The Complexity of Subgame Perfect Equilibria in Quantitative Reachability Games

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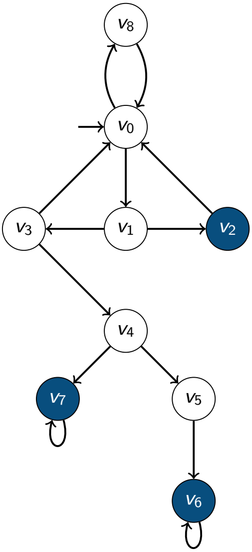
One-player quantitative reachability games

Setting



Quantitative reachability objective

- Target set $F_{\bigcirc} = \{v_2, v_6, v_7\}$;
- For every infinite path (called **play**) ρ, $\rho = \rho_0\rho_1\ldots$,

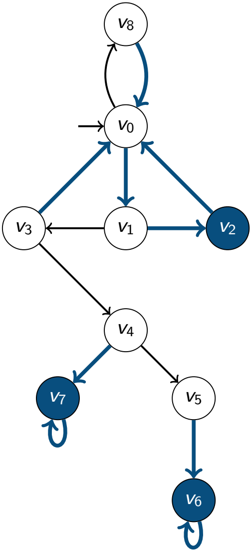
$$\text{Cost}_{\bigcirc}(\rho) = \begin{cases} k & \text{if } k \text{ is the least index st. } \rho_k \in F_i \\ +\infty & \text{otherwise} \end{cases}$$

Ex:

- $\text{Cost}_{\bigcirc}((v_0v_1v_2)^\omega) = 2$;
- $\text{Cost}_{\bigcirc}((v_0v_8)^\omega) = +\infty$.

$\rightsquigarrow$ Player $\bigcirc$ wants to reach $F_{\bigcirc}$ **as soon as possible**!

Constrained existence problem and shortest paths



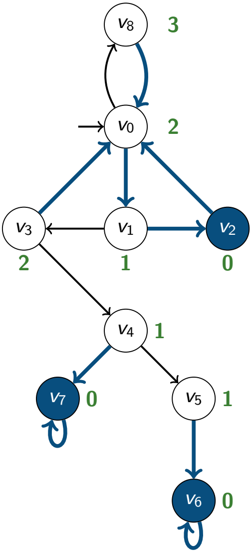
- Strategy: $\sigma_{\bigcirc} : V^* V_{\bigcirc} \to V$;
  Ex: $\sigma_{\bigcirc}$
- Playing according to $\sigma_{\bigcirc} \rightsquigarrow \langle \sigma_{\bigcirc} \rangle_{v_0} = (v_0 v_1 v_2)^\omega$
  ($\langle \sigma_{\bigcirc} \rangle_{v_0}$: the **outcome**)

Constrained existence problem

Given $k \in \mathbb{N} \cup \{+\infty\}$, does there exist a strategy $\sigma_{\bigcirc}$ for Player $\bigcirc$ such that playing according to $\sigma_{\bigcirc}$ ensures a cost less or equal to k?
i.e., $\text{Cost}_{\bigcirc}(\langle \sigma_{\bigcirc} \rangle_{v_0}) \leq k$.

Ex: with $k = 3$: YES with $\sigma_{\bigcirc}$ since $\text{Cost}_{\bigcirc}(\langle \sigma_{\bigcirc} \rangle_{v_0}) = 2$

$\rightsquigarrow$ studying **shortest paths** in the game graph.



How to find shortest paths ?

- Dijkstra algorithm;
- Bellman–Ford algorithm;
- ...

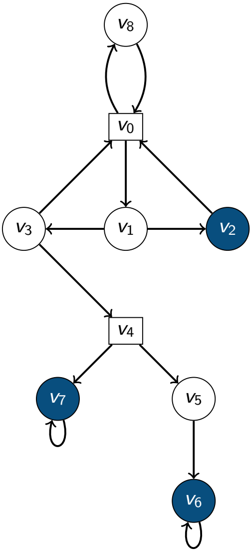
Main idea

- $X(v) = 0$ for all $v \in F_{\bigcirc}$ and $X(v) = +\infty$ otherwise.
- Repeat: $X_{pre} = X$, for each $v \in V \setminus F_{\bigcirc}$: $X(v) = \min_{v' \in E(v)} \{X_{pre}(v') + 1\}$.

