Average conditions for permanence in nonautonomous competitive systems with nonlocal dispersal.

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Workshop in Nonlinear PDEs, Brussels
September 7 – 11, 2015

Definitions and assumptions

By the nonautonomous competitive system of partial differential equations (PDEs) with dispersal we mean

$$\frac{\partial u_i}{\partial t} = \rho_i\Big(\int_\Omega K_i(x,y)u_i(t,y)dy - u_i(t,x)\Big) + f_i(t,x,u_1\dots,u_N)u_i,$$
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- $f_i(t,x,u_1,\ldots,u_N)$ – local per capita growth rate of the i-th species,
- $K(\cdot,\cdot)$ is nonlocal convolution kernel satisfying the following assumption

(A1) $K_i(\cdot,\cdot) : \bar{\Omega} \times \bar{\Omega} \to R$ are C^1- functions, $\int_\Omega K_i(x,y) \leq 1$ for any $x \in \bar{\Omega}$, $\int_\Omega K_i(x,y) \neq 1$ and there is a $\delta_0 > 0$ such that for any $x \in \bar{\Omega}$, $K_i(x,y) > 0$ for $y \in \bar{\Omega}$ and $||x-y|| < \delta_0$

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System (D) is the nonlocal dispersal counterpart of the following Kolmogorov competition system with random dispersal and Dirichlet boundary conditions

$$\begin{cases} \frac{\partial u_i}{\partial t} = \rho_i \Delta u_i + f_i(t,x,u_1\dots,u_N)u_i, \quad t \geq 0, x \in \Omega, \quad i = 1,\dots,N \\ u_i = 0 \quad x \in \partial\Omega \end{cases}$$

Denote by λ_i the principal eigenvalue of the problem

$$\begin{cases} \int_\Omega K_i(x,y)\varphi_i(y)dy - \varphi_i(x) = \lambda_i \varphi_i(x) \quad \text{on} \quad \Omega \\ u_i = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

Notice that principal eigenfunction φ_i corresponding λ_i can be chosen so that $\varphi_i(x) > 0$ for all $x \in \bar{\Omega}$. Throughout this paper we assume that $\lambda_i \geq 0$.

We deal with the positive solutions

Definition

The solution $u(t,x) = (u_1(t,x), \ldots, u_N(t,x))$ of system (D) is *positive* if $u_i(t,x) > 0$ for all $i = 1, \ldots, N$, $t \in (0, \tau_{\max})$ and $x \in \Omega$.

Now we introduce a following assumptions for a function f_i

(A2) $f_i : [0,\infty) \times \bar{\Omega} \times [0,\infty)^N \to \mathbb{R}$ $(1 \leq i \leq N)$ as well as their first derivatives $\frac{\partial f_i}{\partial t}$ $(1 \leq i \leq N)$, $\frac{\partial f_i}{\partial u_j}$ $(1 \leq i, j \leq N)$, $\frac{\partial f_i}{\partial x_k}$ $(1 \leq k \leq n,\ 1 \leq i, j, \leq N)$ are continuous. Moreover, the derivatives $\frac{\partial f_i}{\partial u_j}$ $(1 \leq i, j \leq N)$ are bounded and uniformly continuous on sets of the form $[0,\infty) \times \bar{\Omega} \times B$ where B is a bounded subset of $[0,\infty)^N$.

(A3) The functions $[[0,\infty) \times \bar{\Omega}] \ni (t,x) \mapsto f_i(t,x,0,\dots,0) \in \mathbb{R}]$, $1 \leq i \leq N$ are bounded.

We write

$$\underline{a}_i = \inf\{f_i(t,x,0,\dots,0) : t \geq 0, x \in \bar{\Omega}\}$$

$$\overline{a}_i = \sup\{f_i(t,x,0,\dots,0) : t \geq 0, x \in \bar{\Omega}\}.$$

(A4) $\frac{\partial f_i}{\partial u_j}(t,x,u) \leq 0$ for all $t \geq 0$ $x \in \bar{\Omega}$, $u \in [0,\infty)^N$, $1 \leq i, j, \leq N$, $i \neq j$.

$\frac{\partial f_i}{\partial u_j}(t,x,u_1,\dots,u_N)$ measures the influence of the jth species on the growth rate of the ith species.
The systems which satisfy assumption (A4) we call *competitve*.

(A5) There exist $\underline{b}_{ii} > 0$ such that $\frac{\partial f_i}{\partial u_i}(t,x,u) \leq -\underline{b}_{ii}$ for all $t \geq 0$, $x \in \bar{\Omega}$ $1 \leq i \leq N$, $u \in [0,\infty)^N$.

