Positive radial solutions of a prescribed mean curvature equation in Lorentz-Minkowski space

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Workshop in Nonlinear PDEs BRUXELLES 2015
Presentation plan

1. Introduction
2. Bonheure-De Coster-Derlet results
3. Problems left open by BDD
4. Supercritical case: multiplicity result
5. Subcritical case: nonexistence result
6. Open problems
The equation

We are interested in the following quasilinear equation

$$\nabla \cdot \left[ \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right] + u^p = 0.$$  \hspace{1cm} (1)

It is known as

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The M-C operator

The mean curvature operator $\nabla \cdot \left[ \frac{\nabla(\cdot)}{\sqrt{1 - |\nabla(\cdot)|^2}} \right]$ appears in physical and geometrical contexts and precisely:

- Classical relativity
  Born-Infeld theory of electrodynamics
- Riemannian Geometry
  Maximal or constant mean curvature hypersurfaces

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The problem: ground state solutions

Our object is to determine conditions for the existence of solutions to the problem

\[ \begin{cases} \nabla \cdot \left[ \frac{\nabla u}{\sqrt{1-|\nabla u|^2}} \right] + u^p = 0, \\ u(x) > 0, \quad \text{in } \mathbb{R}^N, \quad N \geq 3, \\ u(x) \to 0, \quad \text{as } |x| \to \infty. \end{cases} \tag{P_+} \]

A solution to \((P_+)\) is usually called ground state.
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When we look for radial ground state solutions, the problem can be reformulated as follows:

is there any $\xi > 0$ such that the Cauchy problem

$$\begin{cases}
\left( \frac{u'}{\sqrt{1-(u')^2}} \right)' + \frac{N-1}{r} \frac{u'}{\sqrt{1-(u')^2}} + |u|^{p-1}u = 0 \\
u'(0) = 0, \\
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has a (global in $\mathbb{R}_+$) positive solution (vanishing at infinity)?
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What is already known

There is only one existence result on problem \((P_+)\)

**Theorem (Bonheure-De Coster-Derlet, 2012)**

If \(p > \frac{N+2}{N-2}\) then \((P_+)\) has a (radial) solution in \(D^{1,2}(\mathbb{R}^N)\).
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**Theorem (Bonheure-De Coster-Derlet, 2012)**

If \( p > \frac{N+2}{N-2} \), then the prescribed mean curvature equation (1) has infinitely many (radial) solutions in \( D^{1,2} (\mathbb{R}^N) \).
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*If* \( p > \frac{N+2}{N-2} \), *then the prescribed mean curvature equation (1) has infinitely many (radial) solutions in* \( D^{1,2}(\mathbb{R}^N) \).*
Supercritical case and subcritical case

Then, when $p > \frac{N+2}{N-2}$,

- There exists a radial ground state for the prescribed mean curvature equation.
- There exist infinitely many solutions to the prescribed mean curvature equation, with no information on the sign.

Nothing is known when $p \leq \frac{N+2}{N-2}$. 

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Existence and uniqueness

Questions

1. Is the ground state solution unique when \( p > \frac{N+2}{N-2} \)?
2. There exist ground state solutions when \( p \leq \frac{N+2}{N-2} \)?
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Multpicity result and decaying properties

Theorem (A., 2015)

If $p > \frac{N+2}{N-2}$, then there exist infinitely many radial solutions to $(\mathcal{P}_+)$ not belonging to $\mathcal{D}^{1,2}(\mathbb{R}^N)$. 
Multiplicity result and decaying properties

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Multiplicity result and decaying properties

**Theorem**

If $p > \frac{N+2}{N-2}$, we have the following decaying estimates for the radial ground states:

- $u \in D^{1,2}(\mathbb{R}^N)$ iff $u(x) = O(1/|x|^{N-2})$ for $|x| \to +\infty$;
- $u \notin D^{1,2}(\mathbb{R}^N)$ iff there exist $c_1, c_2 > 0$ such that
  $$
  \frac{c_1}{|x|^{\frac{2N}{N-1}(p+1)-2N}} \leq u(x) \leq \frac{c_2}{|x|^{p-1}} \text{ definitely for } |x| \to +\infty.
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Moreover, in the second case, there exists no $\alpha > \frac{2}{p-1}$ such that, definitely, $u(r) \leq c/r^\alpha$ for some $c > 0$. 
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If \( p > \frac{N+2}{N-2} \), we have the following decaying estimates for the radial ground states:

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Moreover, in the second case, there exists no \( \alpha > 2/(p-1) \) such that, definitely, \( u(r) \leq c/r^\alpha \) for some \( c > 0 \).
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Proof of multiplicity result

Proof

It is easy to see that any solution of (C) is global and either it is sign-changing or it is a ground state.

Steps:

- We take any initial datum \( \xi > 0 \) sufficiently small (smaller than a suitable \( \bar{\xi} > 0 \)).
- We assume (by contradiction) that there exists a point where the solution vanishes.
- We show a Pucci-Serrin identity is violated.
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Nonexistence result

**Theorem**

*If* $1 < p < \frac{N+2}{N-2}$, *then there exists no radial solution to* $(\mathcal{P}_+)$.

**Proof**

Steps:

- We assume by contradiction the existence of a radial ground state solving $(C)$ for a certain $\bar{\xi} > 0$.
- We prove that all solutions of $(C)$ corresponding to $\xi \in [0, \bar{\xi})$ are ground states.
- We prove the existence of $\tilde{\xi} > 0$ such that all solutions of $(C)$ corresponding to $\xi \in [0, \tilde{\xi})$ are sign changing.
- We conclude.
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If \( 1 < p < \frac{N+2}{N-2} \), then there exists no radial solution to \((P_+)\).

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Open problems

1. Is the [BDD] solution the unique ground state with finite energy (namely in $D^{1,2}$) for $p > \frac{N+2}{N-2}$?

2. Do infinite energy radial ground states decay exactly as $1/r^{p-1}$?

3. What happens if $p = \frac{N+2}{N-2}$?

4. Are there non radial ground state solutions for our equation?
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