On Relevant Equilibria in Reachability Games

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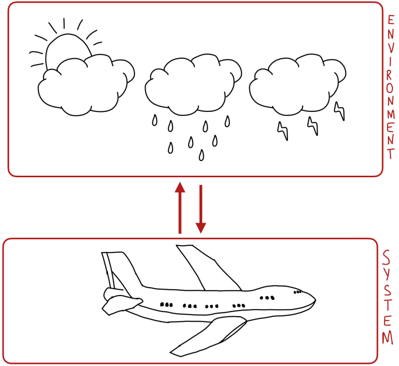
Context

Verification and synthesis

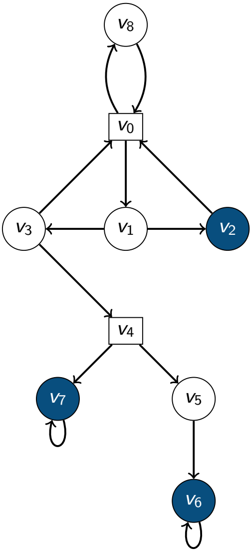


- **Verification: checking** that the **system** satisfies some **specifications**.
- **Synthesis: building** a system which satisfies some specifications by construction.

  ↪ games played on graph.

Two player zero-sum games

Qualitative two-player zero-sum reachability games

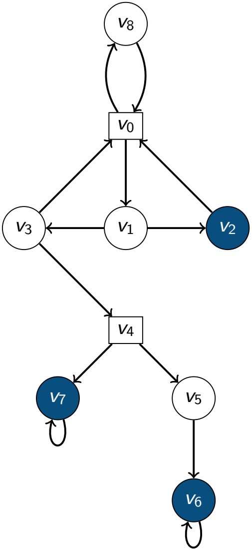


- Player ◯: **the system**
  Goal: **satisfying a property.**
  Here: reaching a vertex of the target set
  $F_{\bigcirc} = \{v_2, v_6, v_7\}$ (**reachability objective**)

- Player □: **the environment**
  Goal: **avoid that.**

> The system satisfies the property
> $\Leftrightarrow$
> Player ◯ has a **winning strategy**.

Qualitative two-player zero-sum reachability games



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  Goal: **satisfying a property.**
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Too restrictive $\rightsquigarrow$ **quantitative** specification.
(Ex: reaching a vertex of the target set within k steps.)

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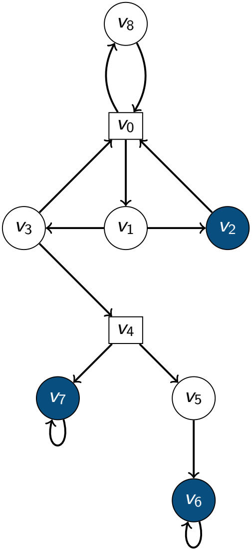

- **Two** players: Player ◯ (Min) and Player □ (Max).

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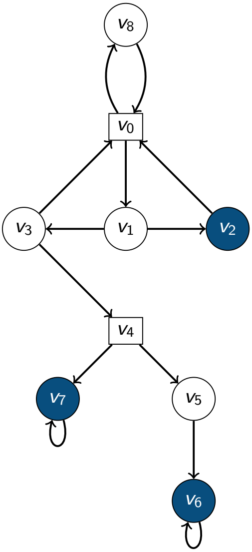
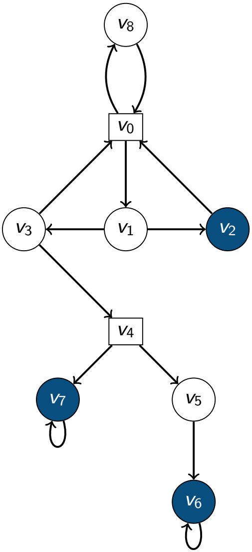


- **Two** players: Player ◯ (Min) and Player □ (Max).

- (**Quantitative** reachability objective) For every infinite path (called **play**) ρ, $\rho = \rho_0 \rho_1 \ldots$,

$$\mathrm{Cost}_{\bigcirc}(\rho) = \begin{cases} k & \text{if } k \text{ is the least index st. } \rho_k \in F_{\bigcirc} \\ +\infty & \text{otherwise} \end{cases}$$

Quantitative two-player zero-sum reachability games



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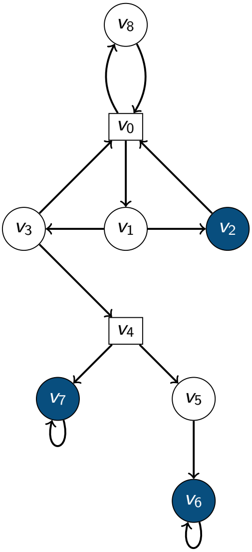
Ex:

- $\mathrm{Cost}_{\bigcirc}((v_0v_1v_2)^\omega) = 2$;
- $\mathrm{Cost}_{\bigcirc}((v_0v_8)^\omega) = +\infty$.