$\rightsquigarrow$ only computing some minimum.

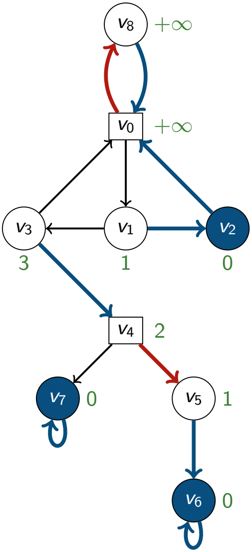
Two-player zero-sum quantitative reachability games

Setting



- **Two** players: Player ◯ (Min) and Player □ (Max).
- Objectives:
  - Player ◯ wants to reach $F_{\bigcirc}$ ASAP;
  - Player □ wants to **avoid** that.

Constrained existence problem, values and optimal strategies

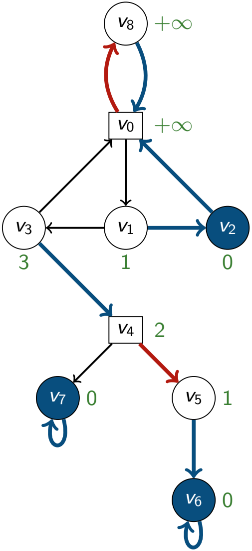


Constrained existence problem

Given $k \in \mathbb{N} \cup \{+\infty\}$, does **there exist** a strategy $\sigma_{\bigcirc}$ such that **for each** strategy $\sigma_{\Box}$: $\text{Cost}_{\bigcirc}(\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_{v_0}) \leq k$.

- From v_0, Player ◯ can ensure a cost of $+\infty$;
- From v_3, Player ◯ can ensure a cost of 3;

$\rightsquigarrow$ **value** of a node
$\rightsquigarrow$ **optimal strategies**



How to compute these values?

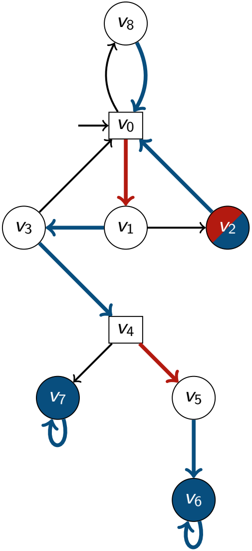
(Ex: [BGHM17])

foreach $v \in V$ **do**
  if $v \in F_{\bigcirc}$ **then**
    $X(v) = 0$
  else
    $X(v) = +\infty$
  end
end
repeat
  $X_{pre} = X$
  foreach $v \in V_{Max} \setminus F_{\bigcirc}$ **do**
    $X(v) = \max_{v' \in E(v)} (1 + X_{pre}(v'))$
  end
  foreach $v \in V_{Min} \setminus F_{\bigcirc}$ **do**
    $X(v) = \min_{v' \in E(v)} (1 + X_{pre}(v'))$
  end
until $X = X_{pre}$
return X.

$\rightsquigarrow$ only computing some **minimum** if it is a node of Player ◯ or some **maximum** if it is a node of Player □.

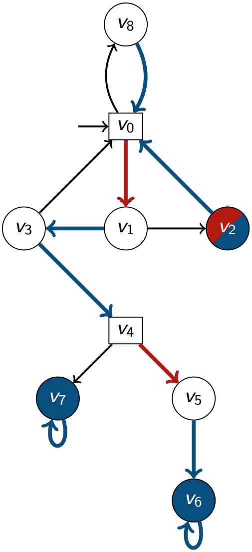
Multiplayer (non zero-sum) quantitative reachability games

Setting



- **Two** (or more) players;
  Ex: Player ◯ and Player □.
- Objectives:
  - Player ◯ wants to reach $F_{\bigcirc} = \{v_2, v_6, v_7\}$ (ASAP);
  - Player □ wants to **reach** $F_{\square} = \{v_2\}$ (ASAP).
  - ⇝ non antagonistic.

