Ground states for the critical NLS on graphs

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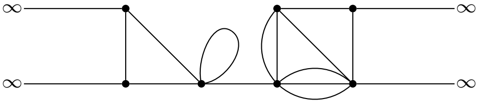
Joint work with Enrico Serra and Paolo Tilli

Proc. Mathematics Technology of Networks 2015,
Calc. Var. PDE 2015, arXiv:1505.03714, w.i.p.

Workshop on Nonlinear PDEs, Bruxelles, 7–11/9/2015

The problem

Let $\mathcal{G}$ be a noncompact metric graph



Consider the functional $E(\,\cdot\,,\mathcal{G}) : H^1(\mathcal{G}) \to \mathbb{R}$ defined by

$$E(u,\mathcal{G}) = \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{6}\|u\|^6_{L^6(\mathcal{G})} = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{1}{6}\int_{\mathcal{G}} |u|^6\,dx$$

and look for Ground States, i.e. global minimizers under the mass (or L^2) constraint.

A metric graph $\mathcal{G}$ is defined by a set V of *vertices* and a set E of *edges*

$$\mathcal{G} = (V, E)$$

$E \ni e = (\mathrm{v}_1, \mathrm{v}_2) \sim$ a line joining v_1 with v_2.

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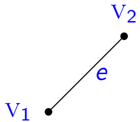
To an edge e joining two vertices v_1 and v_2 we associate either a closed interval

$$I_e = [0, \ell_e]$$

or a halfline

$$I_e = [0, +\infty)$$



$$\longmapsto \qquad [0, \ell_e] \subset \mathbb{R}$$

Functions on graphs

A function $u : \mathcal{G} \to \mathbb{C}$ is a bunch of functions $u = (u_e)_{e \in E}$, with

$$u_e : I_e \to \mathbb{C} \qquad \forall e \in E$$

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The usual function spaces can be defined as

$$L^p(\mathcal{G}) \;=\; \bigoplus_{e\in E} L^p(I_e), \qquad \|u\|^p_{L^p(\mathcal{G})} = \sum_{e\in E} \|u_e\|^p_{L^p(I_e)}$$

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$H^1(\mathcal{G})$ is the set of continuous $u = (u)_e$ such that

$$u_e \in H^1(I_e) \quad \forall e, \qquad \|u\|^2_{H^1(\mathcal{G})} = \sum_{e \in E} \|u_e\|^2_{H^1(I_e)}.$$

Continuity implies no jump at vertices (Kuchment 03, Exner-Berkolaiko 05) let $\mu > 0$ and define the space

$$H^1_\mu(\mathcal{G}) = \{u \in H^1(\mathcal{G}) \; : \; \|u\|^2_{L^2(\mathcal{G})} = \mu\}.$$

We look for

global minimizers, or ground states of mass μ

of the functional

$$E(u,\mathcal{G}) = \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{6}\|u\|^6_{L^6(\mathcal{G})}$$

Thus, defined

$$\mathcal{E}_\mathcal{G}(\mu) \; := \inf_{v \in H^1_\mu(\mathcal{G})} E(v,\mathcal{G}),$$

we are looking for functions $u \in H^1_\mu(\mathcal{G})$ such that

$$E(u,\mathcal{G}) = \mathcal{E}_\mathcal{G}(\mu).$$

N.B.: u can always be chosen as real non-negative valued.

Subcritical NLS on the line ($\mathcal{G} = \mathbb{R}$)

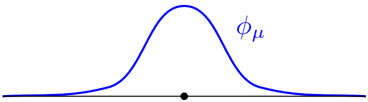
Consider the following functional defined on $H^1(\mathbb{R})$:

$$E_p(u, \mathbb{R}) = \frac{1}{2}\|u'\|_2^2 - \frac{1}{p}\|u\|_p^p$$

$2 < p < 6$ is the subcritical case.

