PSPACE algorithm for SPEs in quantitative reachability games

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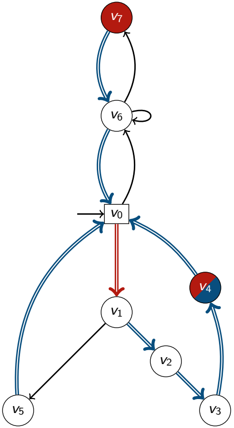
Background

- Quantitative reachability game
- Subgame
- Nash Equilibrium and Subgame Perfect Equilibrium
- Extended game
- Constrained existence problem

Quantitative reachability game



$(\mathcal{G}, v_0) = (\Pi, V, E, (V_i)_{i \in \Pi}, (\text{Cost}_i)_{i \in \Pi}, (F_i)_{i \in \Pi})$

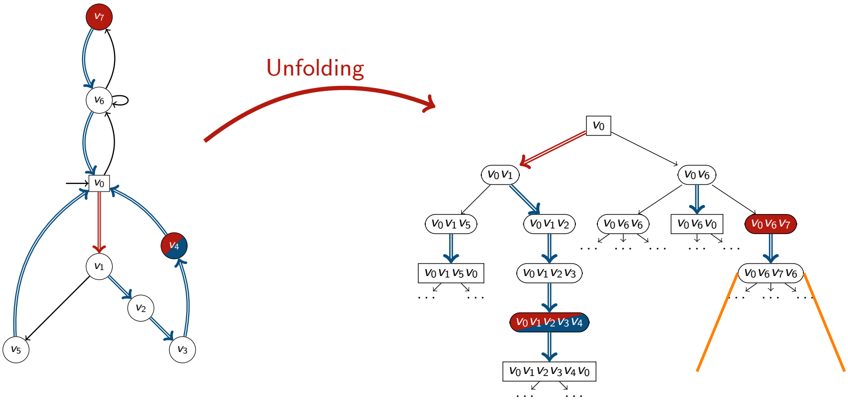
Quantitative reachability objective

- For all $i \in \Pi$, F_i: the target set of player i;
  Ex: $F_{\bigcirc} = \{v_4\}$ and $F_{\square} = \{v_4, v_7\}$
- For all $\rho \in V^\omega$, $\rho = \rho_0 \rho_1 \ldots$, for all $i \in \Pi$:

$$\text{Cost}_i(\rho) = \begin{cases} k & \text{if } k \text{ is the least index st. } \rho_k \in F_i \\ +\infty & \text{otherwise} \end{cases}$$

- Play (resp. history): infinite (resp. finite) path in the graph from v_0;
- Strategy: choice of a player when it is his turn to play;
  Ex: $\sigma_{\bigcirc}$ is a strategy for $P_{\bigcirc}$ and $\sigma_{\square}$ is a strategy for $P_{\square}$;
- Strategy profile: $(\sigma_{\bigcirc}, \sigma_{\square})$ is a strategy profile;
- Outcome: given $(\sigma_{\bigcirc}, \sigma_{\square})$, $\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0} = (v_0 v_1 v_2 v_3 v_4)^\omega$ is the unique consistent play.

Subgame



Let hv be an history $\rightsquigarrow (\mathcal{G}_{\restriction h}, v)$: ⚠ Don't forget the past!

$(\mathcal{G}_{\restriction h}, v) = (\Pi, V, E, (V_i)_{i \in \Pi}, (\mathbf{Cost}_{i\restriction h})_{i \in \Pi}, (F_i)_{i \in \Pi})$

- $\text{Cost}_{i\restriction h}(\rho) = \text{Cost}_i(h\rho)$;
- $\sigma_{i\restriction h}(h'v) = \sigma_i(hh'v)$.