Let $u(t,x) = (u_1(t,x),\dots,u_N(t,x))$, $t \in [0,\tau_{\max})$ be a positive solution of (D).
For each $1 \leq i \leq N$ we define $\xi_i(t)$, $t \in [0,\infty)$ to be the positive solution of the ordinary differential equations

$$\begin{cases} \xi_i'(t) = (\rho_i\lambda_i + \max_{x\in\bar{\Omega}} f_i(t,x,0,\dots,0) - \underline{b}_{ii}\xi_i(t))\xi(t) \\ \xi_i(0) = \sup_{x\in\bar{\Omega}} \frac{u_i(0,x)}{\varphi_i(x)} \end{cases} \tag{1}$$

Lemma 1

If $\xi_i : [0, \tau_{\max}) \to E$ is maximally defined solution of (1) such that $\xi_i(0) > 0$ for $i = 1, \dots, N$ then

(i) $\xi_i(t) > 0$ for $i = 1, \dots, N$, $t \in [0, \tau_{\max})$,

(ii) $\tau_{\max} = \infty$,

(iii) $\limsup_{t\to\infty} \xi_i(t) \le \frac{\rho_i \lambda_i + \overline{a_i}}{\underline{b}_{ii}}$ for $i = 1, \dots, N$

Lemma 2

Assume (A2) - (A5). Then for any positive solution $u(t,x) = (u_1)(t,x), \dots, u_N(t,x))$ of the equation (D) there holds

$$u_i(t,x) \le \xi_i(t)\varphi_i(x) \quad t \in [0, \tau_{\max}), \quad x \in \bar{\Omega}, \quad i = 1, \dots, N$$

where $\xi_i(t)$ is the solution of the IVP (1)

Lemma 3

Assume (A2) - (A5). Then for any maximally defined positive solution $u(t,x) = (u_1(t,x), \ldots, u_N(t,x))$ of (D) we have
(i) $\tau_{\max} = \infty$
(ii) $\limsup_{t\to\infty} u_i(t,x) \leq \frac{\rho_i\lambda_i + \overline{a}_i}{\underline{b}_{ii}}$, $i = 1, \ldots, N$ where the limit is uniform in $x \in \bar{\Omega}$

(A6) There exist $\overline{b}_{ij} \geq 0$ such that $\frac{\partial f_i}{\partial u_j}(t,x,u) \geq -\overline{b}_{ij}$ for all $t \geq 0$, $x \in \bar{\Omega}$ $1 \leq i,j \leq N$, $u \in [0,\infty)^N$.

Averaging

Definition

We define a *lower average* of a function f_i as

$$m[f_i] := \liminf_{t-s\to\infty} \frac{1}{t-s} \int_s^t \min_{x\in\bar{\Omega}} f_i(\tau, x, 0, \dots, 0) d\tau$$

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(A7) $m[f_i] > 0$.

Permanence in systems of PDEs with nonlocal dispersal

Definition

System (D) is *permanent* if there are positive constants $\underline{\delta}_i, \overline{\delta}_i > 0$ such that for each positive solution $u(t,x) = (u_1(t,x), \ldots, u_N(t,x))$ of (D) there holds

$$\underline{\delta}_i \leq \liminf_{t\to\infty} \frac{u_i(t,x)}{\varphi_i(x)} \leq \limsup_{t\to\infty} \frac{u_i(t,x)}{\varphi_i(x)} \leq \overline{\delta}_i \quad 1 \leq i \leq N \qquad \text{(permanence)}$$

where the limit is uniform in $x \in \Omega$.

Average conditions for evolution system of (D)

$$m[f_i] > \rho_i \lambda_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\overline{b}_{ij}(M[f_j] + \rho_j \lambda_j)}{\underline{b}_{jj}} \quad 1 \leq i \leq N \qquad \text{(A)}$$

Theorem 1 [Main Theorem]

Assume **(A1)** through **(A7)**. If **(A)** holds then system (D) is permanent.

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- J. Balbus ,*Permanence in nonautonomous competitive systems with nonlocal dispersal*, submitted for publication.

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The following result will be useful to prove Theorem 1.

Permanence in logistic equation of ODEs

Theorem 2 [Vance - Coddington Estimates]

Let $c\colon [t_0, \infty) \to \mathbb{R}$, where $t_0 \geq 0$, be a bounded continuous function, where $c_* > 0$ and $c^* > 0$ are such that $-c_* \leq c(t) \leq c^*$ for all $t \geq t_0$, and let $d > 0$. Assume moreover that there are $L > 0$ and $\beta > 0$ such that

$$\frac{1}{L} \int_{t}^{t+L} c(\tau)\, d\tau \geq \beta$$

for all $t \geq t_0$.