There exist ground states for any value of μ: the solitons (Zakharov-Shabat 72).

$$\phi_\mu(x) = C\mu^{\frac{2}{6-p}} sech^{\frac{2}{p-2}}(c\mu^{\frac{p-2}{6-p}} x).$$



The dynamical problem is globally well-posed.
Stationary solutions are orbitally stable (Cazenave-Lions 82, Grillakis-Shatah-Strauss 87).

p=6: critical case and critical mass

- There exist ground states for $\mu = \mu_{\mathbb{R}}$ only.
- The dynamical problem is globally well-posed for all initial data with $\mu < \mu_{\mathbb{R}}$, while for $\mu \geq \mu_{\mathbb{R}}$ blow up solutions arise.
- Stationary solutions are orbitally unstable.

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We focus on the first aspect: ground state exist for a precise value of the mass only. More precisely,

$$\mathcal{E}_{\mathbb{R}}(\mu) = \begin{cases} -\infty & \text{if } \mu > \mu_{\mathbb{R}} \\ 0 & \text{if } \mu \leq \mu_{\mathbb{R}} \end{cases} \qquad \left(\mu_{\mathbb{R}} = \pi\sqrt{3}/2\right).$$

the minimum is attained only if $\mu = \mu_{\mathbb{R}}$ and the minimizers are the solitons

$$\phi_\lambda(x) = \sqrt{\lambda}\phi(\lambda x), \qquad \lambda > 0,$$

where $\phi(x) = sech^{1/2}(\frac{2}{\sqrt{3}}x)$

Interpretation of the critical mass $\mu_{\mathbb{R}}$

Using the Gagliardo–Nirenberg inequality

$$\|u\|_6^6 \leq K_{\mathbb{R}} \|u\|_2^4 \, \|u'\|_2^2, \qquad K_{\mathbb{R}} = \sup_{\substack{u \in H^1(\mathbb{R}) \\ u \not\equiv 0}} \frac{\|u\|_6^6}{\|u\|_2^4 \, \|u'\|_2^2}.$$

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one gets

$$E(u, \mathbb{R}) = \frac{1}{6} \left(3\|u'\|_2^2 - \|u\|_6^6 \right) \geq \frac{1}{6} \|u'\|_2^2 \left(3 - K_{\mathbb{R}} \mu^2 \right)$$

If $\mu^2 < 3/K_{\mathbb{R}}$, then $E(u, \mathbb{R}) > 0$, so transforming u as

$$u(x) \ \mapsto \ u_\lambda(x) = \sqrt{\lambda} u(\lambda x), \qquad E(u_\lambda, \mathbb{R}) = \lambda^2 E(u, \mathbb{R})$$

and letting $\lambda \to 0$ (stretching), one immediately sees that

$$\mathcal{E}_{\mathbb{R}}(\mu) = \inf_{u \in H^1_\mu(\mathbb{R})} E(u, \mathbb{R}) = 0.$$

On the other hand, if $\mu^2 > 3/K_{\mathbb{R}}$, and u is close to optimality in the Gagliardo–Nirenberg inequality,

$$E(u,\mathbb{R}) \leq \frac{1}{6}\|u'\|_2^2 \left(3 - (K_{\mathbb{R}} - \varepsilon)\mu^2\right) < 0,$$

and squeezing u as

$$u(x) \; \mapsto \; \sqrt{\lambda} u(\lambda x), \quad \lambda \to +\infty$$

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$$\mathcal{E}_{\mathbb{R}}(\mu) = \inf_{u \in H^1_\mu(\mathbb{R})} E(u, \mathbb{R}) = -\infty.$$

Then we see that

$$\mu_{\mathbb{R}}^2 = \frac{3}{K_{\mathbb{R}}}.$$

The halfline

Gagliardo-Nirenberg estimate holds for the halfline too

$$\|u\|_{L^6(\mathbb{R}^+)}^6 \leq K_{\mathbb{R}^+} \|u\|_{L^2(\mathbb{R}^+)}^4 \, \|u'\|_{L^2(\mathbb{R}^+)}^2, \qquad K_{\mathbb{R}^+} = 4K_{\mathbb{R}^+}$$

Therefore, proceeding as in the case of the line, we have that

$$\mu_{\mathbb{R}^+} = \frac{1}{2}\mu_{\mathbb{R}}$$

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Two key ingredients:

1. Gagliardo-Nirenberg inequalities
2. Stretching-squeezing

Gagliardo-Nirenberg estimates on graphs

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Gagliardo-Nirenberg estimates on graphs

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Given $u \in H^1_\mu(\mathcal{G})$,

1. it is always possible to construct its *monotone rearrangement* $u^* \in H^1_\mu(\mathbb{R}^+)$ s.t.

$$\|u^*\|_{L^p(\mathbb{R}^+)} = \|u\|_{L^p(\mathcal{G})}, \qquad \|(u^*)'\|_{L^p(\mathbb{R}^+)} \leq \|u'\|_{L^p(\mathcal{G})}$$

   then

$$\begin{aligned} \|u\|^6_{L^6(\mathcal{G})} =& \|u^*\|^6_{L^6(\mathbb{R}^+)} \leq K_{\mathbb{R}^+} \|u^*\|^4_{L^2(\mathbb{R}^+)} \|(u^*)'\|^2_{L^2(\mathbb{R}^+)} \\ \leq & K_{\mathbb{R}^+} \|u\|^4_{L^2(\mathcal{G})} \|u'\|^2_{L^2(\mathcal{G})} \end{aligned}$$

2. if $\sharp u^{-1}(t) \geq 2$ for almost every $t \in \mathrm{Ran}\, u$, then the *symmetric rearrangement* $\widehat{u} \in H^1_\mu(\mathbb{R})$ of u satisfies

$$\|\widehat{u}\|_{L^p(\mathbb{R})} = \|u\|_{L^p(\mathcal{G})}, \qquad \|\widehat{u}'\|_{L^p(\mathbb{R})} \leq \|u'\|_{L^p(\mathcal{G})}$$

   Thus

$$\begin{aligned} \|u\|^6_{L^6(\mathcal{G})} =& \|\widehat{u}\|^6_{L^6(\mathbb{R})} \leq K_{\mathbb{R}} \|\widehat{u}\|^4_{L^2(\mathbb{R})} \|\widehat{u}'\|^2_{L^2(\mathbb{R})} \\ \leq & K_{\mathbb{R}} \|u\|^4_{L^2(\mathcal{G})} \|u'\|^2_{L^2(\mathcal{G})} \end{aligned}$$

Furthermore, the optimizer on the line can be arbitrarily well approximated on an infinite edge of $\mathcal{G}$, then

$$K_{\mathbb{R}} \leq K_{\mathcal{G}} \leq K_{\mathbb{R}^+}$$

In analogy to the case of $\mathbb{R}$, we define the critical mass as

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Remark. The critical mass for the graphs interpolates between those of the halfline and of the line.

$$\mu_{\mathbb{R}^+} \le \mu_{\mathcal{G}} \le \mu_{\mathbb{R}}.$$

But is its meaning the same?

Stretching/squeezing

The second crucial ingredient for solving the problem on $\mathbb{R}$ was the mass-preserving transformation

$$u(x) \mapsto u_\lambda(x) = \sqrt{\lambda}u(\lambda x)$$

Notice that energy transforms as

$$E(u_\lambda, \lambda^{-1}\mathcal{G}) = \lambda^2 E(u, \mathcal{G})$$

If $\mathcal{G} = \mathbb{R}$, then $\lambda^{-1}\mathcal{G} = \mathcal{G}$ and the scaling can be used in order to stretch or squeeze the function.

But $\lambda^{-1}\mathcal{G} = \mathcal{G}$ if and only if $\mathcal{G}$ is a star graph.

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So that, the strategy used for the line cannot work for a generic graph. More precisely,

- It is always possible to stretch a function supported on a half-line, obtaining $\mathcal{E}_\mathcal{G}(\mu) = 0$ If $\mu \leq \mu_\mathcal{G}$.
- Conversely, it is not always possible to shrink a function on $\mathcal{G}$, so one cannot immediately conclude that $\mathcal{E}_\mathcal{G}(\mu) = -\infty$ if $\mu > \mu_\mathcal{G}$.