Ex: $(\mathcal{G}_{\restriction v_0 v_6 v_7}, v_6)$

Nash Equilibrium and Subgame Perfect Equilibrium

Nash equilibrium (NE)

A strategy profile $(\sigma_i)_{i \in \Pi}$ is a Nash equilibrium in $(\mathcal{G}, v_0)$ if and only if for all $i \in \Pi$ and all σ'_i:

$$\text{Cost}_i(\langle \sigma \rangle_{v_0}) \leq \text{Cost}_i(\langle \sigma'_i, (\sigma_j)_{j \in \Pi \setminus \{i\}} \rangle_{v_0}).$$

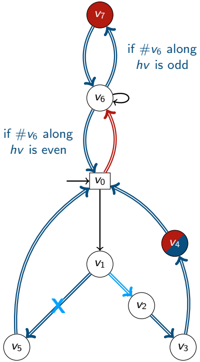
$\rightsquigarrow$ No incentive to deviate unilateraly.

Nash Equilibrium and Subgame Perfect Equilibrium

Subgame perfect equilibrium (SPE)

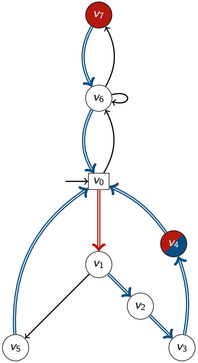
A strategy profile $(\sigma_i)_{i\in\Pi}$ is a subgame perfect equilibrium in $(\mathcal{G}, v_0)$ if and only for all history hv : $(\sigma_{i\restriction h})_{i\in\Pi}$ is a Nash equilibrium in $(\mathcal{G}_{\restriction h}, v)$.

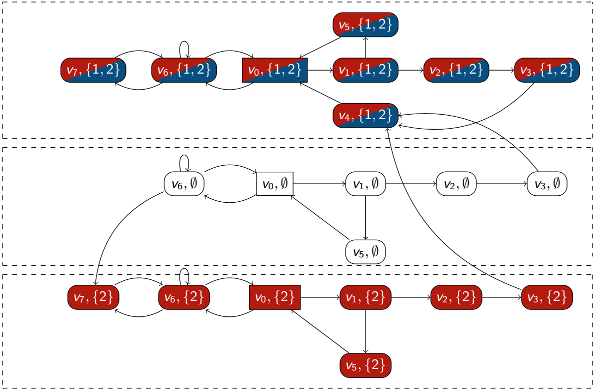
Counter-example:



- NE with outcome $(v_0 v_6 v_7 v_6)^\omega \rightsquigarrow$ Cost $= (+\infty, 2)$.
- In $(\mathcal{G}_{\restriction v_0}, v_1)$, outcome: $v_1 v_5 (v_0 v_6 v_7 v_6)^\omega \rightsquigarrow$ $\text{Cost}_{\restriction v_0} = \text{Cost}(v_0 \cdot v_1 v_5 (v_0 v_6 v_7 v_6)^\omega) = (+\infty, 5)$
- Profitable deviation, outcome: $v_1 v_2 v_3 v_4 \ (v_0 v_6 v_7 v_6)^\omega \rightsquigarrow$ $\text{Cost}_{\restriction v_0} = \text{Cost}(v_0 \cdot v_1 v_2 v_3 v_4 \ (v_0 v_6 v_7 v_6)^\omega) = (4, 4)$

Extended game





- Regions (denoted by I) $\rightsquigarrow \emptyset, \{2\}$ and $\{1,2\}$;
- Total order on regions ($I' \preceq I$ iff $I \in \mathrm{Succ}^*(I')$) $\rightsquigarrow \emptyset \preceq \{2\} \preceq \{1,2\}$;
- Decomposition of plays region by region $\rightsquigarrow$

$$\underbrace{(v_0,\emptyset)(v_1,\emptyset)(v_2,\emptyset)(v_3,\emptyset)}_{\text{region } \emptyset}[\underbrace{(v_4,\{1,2\})(v_0,\{1,2\})(v_1,\{1,2\})(v_2,\{1,2\})(v_3,\{1,2\})}_{\text{region } \{1,2\}}]^\omega$$