Definition of Nash equilibrium



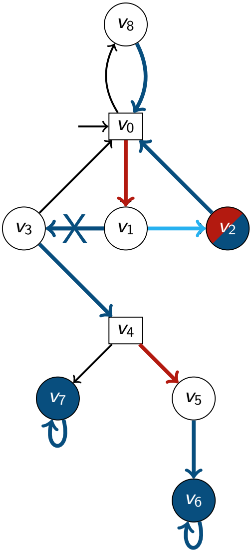
- ~~optimal strategies~~ $\rightsquigarrow$ other solution concept: **Nash equilibrium**.

Nash equilibrium

A **strategy profile** $(\sigma_{\bigcirc}, \sigma_{\square})$ is a Nash equilibrium (NE) if **no** player has an **incentive to deviate unilaterally**.

- Counter-ex: $(\sigma_{\bigcirc}, \sigma_{\square})$:
  - $(\sigma_{\bigcirc}, \sigma_{\square}) \rightsquigarrow \langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0} = v_0 v_1 v_3 v_4 v_5 v_6^{\omega}$;
  - $(\mathrm{Cost}_{\bigcirc}(\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0}), \mathrm{Cost}_{\square}(\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0})) = (5, +\infty)$.

Definition of Nash equilibrium



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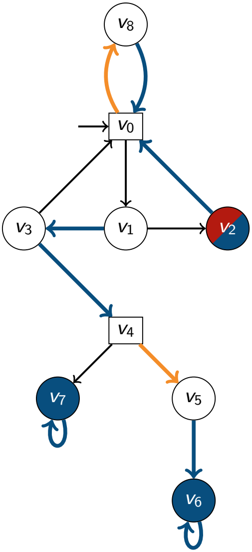
$\rightsquigarrow$ not an NE.

Definition of subgame perfect equilibrium

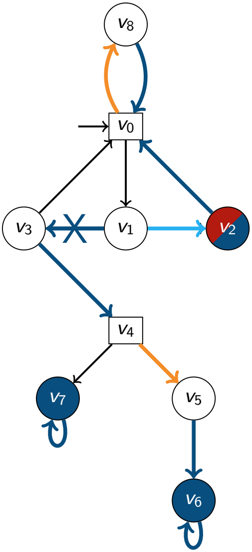


- refined solution concept: **subgame perfect equilibrium**.

Subgame perfect equilibrium

A strategy profile $(\sigma_{\bigcirc}, \sigma_{\Box})$ is a subgame perfect equilibrium (SPE) if it is an NE from each history.

Definition of subgame perfect equilibrium



- refined solution concept: **subgame perfect equilibrium**.

Subgame perfect equilibrium

A strategy profile $(\sigma_{\bigcirc}, \sigma_{\square})$ is a subgame perfect equilibrium (SPE) if it is an NE from each history.

- $(\sigma_{\bigcirc}, \sigma_{\square})$ is an **NE**;
- $(\sigma_{\bigcirc}, \sigma_{\square})$ is **not an SPE**: there is a **profitable deviation** from $v_0 v_1$.

Constrained existence problem

Constrained existence problem (with 2 players)

Given $(k_1, k_2) \in (\mathbb{N} \cup \{+\infty\})^2$, does **there exist** an equilibrium (NE or SPE) $(\sigma_{\bigcirc}, \sigma_{\Box})$ such that:
i.e.,

$$\mathrm{Cost}_{\bigcirc}(\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_{v_0}) \leq k_1$$
and
$$\mathrm{Cost}_{\Box}(\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_{v_0}) \leq k_2$$

- For **NEs**, the constrained existence problem with n players is **NP-complete**. [BBGT19]
- For **SPEs**, the constrained existence problem with n players is **PSPACE-complete**. **(our contribution)**

Our approach

Outcome characterization of an equilibrium

Let ρ be a play,

there exists an equilibrium $(\sigma_{\bigcirc}, \sigma_{\square})$ (an NE or an SPE) such that $\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0} = \rho$

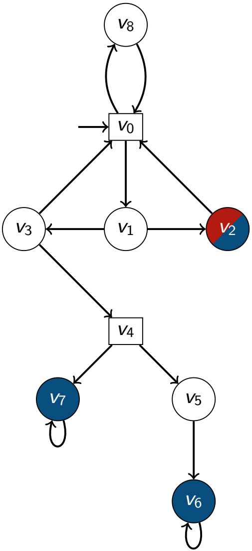
if and only if

ρ satisfies a "good" property.

