Existence of constant-sign and changing-sign ground states for some $p\&q$ elliptic problems with vanishing potentials and critical nonlinearities

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In **[1]** we deal with the existence of at least three (positive, negative and nodal) nontrivial ground state solutions for the class of $p\&q$ problems given by

$$-\operatorname{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u)+V(x)b(|u|^p)|u|^{p-2}u = K(x)f(u), \quad \text{in } \mathbb{R}^N, \quad (\mathcal{P})$$

where $N \geq 3$, $2 \leq p < N$, a, b, f are C^1 real functions satisfying suitable growth and monotonicity conditions and V, K are positive continuous potentials which include potentials vanishing at infinity (*"zero mass case"*). The choice of this class allows us to exploit some compact embeddings in weighted Sobolev spaces (Hardy-type inequalities) involving V and K introduced in

B. Opic and A. Kufner, *Hardy-Type Inequalities*, Pitman Res. Notes Math. Ser., vol. 219, Longman Scientific and Technical, Harlow, 1990

and to overcome the loss of compactness of the problem due to the unboundedness of the domain and the critical growth at infinity of the nonlinearity.
By means of Mountain Pass Theorem and Harnack's inequality we get problem $(\mathcal{P})$ possesses a positive and a negative solution with energy levels equal to the Mountain Pass levels related to the energy functional J associated to the problem, that is, a positive and a negative ground state solution to $(\mathcal{P})$.

[1] S. Barile, G.M. Figueiredo, *Existence of least energy positive, negative and nodal solutions for a class of $p\&q$-problems with potentials vanishing at infinity*, J. Math. Anal. Appl. 427 (2015).

Furthermore, we prove the existence of a least energy solution w of $(\mathcal{P})$ which is nodal or changing-sign in $\mathbb{R}^N$, i.e. $w = w^+ + w^-$ with $w^+ = \max\{w, 0\} \neq 0$, $w^- = \min\{w, 0\} \neq 0$ in $\mathbb{R}^N$ and the supports of w^+ and w^- are disjoint. Indeed, by means of a minimization argument we show the existence of a $w \in \mathcal{M}$ such that

$$J(w) = \min_{v \in \mathcal{M}} J(v), \quad \mathcal{M} = \left\{ v \in \mathcal{N} : v^{\pm} \neq 0, \langle J'(v^{\pm}), v^{\pm} \rangle = 0 \right\}$$

with $\mathcal{M}$ the subset of the Nehari manifold $\mathcal{N}$ containing all changing-sign solutions of (P). The main difficulty facing this problem is due to the fact that $\mathcal{M}$ is not a submanifold of the functional space on which we work, thus we cannot talk about vector fields on $\mathcal{M}$ and deformations cannot be easily construct on $\mathcal{M}$. However, following the arguments in

T. Bartsch, T. Weth and M. Willem, *Partial symmetry of least energy nodal solutions to some variational problems*, J. Anal. Math. 96 (2005)

based on a suitable quantitative deformation lemma (without Palais-Smale condition), we are able to prove that every minimizer on $\mathcal{M}$ of $J|_{\mathcal{M}}$ is a critical point of J. Finally we show that such solution has precisely two nodal domains or changes sign exactly once in $\mathbb{R}^N$.

These results extend previous works to a larger class of $p\&q$ type problems that include $-\Delta_p + V(x)$ or $-\Delta_p - \Delta_q + V(x)$, $2 \leq p < q < N$, whose interest has increased considerably in the literature of the last years due to applications in physics and related sciences and to mathematical techniques used such as variational and topological arguments.