Constrained existence problem

Constrained existence problem

Let $(\mathcal{G}, v_0)$ be a quantitative reachability game and $x \in (\mathbb{N} \cup \{+\infty\})^{|\Pi|}$ be a threshold, we want to decide if there exists an SPE $(\sigma_i)_{i\in\Pi}$ such that for all $i \in \Pi$:

$$\mathsf{Cost}_i(\langle(\sigma_i)_{i\in\Pi}\rangle_{v_0}) \leq x_i.$$

Result

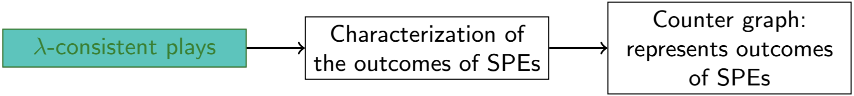
The constrained existence problem is PSPACE-complete.

- PSPACE-hardness: polynomial reduction from the QBF problem;
- PSPACE-easyness: subject of this talk.

PSPACE algorithm

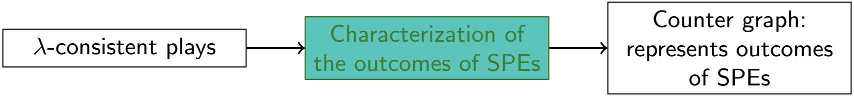
- Overview
- λ^*-consistent plays
- Complexity of the computation

Overview



- Labeling function $\lambda : V^X \to \mathbb{N} \cup \{+\infty\}$
  $\hookrightarrow$ gives constraints on the cost of a play
- If ρ satisfies these constraints, ρ is $\lambda-$consistent.

Overview



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- Definition of λ^* such that
  $\{\rho \in \text{Plays} \mid \rho \text{ is } \lambda^*\text{consistent}\} \Leftrightarrow \{\rho \in \text{Plays} \mid \rho \text{ is the outcome of an SPE }\}$.

Overview

λ-consistent plays → Characterization of the outcomes of SPEs → Counter graph: represents outcomes of SPEs

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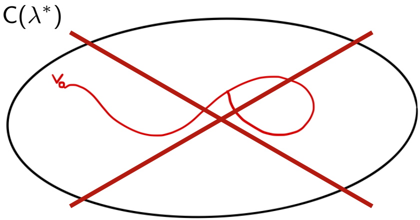
- $C(\lambda^*)$: the extended game extended with counters keeping track of the constraints given by λ^*.
  $\hookrightarrow$ an infinite path in $C(\lambda^*) \Leftrightarrow$ the outcome of an SPE.
- $|C(\lambda^*)|$: depends on $\mathrm{mR}(\lambda^*) = \max\{\lambda^*(v) \mid v \in V^X \text{ and } \lambda^*(v) \neq +\infty\}$
- $\mathrm{mR}(\lambda^*) \leq \mathcal{O}(|V|^{(|\Pi|+1)\cdot(|\Pi|+|V|)})$
  $\hookrightarrow$ we don't know the **exact size** of the counter graph.

PSPACE = NPSPACE

Intuition: guessing an infinite path **node by node** in C(λ^*) which satisfies the constraints given by the constrained existence problem.

⚠ need to know:

- the exact size of C(λ^*).
- $\lambda^* : V^X \to \mathbb{N} \cup \{+\infty\} \rightsquigarrow$ an exponential number of values.



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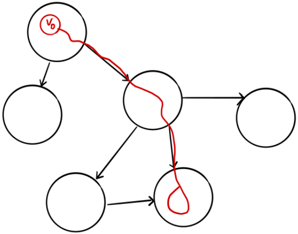
$\rightsquigarrow$ decomposition region by region.



