Positive radial solutions of a prescribed mean curvature equation in Lorentz-Minkowski space

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Presentation plan

1. Introduction
2. Bonheure-De Coster-Derlet results
3. Problems left open by BDD
4. Supercritical case: multiplicity result
5. Subcritical case: nonexistence result
6. Open problems

The equation

We are interested in the following quasilinear equation

$$\nabla \cdot \left[\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right] + u^p = 0. \qquad (1)$$

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The mean curvature operator $\nabla \cdot \left[\frac{\nabla(\cdot)}{\sqrt{1-|\nabla(\cdot)|^2}}\right]$ appears in physical and geometrical contexts and precisely

- Classical relativity
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The problem: ground state solutions

Our object is to determine conditions for the existence of solutions to the problem

$$\begin{cases} \nabla \cdot \left[\frac{\nabla u}{\sqrt{1-|\nabla u|^2}} \right] + u^p = 0, \\ u(x) > 0, \quad \text{in } \mathbb{R}^N, \ N \geq 3, \\ u(x) \to 0, \quad \text{as } |x| \to \infty. \end{cases} \qquad (\mathcal{P}_+)$$

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is there any $\xi > 0$ such that the Cauchy problem

$$\begin{cases} \left(\frac{u'}{\sqrt{1-(u')^2}}\right)' + \frac{N-1}{r}\frac{u'}{\sqrt{1-(u')^2}} + |u|^{p-1}u = 0 \\ u'(0) = 0, \\ u(0) = \xi \end{cases} \qquad (\mathcal{C})$$

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There is only one existence result on problem $(\mathcal{P}_+)$

Theorem (Bonheure-De Coster-Derlet, 2012)

If $p > \frac{N+2}{N-2}$ then $(\mathcal{P}_+)$ has a (radial) solution in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

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Supercritical case and subcritical case

Then, when $p > \frac{N+2}{N-2}$,

- There exists a radial ground state for the prescribed mean curvature equation
- There exist infinitely many solutions to the prescribed mean curvature equation, with no information on the sign

Nothing is known when $p \leq \frac{N+2}{N-2}$.

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Existence and uniqueness

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Multiplicity result and decaying properties

Theorem (A., 2015)

If $p > \frac{N+2}{N-2}$, then there exist infinitely many radial soutions to $(\mathcal{P}_+)$ not belonging to $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

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Multiplicity result and decaying properties

Theorem

If $p > \frac{N+2}{N-2}$, we have the following decaying estimates for the radial ground states:

1. $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ *iff* $u(x) = O(1/|x|^{N-2})$ *for* $|x| \to +\infty$;
2. $u \notin \mathcal{D}^{1,2}(\mathbb{R}^N)$ *iff there exist* $c_1, c_2 > 0$ *such that* $c_1/|x|^{\frac{2N}{(N-1)(p+1)-2N}} \le u(x) \le c_2/|x|^{\frac{2}{p-1}}$ *definitely for* $|x| \to +\infty$.

Moreover, in the second case, there exists no $\alpha > 2/(p-1)$ such that, definitely, $u(r) \le c/r^\alpha$ for some $c > 0$.

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Proof of multiplicity result

Proof

It is easy to see that any solution of $(\mathcal{C})$ is global and either it is sign-changing or it is a ground state.

Steps:

- we take any initial datum $\xi > 0$ sufficiently small (smaller than a suitable $\bar{\xi} > 0$);
- we assume (by contradiction) that there exists a point where the solution vanishes;
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Nonexistence result

Theorem

If $1 < p < \frac{N+2}{N-2}$, then there exists no radial solution to $(\mathcal{P}_+)$.

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Steps:

- we assume by contradiction the existence of a radial ground state solving $(\mathcal{C})$ for a certain $\bar{\xi} > 0$,
- we prove that all solutions of $(\mathcal{C})$ corresponding to $\xi \in]0, \bar{\xi}]$ are ground states,
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Open problems

1. Is the [BDD] solution the unique ground state with finite energy (namely in $\mathcal{D}^{1,2}$) for $p > \frac{N+2}{N-2}$?
2. Do infinite energy radial ground states decay exactly as $1/r^{\frac{2}{p-1}}$?
3. What happens if $p = \frac{N+2}{N-2}$?
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