Results

Theorem (1)

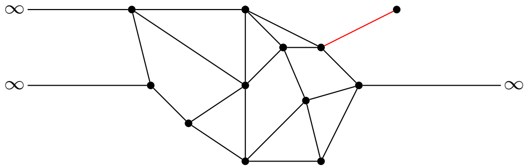
Let $\mathcal{G}$ be a noncompact graph having at least one terminal edge. Then

$$\mu_{\mathcal{G}} = \mu_{\mathbb{R}^+}$$

and

$$\mathcal{E}_{\mathcal{G}}(\mu) = \begin{cases} -\infty & \text{if} \quad \mu > \mu_{\mathbb{R}^+} \\ 0 & \text{if} \quad \mu \leq \mu_{\mathbb{R}^+}. \end{cases}$$

A ground state exists if and only if $\mu = \mu_{\mathbb{R}^+}$ and $\mathcal{G}$ is a half-line.



Idea: The terminal edge can host half a (squeezed) soliton by itself.

Theorem (2)

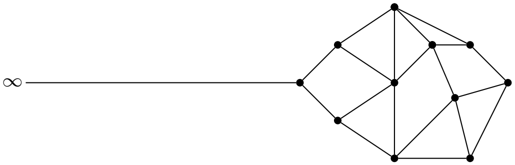
Let $\mathcal{G}$ be a noncompact graph having exactly one half-line and no terminal edge. Then

$$\mu_{\mathcal{G}} = \mu_{\mathbb{R}^+}$$

and

$$\mathcal{E}_{\mathcal{G}}(\mu) \begin{cases} = -\infty & \text{if} \quad \mu > \mu_{\mathbb{R}} \\ < 0 & \text{if} \quad \mu_{\mathbb{R}^+} < \mu \leq \mu_{\mathbb{R}} \\ = 0 & \text{if} \quad \mu \leq \mu_{\mathbb{R}^+}. \end{cases}$$

A ground state exists if and only if $\mu_{\mathbb{R}^+} < \mu \leq \mu_{\mathbb{R}}$.



Idea: the graph can host a (stretched) soliton that sees the halfline only.

A topological hypothesis

Consider the following assumption.

(H) Every $x \in \mathcal{G}$ lies on a *trail* that contains two half-lines

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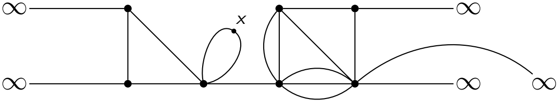

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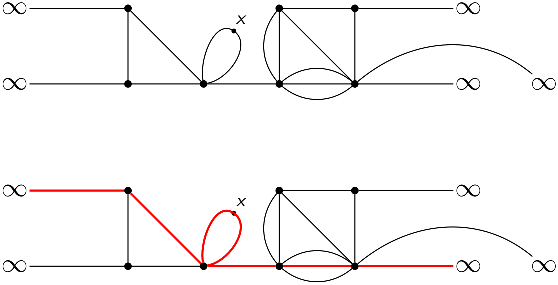


Theorem (3)

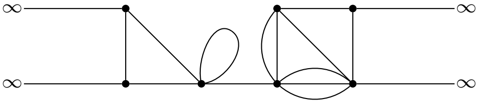
Let $\mathcal{G}$ be a noncompact graph satisfying assumption (H). Then

$$\mu_{\mathcal{G}} = \mu_{\mathbb{R}}$$

and

$$\mathcal{E}_{\mathcal{G}}(\mu) = \begin{cases} -\infty & \text{if} \quad \mu > \mu_{\mathbb{R}} \\ 0 & \text{if} \quad \mu \leq \mu_{\mathbb{R}}. \end{cases}$$

A ground state exists if and only if $\mu = \mu_{\mathbb{R}}$ and $\mathcal{G}$ is a "tower of bubbles".