Computation of λ^* region by region

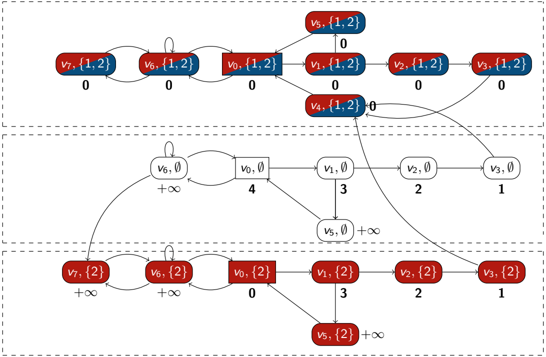
For all region I,
$\{\lambda^*(v) \mid v$ a vertex in the region $I\}$ and $\mathrm{mR}(\lambda^*_{\geq I})$ can be computed in PSPACE.

λ^*-consistent plays and characterization

λ-consistent play

Let $\lambda : V^X \to \mathbb{N} \cup \{+\infty\}$ be a labeling function. Let ρ be a play, we say that $\rho = \rho_0\rho_1 \dots$ is *λ-consistent* ($\rho \models \lambda$) if for all $n \in \mathbb{N}$ and $i \in \Pi$ such that $\rho_n \in V_i$:

$$\mathrm{Cost}_i(\rho_{\geq n}) \leq \lambda(\rho_n).$$



- $\rho = [(v_0, \emptyset)(v_1, \emptyset)(v_5, \emptyset)]^\omega \not\models \lambda$. $\mathrm{Cost}(\rho) = (+\infty, +\infty) \rightsquigarrow \mathrm{Cost}_\Box(\rho) > 4$.

λ^*-consistent plays and characterization

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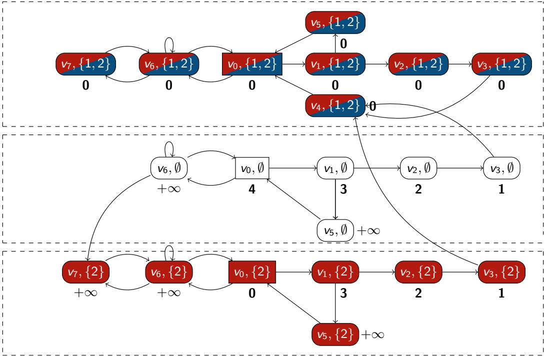
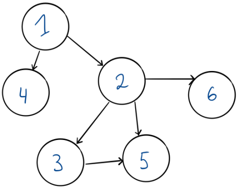


- $(v_0, \emptyset)(v_1, \emptyset)(v_2, \emptyset)(v_3\emptyset)[(v_4, \{1,2\})(v_0, \{1,2\})(v_1, \{1,2\})(v_2, \{1,2\})(v_3, \{1,2\})]^\omega \models \lambda$.

Iteratively: $\lambda^0 \rightsquigarrow \lambda^1 \rightsquigarrow \ldots \rightsquigarrow \lambda^k \rightsquigarrow \lambda^{k+1} \rightsquigarrow \ldots \rightsquigarrow \lambda^*$

Initialization: for all $(v, I) \in V^X$, if $v \in V_i$, $\lambda^0(v, I) = \begin{cases} 0 & \text{if } i \in I \\ +\infty & \text{otherwise} \end{cases}$.

Region by region:



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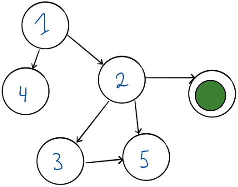

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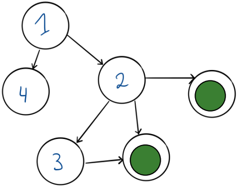

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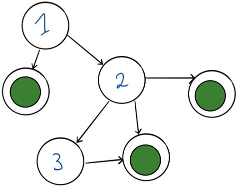

λ^*-consistent plays

How to compute λ^*

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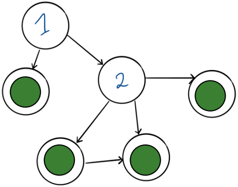