Quantitative two-player zero-sum reachability games



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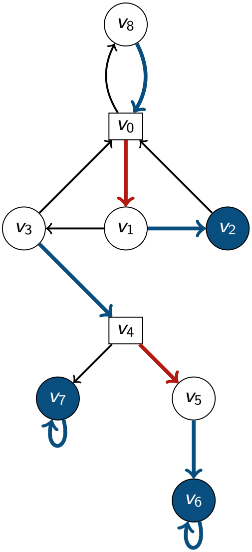
- $\text{Cost}_{\bigcirc}((v_0 v_8)^\omega) = +\infty$.

- Objectives:
  - Player ○ wants to reach $F_{\bigcirc}$ ASAP;
  - Player □ wants to **avoid** that.

Quantitative two-player zero-sum reachability games



- Strategy: $\sigma_i : V^* V_i \to V$;
  Ex: $\sigma_{\bigcirc}$ and $\sigma_{\Box}$
- A strategy profile: $(\sigma_{\bigcirc}, \sigma_{\Box}) \rightsquigarrow$
  $\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_{v_0} = (v_0 v_1 v_2)^\omega$ (called **outcome**)

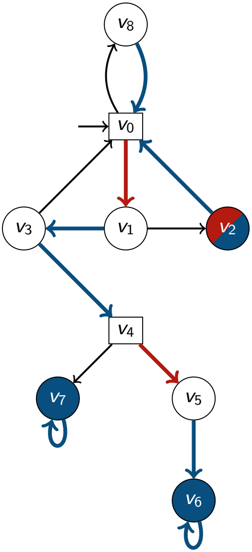
What cost can Player ◯ ensure?

- From v_0, Player ◯ can ensure a cost of $+\infty$;
- From v_3, Player ◯ can ensure a cost of 3;

$\rightsquigarrow$ **value** of a vertex
$\rightsquigarrow$ ~~winning strategy~~ $\rightsquigarrow$ **optimal strategies**.

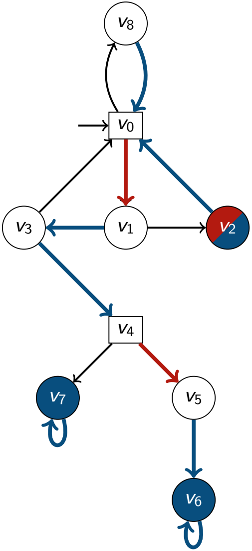
Multiplayer (non zero-sum) quantitative reachability games

Setting



- **Two** (or more) players;
  Ex: Player ◯ and Player □.
- Objectives:
  - Player ◯ wants to reach $F_{\bigcirc} = \{v_2, v_6, v_7\}$ (ASAP);
  - Player □ wants to **reach** $F_{\square} = \{v_2\}$ (ASAP).
  - $\rightsquigarrow$ non antagonistic.

Definition of Nash equilibrium

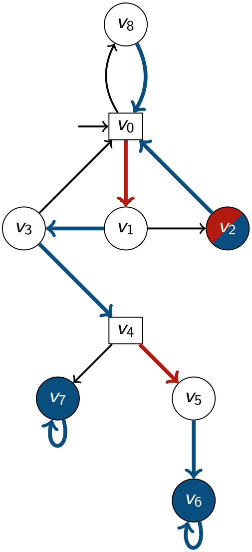


- ~~optimal strategies~~ ⇝ other solution concept: **Nash equilibrium**.

Nash equilibrium

A **strategy profile** $(\sigma_{\bigcirc}, \sigma_{\Box})$ is a Nash equilibrium (NE) if **no** player has an **incentive to deviate unilaterally**.

Definition of Nash equilibrium



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- Counter-ex: $(\sigma_{\bigcirc}, \sigma_{\square})$:
  - $(\sigma_{\bigcirc}, \sigma_{\square}) \rightsquigarrow \langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0} = v_0 v_1 v_3 v_4 v_5 v_6^{\omega}$;
  - $(\mathrm{Cost}_{\bigcirc}(\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0}), \mathrm{Cost}_{\square}(\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0})) = (5, +\infty)$.