Assumptions on a

The hypotheses on function $a : \mathbb{R}^+ \to \mathbb{R}^+$ of C^1 class are the following:

(a_1) There exist constants $\xi_i > 0$, $i = 0, 1, 2, 3$, $2 \leq p \leq q < N$ such that

$$\xi_0 + \xi_1 t^{(q-p)/p} \leq a(t) \leq \xi_2 + \xi_3 t^{(q-p)/p}, \quad \text{for all } t \geq 0.$$

(a_2) There exist positive real constants α and θ such that, for all $t \geq 0$,

$$\frac{1}{\alpha} a(t)t \leq A(t) = \int_0^t a(s)ds, \quad \text{with } \tfrac{q}{p} \leq \alpha < \tfrac{\theta}{p},$$

with θ also defined in (f_3) such that $q < \theta < q^*$ with $q^* = \frac{Nq}{N-q}$.

(a_3) The map $t \mapsto \dfrac{a(t)}{t^{(q-p)/p}}$ is decreasing for all $t > 0$, or, equivalently, the map a and its derivative a' satisfy

$$a'(t)t \leq \frac{(q-p)}{p} a(t) \text{ for all } t > 0.$$

(a_4) The map $t \to a(t)t^{(p-2)/p}$ is increasing for every $t > 0$.

Assumptions on b

The assumptions on $b : \mathbb{R}^+ \to \mathbb{R}^+$ of C^1 class are the following:

(b_1**)** There exist constants $\sigma_i > 0$, $i = 0, 1, 2, 3$, such that

$$\sigma_0 + \sigma_1 t^{(q-p)/p} \leq b(t) \leq \sigma_2 + \sigma_3 t^{(q-p)/p}, \quad \text{for all } t \geq 0.$$

(b_2**)** There exists a positive constant α (also defined in (a_2)) such that

$$\frac{1}{\alpha} b(t)t \leq B(t) = \int_0^t b(s)ds, \quad \text{for all } t \geq 0.$$

(b_3**)** The map $t \mapsto \frac{b(t)}{t^{(q-p)/p}}$ is decreasing for all $t > 0$, or equivalently, the map b and its derivative b' satisfy

$$b'(t)t \leq \frac{(q-p)}{p} b(t) \text{ for all } t > 0.$$

(b_4**)** The map $t \to b(t)t^{(p-2)/p}$ is increasing for $t > 0$.

Assumptions on V and K

On functions $V, K : \mathbb{R}^N \to \mathbb{R}$ continuous on $\mathbb{R}^N$ we assume the following general conditions. Indeed, we say that $(V, K) \in \mathcal{K}$ if

(VK_0) $V(x), K(x) > 0$ for all $x \in \mathbb{R}^N$ and $K \in L^\infty(\mathbb{R}^N)$.

(VK_1) If $\{A_n\}_n \subset \mathbb{R}^N$ is a sequence of Borel sets such that the Lebesgue measure $\text{meas}(A_n) \leq R$, for all $n \in \mathbb{N}$ and some $R > 0$, then

$$\lim_{r\to+\infty} \int_{A_n \cap B_r^c(0)} K(x) = 0, \quad \text{uniformly in } n \in \mathbb{N}.$$

(VK_2) $\frac{K}{V} \in L^\infty(\mathbb{R}^N)$

or

(VK_3) there exists $m \in (q, q^*)$ such that

$$\frac{K(x)}{V(x)^{\frac{q^*-m}{q^*-p}}} \to 0 \quad \text{as } |x| \to +\infty.$$

Observe that the hypotheses on the functions V and K characterize problem $(\mathcal{P})$ as zero mass. This class of problems has been studied for $a(t) = b(t) = 1$ and $p = 2$ by many authors, see for instance, **B. Opic, A. Kufner**, Pitman Res. Notes Math. (1990), **A. Ambrosetti, V. Felli, A. Malchiodi**, JEMS (2005), **A. Ambrosetti, Z.Q. Wang**, Diff. Integr. Eq. (2005), **D. Bonheure, J. Van Schaftingen**, Ann.Mat. Pura Appl. (2010), **C.O. Alves, M.A.S. Souto**, J. Diff. Eq. (2013), and references therein.