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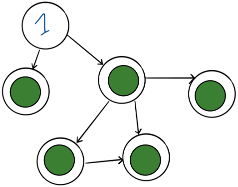

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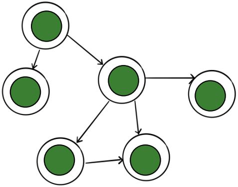
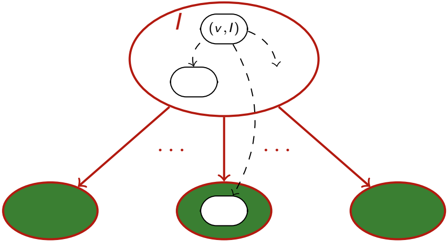
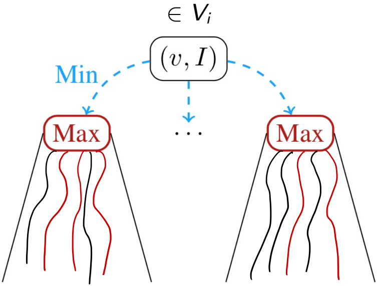


λ^*-consistent plays

How to compute λ^*

A region I is fixed:



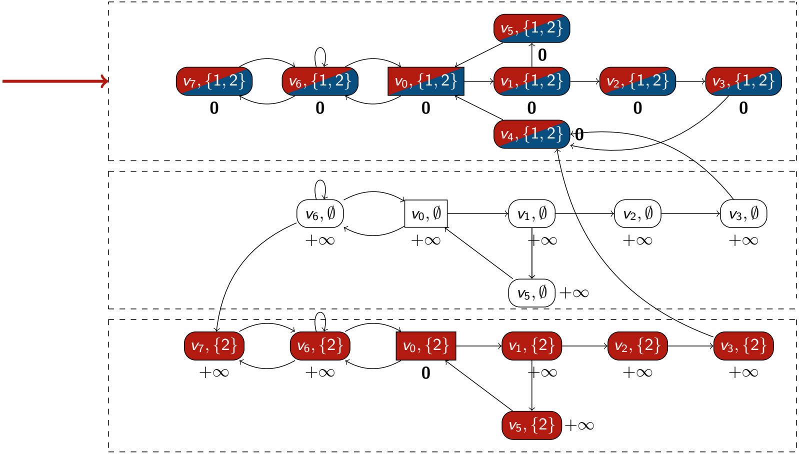


For all (v, I), assuming $v \in V_i$,

$$\lambda^{k+1}(v, I) = \begin{cases} 0 & \text{if } i \in I \\ 1 + \min_{(v', I') \in \text{Succ}(v, I)} \max\{\text{Cost}_i(\rho) \mid \rho_0 = (v', I') \text{ and } \rho \models \lambda^k\} & \text{otherwise} \end{cases}.$$

For all (v', I') with $I' \neq I$, $\lambda^{k+1}(v', I') = \lambda^k(v', I')$

$\rightsquigarrow$ Until a local fixpoint has been reached.



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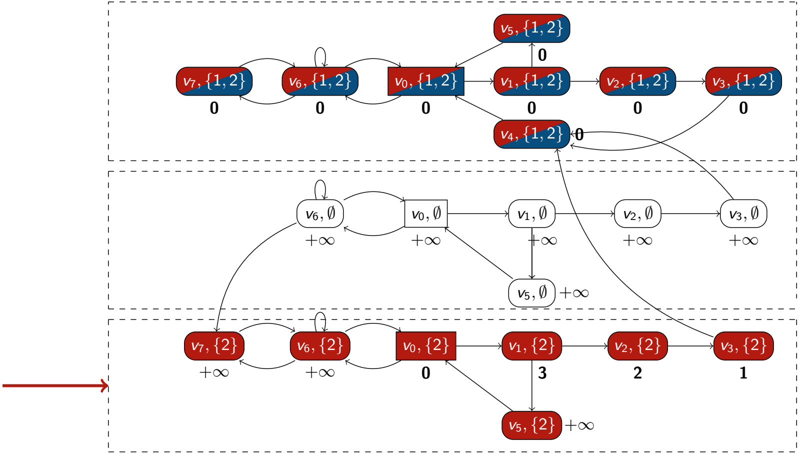