Theorem 2 [Vance - Coddington Estimates] continued

Then for any solution $\zeta(t)$ of the initial value problem

$$\begin{cases} \zeta' = (c(t) - d\zeta)\zeta \\ \zeta(t_0) = \zeta_0, \end{cases}$$

where $\zeta_0 > 0$, there holds

$$\frac{\beta}{d} e^{-L(c_* + \beta)} \leq \liminf_{t\to\infty} \zeta(t) \leq \limsup_{t\to\infty} \zeta(t) \leq \frac{c^*}{d}.$$

(permanence-logistic)

Theorem 2 [Vance - Coddington Estimates] continued

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- R. R. Vance and E. A. Coddington, *A nonautonomous model of population growth*, J. Math. Biol. **27** (1989), no. 5, 491–506.

proof of Theorem 1

The right-hand side of the inequality (permanence) is satisfied by Lemma 3 (ii). Fix a positive solution $u(t,x) = (u_1(t,x), \ldots, u_N(t,x))$ of system (D). Let $\xi_i(t)$, $1 \leq i \leq N$, $t \geq 0$, be the solutions of (1). Fix $1 \leq i \leq N$.

sketch of the proof of Theorem 1 [continued]

Let $\eta_i(t)$, $t \geq t_0$, be the positive solution of the following problem

$$\begin{cases} \eta_i'(t) = (\rho_i \lambda_i + \min\limits_{x \in \bar{\Omega}} f_i(t, x, 0, \ldots, 0) - \overline{b}_{ii}\eta_i - \sum\limits_{\substack{j=1 \\ j \neq i}}^{N} \overline{b}_{ij}\xi_j(t))\eta_i(t) \\ \eta_i(t_0) = \inf\limits_{x \in \Omega} \frac{u_i(t_0, x)}{\varphi_i(x)}. \end{cases}$$

It is easy to see that $u_i(t, x) \geq \eta_i(t)\varphi_i(x)$ for all $t \geq t_0$ and $x \in \bar{\Omega}$.

sketch of the proof of Theorem 1 [continued]

Now we apply Theorem 2 to (D) where

$$c(t) = \min_{x\in\bar{\Omega}} f_i(t,x,0,\ldots,0) - \lambda_i\mu_i - \sum_{\substack{j=1\\ j\neq i}}^{N} \overline{b}_{ij}(\varepsilon_0)\xi_j(t) \quad \text{i} \quad d = \overline{b}_{ii}(\varepsilon_0).$$

sketch of the proof Theorem 1 [continued]

To prove the permanence of system (D) we show that the parameters in Theorem 1 do not depend on the solution $u(t, x)$, for sufficiently large t. □

Practical persistence

Now we replace conditions (A) with

$$m[f_i] > \rho_i \lambda_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\overline{b}_{ij}(\overline{a}_j + \rho_i \lambda_j)}{\underline{b}_{jj}}$$

Then we can give the lower estimates on the numbers δ_i (in the definition of permanence) in terms of the parameters of system (D):

$$\delta_i \geq \frac{\beta}{\overline{b}_{ii}} \exp\big(-L(m[f_i] - \underline{a}_i)\big).$$

System with nonlocal dispersal in habitats with non-flux boundary

In this section we consider the competition system with nonlocal dispersal in environments with non - flux boundary

$$\frac{\partial u_i}{\partial t} = \rho_i \int_\Omega K_i(x,y)[u_i(t,y) - u_i(t,x)]dy + f_i(t,x,u_1,\ldots,u_N)u_i,$$
$$t \geq 0, x \in \bar{\Omega}, i = 1,\ldots,N. \tag{DN}$$

Note that system (DN) is the nonlocal dispersal counterpart of the following Kolmogorov competing system with random dispersal and Neumann boundary conditions

$$\begin{cases} \frac{\partial u_i}{\partial t} = -\rho_i \Delta u_i + f_i(t,x,u_1,\ldots,u_N)u_i & t \geq 0, x \in \Omega \\ \frac{\partial u_i}{\partial n} = 0 \quad x \in \partial\Omega. \end{cases}$$

We note that principal eigenvalue of the nonlocal dispersal operator on domain with nonflux boundary

$$\begin{cases} \int_\Omega K_i(x,y)(\varphi_i(x)-\varphi_i(y))dy = \lambda\varphi(x) \quad \text{on} \quad \Omega \\ \frac{\partial u_i}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

is zero.

Definition

System (DN) is *permanent* if there are positive constants $\underline{\delta}_i, \overline{\delta}_i > 0$ such that for each positive solution $u(t,x) = (u_1(t,x), \ldots, u_N(t,x))$ of (DN) there holds

$$\underline{\delta}_i \leq \liminf_{t\to\infty} u_i(t,x) \leq \limsup_{t\to\infty} u_i(t,x) \leq \overline{\delta}_i \quad 1 \leq i \leq N$$

where the limit is uniform in $x \in \Omega$.

Average conditions for evolution system of (DN)

$$m[f_i] > \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\overline{b}_{ij} M[f_j]}{\underline{b}_{jj}}.$$

If for some $1 \leq i \leq N$ we assume the following stronger inequality

$$m[f_i] > \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\overline{b}_{ij} \overline{a}_j}{\underline{b}_{jj}}$$

then we can give the lower estimates on the numbers δ_i in the terms of the parameters of system (DN)