}, *2$ are reversible through 0 and get replaced by $*$. Similarly, if $G^{\boldsymbol{L}} \cap \{0, \overline{\alpha}, z\} \neq \emptyset$, then Right options $*, \overline{\alpha}, s, \overline{s}, z, *2$ are reversible through 0 and get replaced by $*$.

Consider first the case when $G^{\boldsymbol{R}} \cap \{0, \alpha, \overline{z}\} = \emptyset$ and $G^{\boldsymbol{L}} \cap \{0, \overline{\alpha}, z\} = \emptyset$. Then $G^{\boldsymbol{L}} \cap \{\alpha, s, \overline{s}, \overline{z}, *2\} = \emptyset$ and $G^{\boldsymbol{R}} \cap \{\overline{\alpha}, s, \overline{s}, z, *2\} = \emptyset$. So $G = 0$ or $\{*|*\}$ which reduces to 0.

Now, suppose $G^{\boldsymbol{R}} \cap \{0, \alpha, \overline{z}\} \neq \emptyset$ but $G^{\boldsymbol{L}} \cap \{0, \overline{\alpha}, z\} = \emptyset$. Then $G^{\boldsymbol{R}} \cap \{\overline{\alpha}, s, \overline{s}, z, *2\} = \emptyset$. Recall that since $o^-(G) = \mathcal{N}$, $G^{\boldsymbol{R}} \cap \{*, \overline{\alpha}, \overline{s}\} \neq \emptyset$. So $G^{\boldsymbol{R}} \in \{\{0, *\}, \{*, \alpha\}, \{0, *, \alpha\}, \{*, \overline{z}\}\}$. On the other hand, $G^{\boldsymbol{L}}$ can be any antichain containing one of $\{*, \alpha, s\}$ and possibly some of $\{\overline{s}, \overline{z}, *2\}$. Thus $G^{\boldsymbol{L}} \in \{\{*\}, \{\alpha\}, \{*, \alpha\}, \{*, *2\}, \{\alpha, *2\}, \{*, \alpha, *2\}, \{s\}, \{\alpha, \overline{s}\}, \{*, \overline{z}\}\}$. When $G^{\boldsymbol{L}} \cap \{0, \overline{\alpha}, z\} \neq \emptyset$ and $G^{\boldsymbol{R}} \cap \{0, \alpha, \overline{z}\} = \emptyset$, we get $G^{\boldsymbol{L}}$ and $G^{\boldsymbol{R}}$ by conjugating the previous $G^{\boldsymbol{R}}$ and $G^{\boldsymbol{L}}$ respectively.

Finally, when $G^{\boldsymbol{R}} \cap \{0, \alpha, \overline{z}\} \neq \emptyset$ and $G^{\boldsymbol{L}} \cap \{0, \overline{\alpha}, z\} \neq \emptyset$, no option is reversible. Therefore, the antichains for $G^{\boldsymbol{R}}$ are those containing at least one of $\{0, \alpha, \overline{z}\}$ and one of $\{*, \overline{\alpha}, \overline{s}\}$. There are 27 of them, namely:

$$\begin{cases} 18 \text{ containing some subset of } \{*, \alpha\}, \text{ some subset of } \{0, \overline{\alpha}\} \text{ and possibly } \{*2\} \\ \{s, z\} \\ \{0, s\}, \{\overline{\alpha}, s\} \text{ and } \{0, \overline{\alpha}, s\}, \\ \{*, z\}, \{\alpha, z\} \text{ and } \{*, \alpha, z\}, \\ \{0, \alpha, \overline{s}\} \\ \{*, \overline{\alpha}, \overline{z}\} \end{cases}$$

The antichains for $G^{\boldsymbol{L}}$ are the conjugates of the antichains for $G^{\boldsymbol{R}}$. □

By Theorem 3.8, adding the number of games with outcome $\mathcal{P}$, $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{N}$, we get:

Theorem 4.8 *There are* 1268 *dicots born by day 3 non equivalent modulo* $\mathcal{D}$*.*

For comparison with what is happening in different universes, we give in Table 3 the number of non equivalent games born by a given day within a given universe.

Games born by day	0	1	2	3
Impartial in $\mathcal{I}$	1	2	3	5
Dicots in $\mathcal{D}$	1	2	9	1268
Dicots in $\mathcal{U}$	1	2	10	7541
Dicot trees	1	2	10	1046530
Games in $\mathcal{U}$	1	4	256	$< 2^{183}$

Table 3: Number of non equivalent games born by a given day.

5 Sums of dicots can have any outcome

In the previous section, we proved that modulo the universe of dicots, there were much fewer distinguishable dicots under misère convention. A natural question that arises is whether in this setting, one could sometimes deduce from the outcomes of two games the outcome of their sum. This occurs in normal convention in particular with games with outcome $\mathcal{P}$. In this section, we show that this is not possible with dicots. We first prove that the misère outcome of a dicot is not related to its normal outcome.