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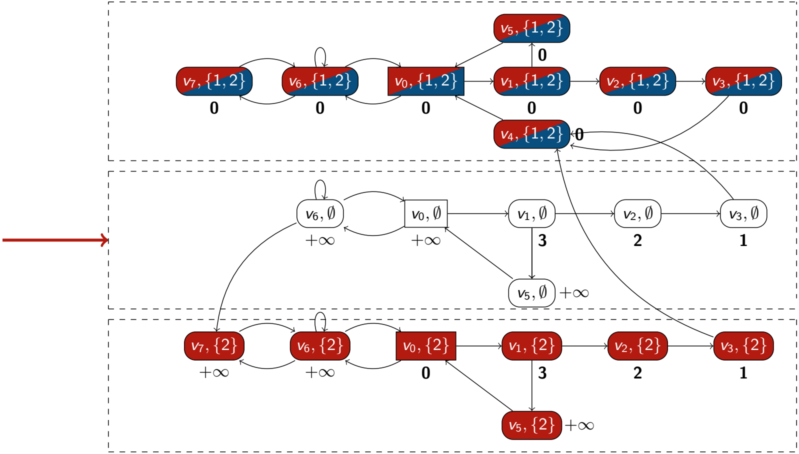

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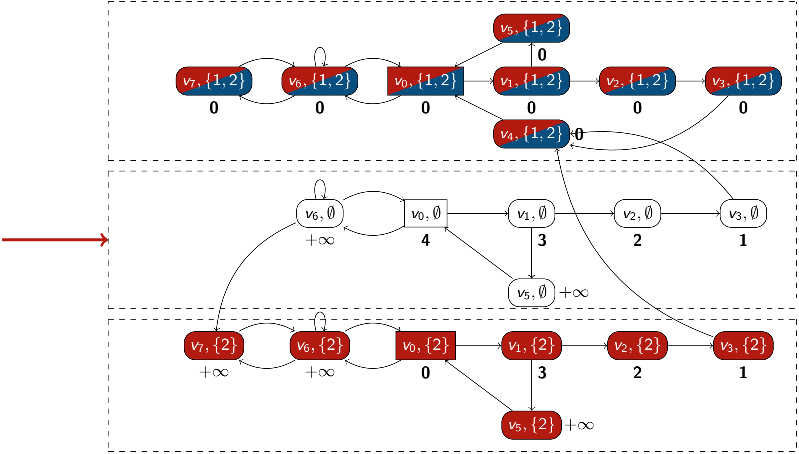

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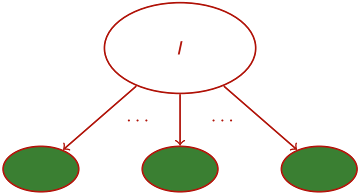
Complexity of the computation

Iterative Computation of λ^* region by region

For all $k \in \mathbb{N}$, for all region I,$\{\lambda^k(v, I) \mid (v, I)$ a vertex in the region $I\}$ and $\mathrm{mR}(\lambda^k_{\geq I})$ can be computed in PSPACE.



We know $\{\lambda^k(v, I) \mid (v, I)$ is a node of region $I\}$.

For all $I' > I$, we can compute $\{\lambda^k(v, I') \mid (v, I')$ is a node of region $I'\}$ and $\mathrm{mR}(\lambda^k_{\geq I})$ in PSPACE.