Theorem (4)

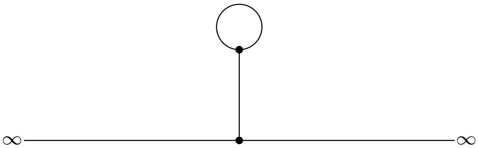
There exist noncompact graphs $\mathcal{G}$, without terminal edges, with more than one half-line and that do not satisfy assumption (H), such that

$$\mu_{\mathbb{R}^+} < \mu_{\mathcal{G}} < \mu_{\mathbb{R}}$$

and

$$\mathcal{E}_{\mathcal{G}}(\mu) \begin{cases} = -\infty & \text{if} \quad \mu > \mu_{\mathbb{R}} \\ < 0 & \text{if} \quad \mu_{\mathcal{G}} < \mu \leq \mu_{\mathbb{R}} \\ = 0 & \text{if} \quad \mu \leq \mu_{\mathcal{G}}. \end{cases}$$

A ground state exists if and only if $\mu_{\mathcal{G}} \leq \mu \leq \mu_{\mathbb{R}}$.



Comments.

- In theorems 1 and 3 the situation is similar to that of $\mathbb{R}$: a ground state exists for a single value of the mass and moreover the graph is forced to have a particular structure (half-line, tower of bubbles...).

- On the other hand theorems 2 and 4 describe a completely new phenomenon: ground states exist for all values of μ in a nontrivial interval. This is due to the different topology of certain graphs with respect to that of $\mathbb{R}$.

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We now sketch a "cumulative" proof of (part of) theorems 2 and 4.

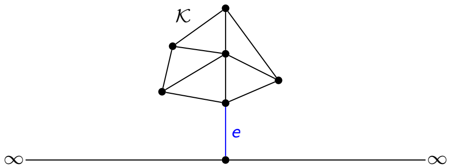
Proof (ideas) in the case $\mu_{\mathcal{G}} < \mu < \mu_{\mathbb{R}}$, so that $\mathcal{E}_{\mathcal{G}}(\mu) < 0$.

First we obtain bounds for a minimizing sequence u_n in the relevant norms.

Then we pass to the limit in $E(u_n, \mathcal{G})$.

Bounds

1. Since $\mathcal{G}$ does not satisfy (H), there exists a cut-edge e and a compact connected component $\mathcal{K}$ of $\mathcal{G} \setminus e$.



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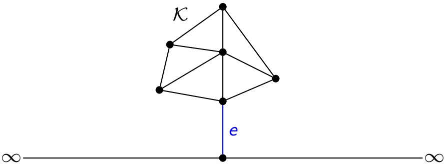
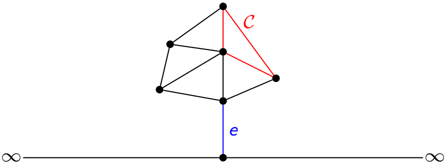


2. $\mathcal{K}$ cannot be a vertex, otherwise e would be a terminal edge, excluded by assumption.

3. $\mathcal{K}$ must contain a cycle $\mathcal{C}$, otherwise $\mathcal{K}$ would be a tree, and have a terminal edge.



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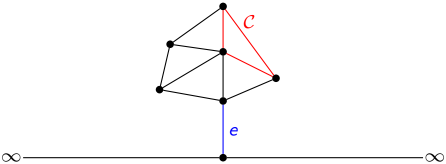

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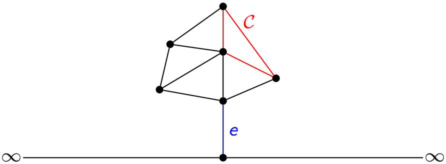

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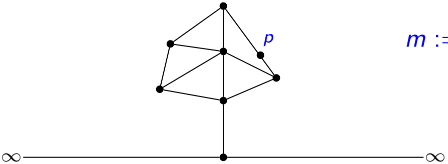
5. For every $u \in H^1_\mu(\mathcal{G})$, let p be an absolute minimum point for u on $\mathcal{C}$, and let $m = u(p)$.



$m := u(p)$

6. Note that $m^2|\mathcal{C}| \leq \int_{\mathcal{C}} |u|^2\, dx \leq \int_{\mathcal{G}} |u|^2\, dx = \mu$, so

$$m^2 \leq \mu|\mathcal{C}|^{-1} =: c\mu.$$

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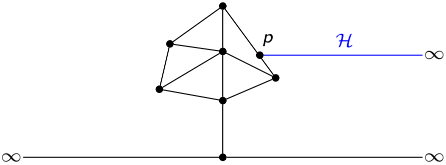
7. We attach a half-line $\mathcal{H}$ to $\mathcal{G}$ at p, and call $\mathcal{G}'$ the new graph, that now satisfies (H).