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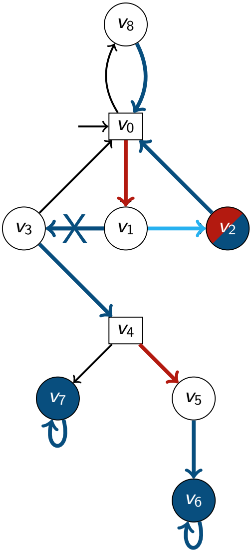

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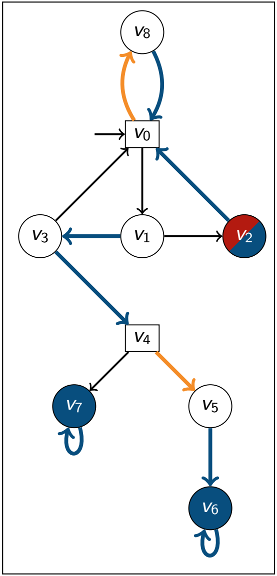
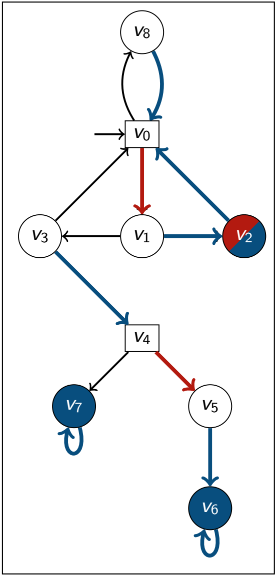
$\rightsquigarrow$ not an NE.

Different NEs may coexist



- $\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0} = (v_0 v_8)^\omega$
- Cost : $(+\infty, +\infty)$
- **NO player** visits his target set ...

- $\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0} = (v_0 v_1 v_2)^\omega$
- Cost : $(2, 2)$
- **BOTH players** visit their target set !



What is (for us) a relevant **Nash equilibrium** ?

Studied problems

1. **(Threshold decision problem)**

2. **(Social welfare decision problem)**
3. **(Pareto optimal decision problem)**

Studied problems

1. **(Threshold decision problem)** Given $(k_1, \ldots, k_n) \in (\mathbb{N} \cup \{+\infty\})^n$, does there exist an NE $(\sigma_1, \ldots, \sigma_n)$ such that, for all $1 \leq i \leq n$:

$$\mathsf{Cost}_i(\langle \sigma_1, \ldots, \sigma_n \rangle_{v_0}) \leq k_i.$$

Studied problems

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For NEs, in multiplayer quantitative reachability games, Problem 1 is **NP-complete**.

Key idea

Outcome characterization of a Nash equilibrium

Let ρ be a play,

there exists an NE $(\sigma_1, \dots, \sigma_n)$ such that $\langle \sigma_1, \dots, \sigma_n \rangle_{v_0} = \rho$

if and only if

ρ satisfies a "good" property.

Key idea

Outcome characterization of a Nash equilibrium

Let ρ be a play,

there exists an NE $(\sigma_1, \ldots, \sigma_n)$ such that $\langle \sigma_1, \ldots, \sigma_n \rangle_{v_0} = \rho$

if and only if

ρ satisfies a "good" property.

$\rightsquigarrow$ Does there exist a play ρ such that:

- for each player i, $\mathrm{Cost}_i(\rho) \leq k_i$;
- ρ satisfies a "good" property?

Algorithm (For NE)

1. it guesses a lasso of polynomial length;
2. it verifies that the cost profile of this lasso satisfies the conditions given by the problem;
3. it verifies that the lasso is the outcome of an NE.

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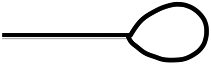
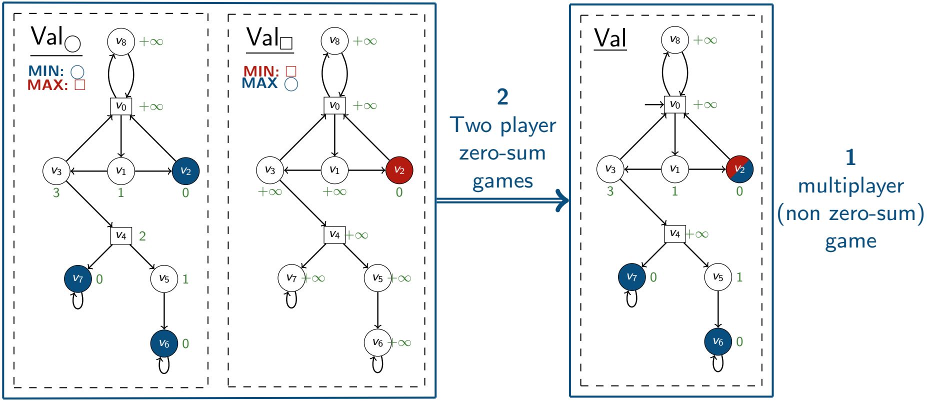
NP-algorithm for Problem 1:

- **Step 1:** if there exists an NE which satisfies the constraints, there exists one which also satisfies the constraints and such that its outcome is a **lasso** ($h\ell^\omega$) with a **polynomial length** ($|h\ell|$).