Most difficult case :

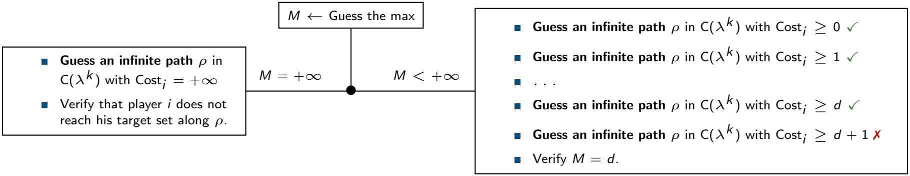
$$\lambda^{k+1}(v, i) = 1 + \min_{(v', I') \in \mathrm{Succ}(v, I)} \max\{\mathrm{Cost}_i(\rho) \mid \rho_0 = (v', I') \text{ and } \rho \models \lambda^k\}.$$

Complexity of the computation

Computation of the max cost

$$\max\{\mathrm{Cost}_i(\rho) \mid \rho_0 = (v', I') \text{ and } \rho \models \lambda^k\}$$

$\rightsquigarrow$ Counter graph $\mathrm{C}(\lambda^k)$ with $\mathrm{mR}(\lambda^k)$ and max at most exponential in the input.

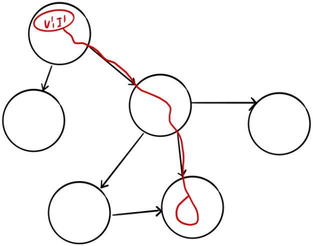


Main idea: guessing paths in $\mathrm{C}(\lambda^k)$

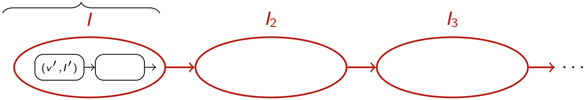
Complexity of the computation

Computation of the max cost

$$\max\{\mathrm{Cost}_i(\rho) \mid \rho_0 = (v', I') \text{ and } \rho \models \lambda^k\}$$



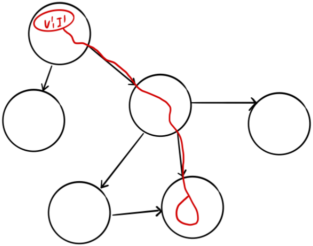
$\{\lambda^k(v, I) \mid (v, I) \text{ a node of } I\}$ is known



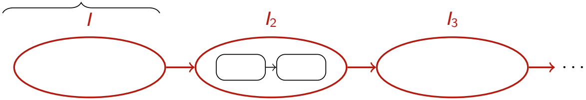
Complexity of the computation

Computation of the max cost

$$\max\{\text{Cost}_i(\rho) \mid \rho_0 = (v', I') \text{ and } \rho \models \lambda^k\}$$



$\{\lambda^k(v, I) \mid (v, I) \text{ a node of } I\}$ is known

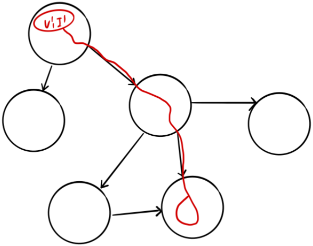


$\{\lambda^k(v, I_2) \mid (v, I_2) \text{ a node of } I_2\}$
can be computed in PSPACE

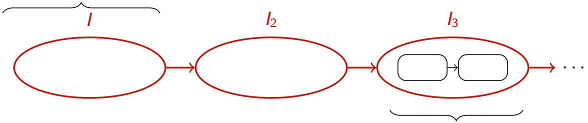
Complexity of the computation

Computation of the max cost

$$\max\{\text{Cost}_i(\rho) \mid \rho_0 = (v', I') \text{ and } \rho \models \lambda^k\}$$



$\{\lambda^k(v, I) \mid (v, I) \text{ a node of } I\}$ is known



$\{\lambda^k(v, I_3) \mid (v, I_3) \text{ a node of } I_3\}$
can be computed in PSPACE