$\rightsquigarrow$ Does there exist a play ρ such that:

- $\text{Cost}_{\bigcirc}(\rho) \leq k_1$ and $\text{Cost}_{\square}(\rho) \leq k_2$;
- ρ satisfies a "good" property?

Outcome characterization of equilibria

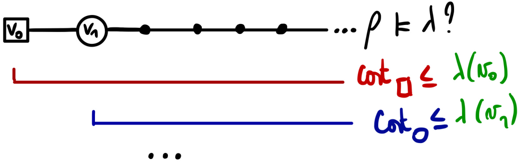


What is this good property (for NEs and SPEs)?

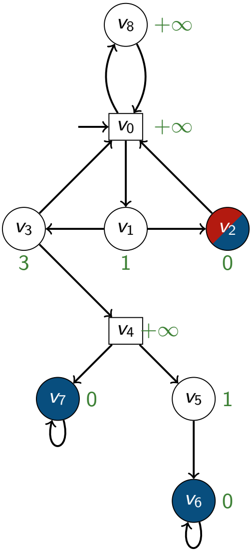
$\rightsquigarrow$ being λ-**consistent**.

λ-consistent play

- $\lambda : V \to \mathbb{N} \cup \{+\infty\}$;
- $\rho = \rho_0\rho_1 \ldots \vDash \lambda$ if and only if for all for all player i and all $k \in \mathbb{N}$ such that $i \notin \text{Visit}(\rho_0 \ldots \rho_k)$ and $\rho_k \in V_i$: $\text{Cost}_i(\rho_{\geq k}) \leq \lambda(\rho_k)$.



Outcome characterization of equilibria



- $\lambda : V \to \mathbb{N} \cup \{+\infty\}$;
- $v_0 v_1 v_3 v_4 v_5 v_6^\omega \not\models \lambda$:
  - $\text{Cost}_\square(v_0 v_1 v_3 v_4 v_5 v_6^\omega) = +\infty \leq +\infty$ ✓
  - $\text{Cost}_\bigcirc(v_1 v_3 v_4 v_5 v_6^\omega) = 4 \not\leq 1$ ✗
- $(v_0 v_8)^\omega \models \lambda$: Cost $= (+\infty, +\infty)$;

How to find the good λ ?
(one for NEs and one for SPEs)

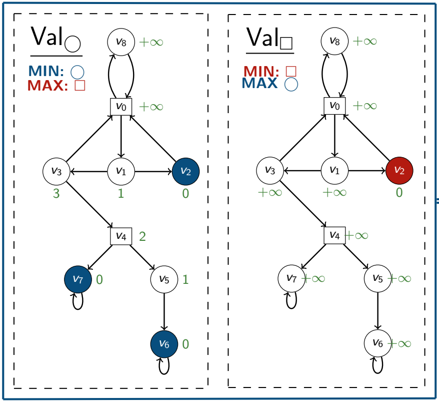
Main idea: $\lambda(v)$: the maximal number of steps within which the player who owns this vertex should reach his target set along ρ, starting from v.

For Nash equilibria:

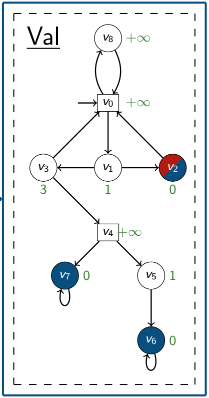
Outcome characterization of NE [BBGT19]

A play ρ is the outcome of an NE
if and only if
ρ is Val-consistent.

$$\mathsf{Val}(v) = \begin{cases} \mathsf{Val}_{\bigcirc}(v) & \text{if } v \in V_{\bigcirc} \\ \mathsf{Val}_{\square}(v) & \text{if } v \in V_{\square} \end{cases}.$$