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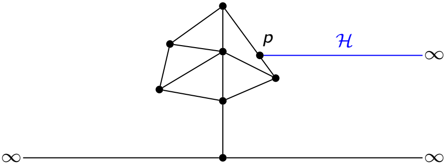

8. We extend u to $\mathcal{G}'$ by setting

$$w(x) = \begin{cases} u(x) & \text{if} \quad x \in \mathcal{G} \\ me^{-x/2\varepsilon} & \text{if} \quad x \in \mathcal{H} \end{cases}$$

9. Clearly $w \in H^1(\mathcal{G}')$ and

$$\|w\|_2^2 = \mu + \varepsilon m^2 \leq \mu(1 + \varepsilon c)$$

$$\|w'\|_2^2 = \|u'\|_2^2 + \frac{m^2}{\varepsilon} \leq \|u'\|_2^2 + \frac{c\mu}{\varepsilon}$$

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$$\|w'\|_2^2 = \|u'\|_2^2 + \frac{m^2}{\varepsilon} \le \|u'\|_2^2 + \frac{c\mu}{\varepsilon}$$

10. If $E(u, \mathcal{G}) < 0$, then $3\|u'\|_2^2 < \|u\|_6^6$ so that by the Gagliardo-Nirenberg inequality

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$$= 3 \frac{\mu^2}{\mu_{\mathbb{R}}^2} \, (1+\varepsilon c)^2 \left(\|u'\|_2^2 + \frac{c\mu}{\varepsilon} \right).$$

Thus,

$$\|u'\|_2^2 \leq \frac{\mu^2}{\mu_{\mathbb{R}}^2}(1+\varepsilon c)^2 \left(\|u'\|_2^2 + \frac{\mu}{\varepsilon}c\right)$$

for every $u \in H^1_\mu(\mathcal{G})$ such that $E(u,\mathcal{G}) < 0$.

Let

$$\theta = \frac{\mu^2}{\mu_{\mathbb{R}}^2}(1+\varepsilon c)^2$$

and note that $0 < \theta < 1$ if ε is chosen small.

Then

$$(1-\theta)\|u'\|_2^2 \leq \theta \frac{c\mu}{\varepsilon}.$$

Conclusion: there exists $C > 0$ such that

$$\begin{cases} u \in H^1_\mu(\mathcal{G}) \\ \\ E(u, \mathcal{G}) < 0 \end{cases} \quad \Longrightarrow \quad \|u'\|_2 \leq C.$$

It is then easy to obtain further estimates like

$$\|u\|_6 \leq C, \quad \|u\|_\infty \leq C, \ldots .$$

from which we also see that

$$\mathcal{E}_{\mathcal{G}}(\mu) > -\infty.$$

Limit

Let $u_n \in H^1_\mu(\mathcal{G})$ be a minimizing sequence:

$$E(u_n, \mathcal{G}) \to \mathcal{E}_{\mathcal{G}}(\mu) < 0.$$

By the previous estimates we can assume that

$$u_n \rightharpoonup u \quad \text{in } H^1(\mathcal{G})$$

$$u_n \to u \quad \text{in } L^q_{\mathrm{loc}}(\mathcal{G}) \quad \forall q \in [1 + \infty]$$

$$u_n(x) \to u(x) \quad a.e.$$

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5. Hence,
$$E(u_n, \mathcal{G}) \geq E(u, \mathcal{G}) + o(1),$$
that is,
$$E(u, \mathcal{G}) = \mathcal{E}_{\mathcal{G}}(\mu).$$

Conclusion. If u has mass $m < \mu$, then the function $v = \sqrt{\mu/m}\, u$ has mass μ and

$$E(v, \mathcal{G}) < \frac{\mu}{m} E(u, \mathcal{G}) < \mathcal{E}_{\mathcal{G}}(u),$$

a contradiction.

Therefore u has mass μ and is the required ground state.