Theorem 5.1 *Let* $\mathcal{A}, \mathcal{B}$ *be any outcomes in* $\{\mathcal{P}, \mathcal{L}, \mathcal{R}, \mathcal{N}\}$*. There exists a dicot* G *with normal outcome* $o^+(G) = \mathcal{A}$ *and misère outcome* $o^-(G) = \mathcal{B}$*.*

Proof. In Figure 5, we give for any $\mathcal{A}, \mathcal{B} \in \{\mathcal{P}, \mathcal{L}, \mathcal{R}, \mathcal{N}\}$ a dicot G such that $o^+(G) = \mathcal{A}$ and $o^-(G) = \mathcal{B}$. □

Theorem 5.2 *Let* $\mathcal{A}, \mathcal{B}$ *and* $\mathcal{C}$ *be any outcomes in* $\{\mathcal{P}, \mathcal{L}, \mathcal{R}, \mathcal{N}\}$*. There exist two dicots* G_1 *and* G_2 *such that* $o^-(G_1) = \mathcal{A}$*,* $o^-(G_2) = \mathcal{B}$ *and* $o^-(G_1 + G_2) = \mathcal{C}$*.*

Proof. In Figure 6, we give for any $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \{\mathcal{P}, \mathcal{L}, \mathcal{R}, \mathcal{N}\}$ two games G_1 and G_2 such that $o^-(G_1) = \mathcal{A}$, $o^-(G_2) = \mathcal{B}$ and $o^-(G_1 + G_2) = \mathcal{C}$. □

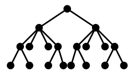
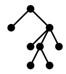
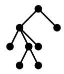
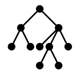
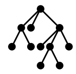
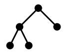
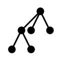
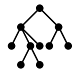
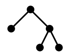
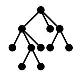
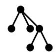
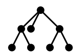
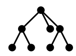
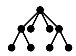
Normal → Misère ↓	$\mathcal{P}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{N}$
$\mathcal{P}$				
$\mathcal{L}$				
$\mathcal{R}$				
$\mathcal{N}$				

Figure 5: Normal and misère outcomes of some dicots

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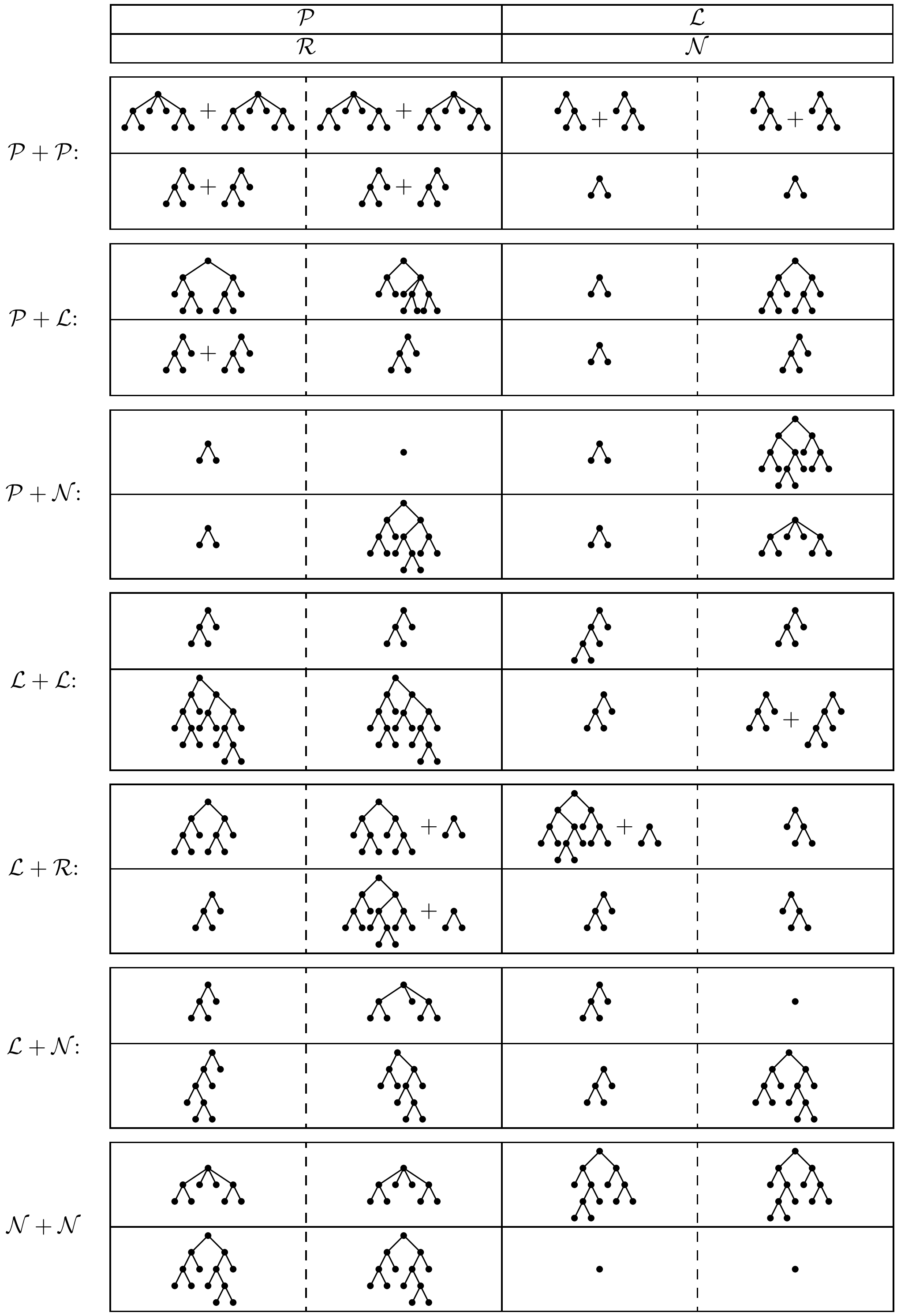

Figure 6: Sums of dicots can have any outcome