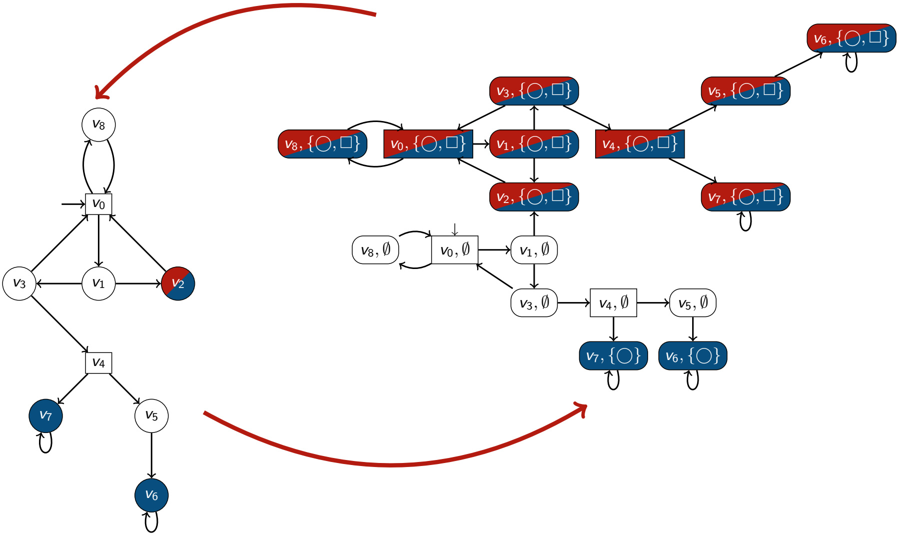
2
Two player
zero-sum
games



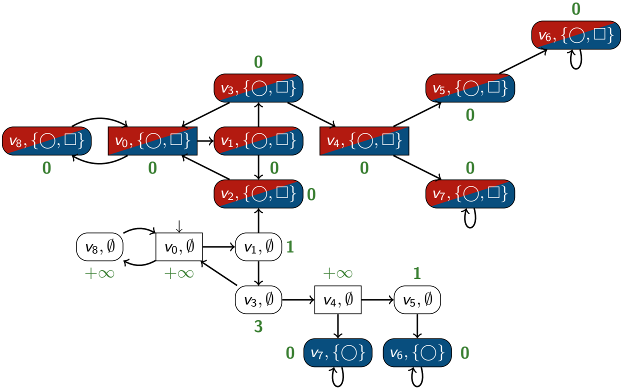
1
two player
(non zero-sum)
game

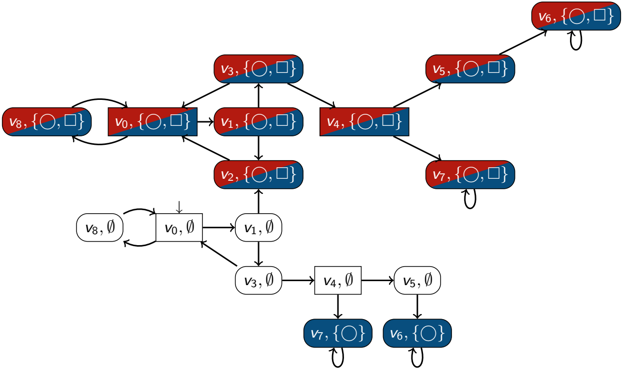
For subgame perfect equilibria (our contribution):

1) NE from **each** history ⇝ extended game.



2) ~~Val~~ : $((v_0,\emptyset)(v_8,\emptyset))^\omega \models \text{Val}$ but can't be the outcome of an SPE.





We have to find $\lambda : V \to \mathbb{N} \cup \{+\infty\}$ such that:

- λ characterizes exactly the set of outcomes of SPEs;
- the values of λ can be computed in polynomial space.