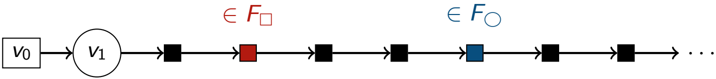


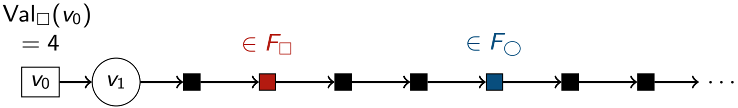
- **Step 2:** can be done in **polynomial time.**
- **Step 3:** checking the "good" property along the lasso of polynomial length can be done in **polynomial time.**

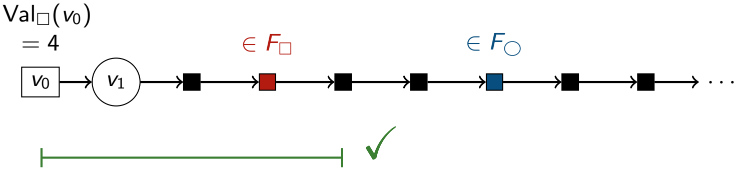
What is this “good” property ?

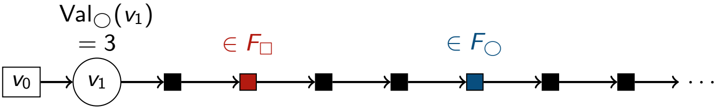


Values in quantitative two-player zero-sum games can be computed in **polynomial time** (see for example [BGHM17])

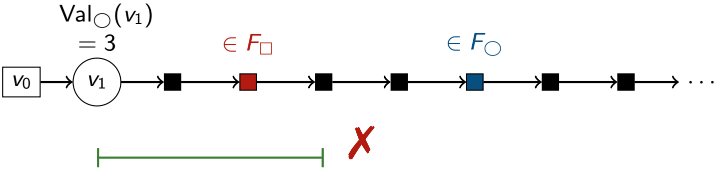

v_0
v_1
$\in F_\Box$
$\in F_\bigcirc$
...

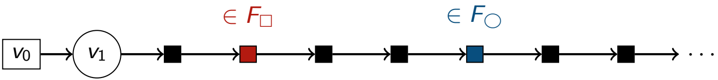

$\mathrm{Val}_{\square}(v_0)$
$= 4$
$\in F_{\square}$
$\in F_{\bigcirc}$
v_0
v_1
$\cdots$


$\mathrm{Val}_{\square}(v_0)$
$= 4$
$\in F_{\square}$
$\in F_{\bigcirc}$
v_0
v_1
$\cdots$
✓



$\mathsf{Val}_{\bigcirc}(v_1)$
$= 3$
$\in F_{\square}$
$\in F_{\bigcirc}$
v_0
v_1
$\cdots$

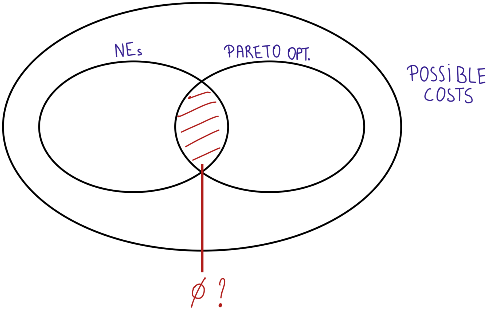

Val○(v1)
= 3
∈ F□
∈ F○
v0
v1
...



- ✓ ✓ ✓ ✓ ✓ . . . ⇝ outcome of an NE;
- ✓ ✓ ✗ ⇝ ~~outcome of an NE~~.

Conclusion and additional results

1. **(Threshold decision problem)**
2. **(Social welfare decision problem)**
3. **(Pareto optimal decision problem)**



Results

Complexity	Qual. Reach.		Quant. Reach.	
	NE	SPE	NE	SPE
Prob. 1	NP-c [CFGR16]	PSPACE-c[BBGR18]	NP-c	PSPACE-c[BBG+19]
Prob. 2	NP-c	PSPACE-c	NP-c	PSPACE-c
Prob. 3	NP-h/Σ_2^P	PSPACE-c	NP-h/Σ_2^P	PSPACE-c

Memory	Qual. Reach.		Quant. Reach.	
	NE	SPE	NE	SPE
Prob. 1	Poly.[CFGR16]	Expo.[BBGR18]	Poly.	Expo.
Prob. 2	Poly.	Expo.	Poly.	Expo.
Prob. 3	Poly.	Expo.	Poly.	Expo.

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