$\rightsquigarrow \lambda^*$: the fixpoint of this algorithm:

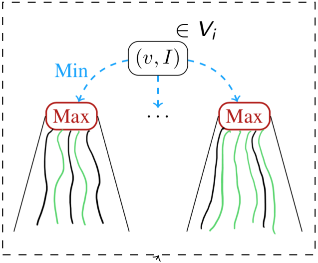
Computation of λ^*

$k \leftarrow 0$
foreach $v \in V$ *(with* $(v, I) \in V_i$ *for some player* i*)* **do**
  if $i \in I$ **then**
    $\lambda^0(v, I) = 0$
  else
    $\lambda^0(v, I) = +\infty$
  end
end
repeat
  $k \leftarrow k + 1$
  foreach $v \in V$ *(with* $(v, I) \in V_i$ *for some player* i*)* **do**
    if $i \in I$ **then**
      $\lambda^k(v, I) = 0$
    else
      $\lambda^{k+1}(v, I) = 1 + \min_{(v', I') \in E((v,I))} \max\{\text{Cost}_i(\rho) \mid \rho_0 = (v', I') \text{ and } \rho \vDash \lambda^k\}$
    end
  end
until $\lambda^k = \lambda^{k-1}$
return λ^k.

$\rightsquigarrow \lambda^*$: the fixpoint of this algorithm:

Computation of λ^*

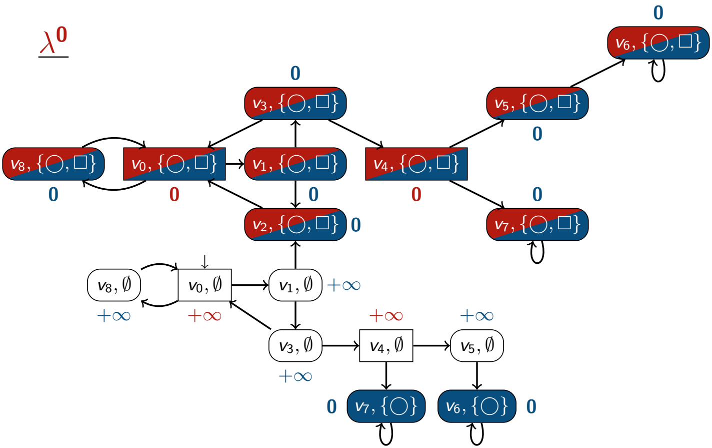
$\underline{\lambda^{\mathbf{k}} \rightsquigarrow \lambda^{\mathbf{k+1}}}$



$$\lambda^{k+1}(v, I) = 1 + \min_{(v', I') \in E((v,I))} \max\{\text{Cost}_i(\rho) \mid \rho_0 = (v', I') \text{ and } \rho \vDash \lambda^k\}$$

$\rightsquigarrow$ alternation of a **minimum** and a **maximum** whoever the player.

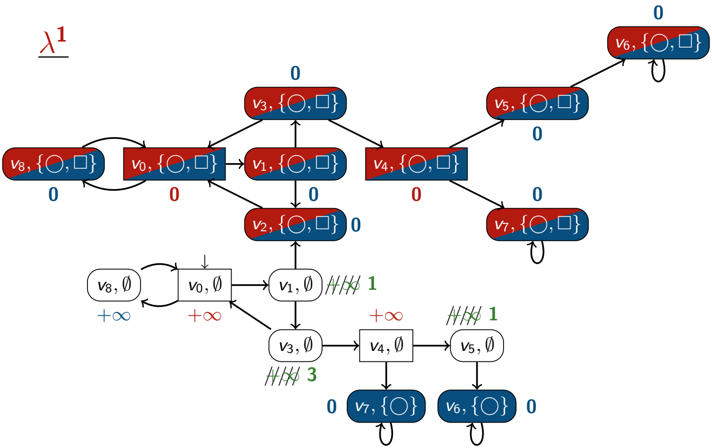
Example



λ^0
$v_6, \{\bigcirc, \square\}$
0
$v_3, \{\bigcirc, \square\}$
$v_5, \{\bigcirc, \square\}$
$v_8, \{\bigcirc, \square\}$
$v_0, \{\bigcirc, \square\}$
$v_1, \{\bigcirc, \square\}$
$v_4, \{\bigcirc, \square\}$
$v_2, \{\bigcirc, \square\}$
$v_7, \{\bigcirc, \square\}$
$v_8, \emptyset$
$v_0, \emptyset$
$v_1, \emptyset$
$+\infty$
$v_3, \emptyset$
$v_4, \emptyset$
$v_5, \emptyset$
$v_7, \{\bigcirc\}$
$v_6, \{\bigcirc\}$

Example

$$\lambda^{k+1}(v, I) = 1 + \min_{(v',I') \in E((v,I))} \max\{\text{Cost}_i(\rho) \mid \rho_0 = (v', I') \text{ and } \rho \models \lambda^k\}$$



Example

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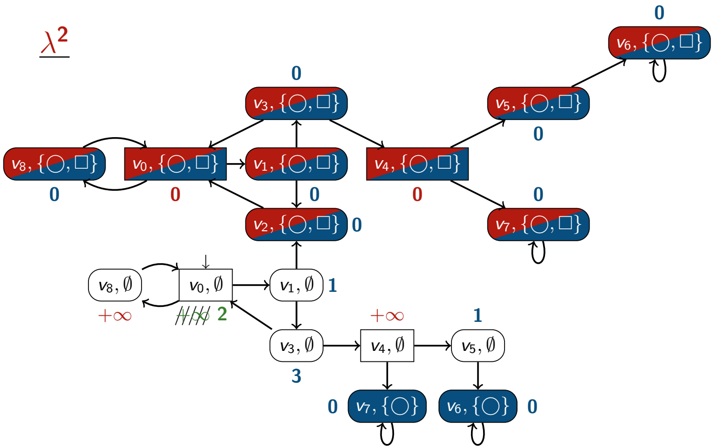
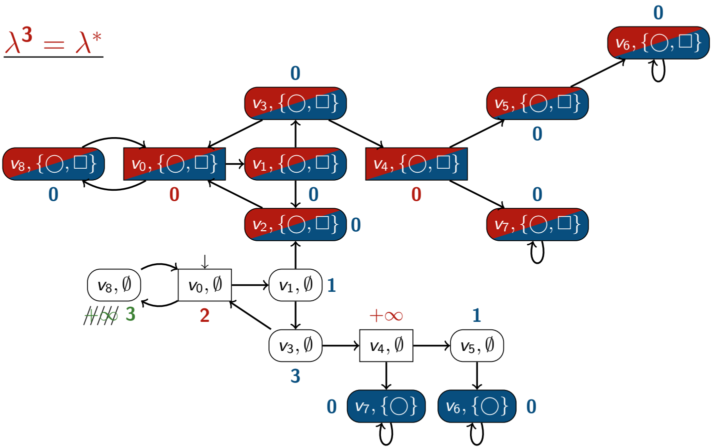


Example

$$\lambda^{k+1}(v, I) = 1 + \min_{(v',I') \in E((v,I))} \max\{\text{Cost}_i(\rho) \mid \rho_0 = (v', I') \text{ and } \rho \models \lambda^k\}$$

$\lambda^3 = \lambda^*$



Outcome characterization of SPEs

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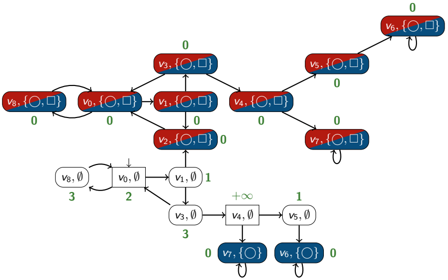

- $((v_0, \emptyset)(v_8, \emptyset))^\omega \not\models \lambda^*$: Cost $= (+\infty, +\infty)$;
  - $\mathrm{Cost}_\square(((v_0, \emptyset)(v_8, \emptyset))^\omega) = +\infty \not\leq 2$ ✗

Conclusion

Conclusion

PSPACE-c of the constrained existence problem for SPEs in multiplayer quantitative reachability games;

$\hookrightarrow$ characterization of the outcomes of SPEs;

$\hookrightarrow$ **finding a "good" labeling function λ^* (!!)**

- one which exactly characterizes the outcomes of SPEs;
- and allows us to obtain the PSPACE easyness of our problem.

References

Thomas Brihaye, Véronique Bruyère, Aline Goeminne, and Nathan Thomasset, On relevant equilibria in reachability games, CoRR **abs/1907.05481** (2019).

Thomas Brihaye, Gilles Geeraerts, Axel Haddad, and Benjamin Monmege, Pseudopolynomial iterative algorithm to solve total-payoff games and min-cost reachability games, Acta Informatica **54** (2017), no. 1, 85–125.