Assumptions on f

Moreover, we assume the following growth conditions in the origin and at infinity for the C^1 function $f : \mathbb{R} \to \mathbb{R}$:

(f_1) $\lim_{|t| \to 0^+} \frac{f(t)}{|t|^{p-1}} = 0$ if (VK_2) holds

or

($\widetilde{f}_1$) $\lim_{|t| \to 0^+} \frac{f(t)}{|t|^{m-1}} = 0$ if (VK_3) holds with $m \in (q, q^*)$ defined above.

(f_2) f has a "quasicritical growth", namely, $\lim_{|t| \to +\infty} \frac{f(t)}{|t|^{q^*-1}} = 0$.

(f_3) There exists $\theta \in (q, q^*)$ so that

$$0 < \theta F(t) = \theta \int_0^t f(s) ds \leq f(t)t, \text{ for all } |t| > 0,$$

where θ is the same in (a_2).

(f_4) The map f and its derivative f' satisfy

$$f'(t) > (q-1)\frac{f(t)}{t} \text{ for all } t \neq 0.$$

Clearly, **(f_4)** implies that the map

$$t \mapsto \frac{f(t)}{|t|^{q-1}} \text{ is strictly increasing for all } |t| > 0.$$

THEOREM *Suppose that* $(V, K) \in \mathcal{K}$, $a, b \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ *satisfy* $(a_1) - (a_4)$ *and* $(b_1) - (b_4)$ *and* $f \in C^1(\mathbb{R}, \mathbb{R})$ *verifies* (f_1) *or* $(\widetilde{f}_1)$ *and* $(f_2) - (f_4)$.
Then, problem $(\mathcal{P})$ *possesses a positive and a negative ground state weak solution, namely, a positive and a negative solution with energy equal to the mountain pass level associated to the energy functional.*
Moreover, problem $(\mathcal{P})$ *admits a least energy nodal (or sign-changing) weak solution which has precisely two nodal domains.*

By adapting some arguments from **G. Li**, *Some properties of weak solutions of nonlinear scalar field equations*, Annales Acad. Sci. Fennicae (1989), we expect the weak solutions found belong to $L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}(\mathbb{R}^N)$ for some $0 < \alpha < 1$.

Problem 1: Let $a(t) = b(t) = 1$. Then conditions $(a_1) - (a_4)$ and $(b_1) - (b_4)$ hold with $q = p$ and $\xi_0 + \xi_1 = \xi_2 + \xi_3 = 1$ and $\sigma_0 + \sigma_1 = \sigma_2 + \sigma_3 = 1$. Hence, our Theorem is valid for the problem

$$-\Delta_p u + V(x)|u|^{p-2}u = K(x)f(u) \quad \text{in} \quad \mathbb{R}^N.$$

Problem 2: Let $a(t) = b(t) = 1 + t^{\frac{q-p}{p}}$. Then, a satisfies $(a_1) - (a_4)$ and b satisfies $(b_1) - (b_4)$ with $\xi_i = \sigma_i = 1$, for $i = 0, 1, 2, 3$. Hence, the above result can be applied to the problem

$$-\Delta_p u - \Delta_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = K(x)f(u) \quad \text{in } \mathbb{R}^N.$$

Problem 3: Let $a(t) = 1 + \frac{1}{(1+t)^{\frac{p-2}{p}}}$ and $b(t) = 1$. Note that, a satisfies $(a_1) - (a_4)$ and b satisfies $(b_1) - (b_4)$ with $q = p$, $\xi_0 = 1$, $\xi_1 = \xi_3 = 0$, $\xi_2 = 2$ and $\sigma_0 + \sigma_1 = \sigma_2 + \sigma_3 = 1$. In this case, our result is valid for the problem

$$- \text{ div } (|\nabla u|^{p-2}\nabla u + \frac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}}) + V(x)|u|^{p-2}u = K(x)f(u) \quad \text{in } \mathbb{R}^N.$$

Problem 4: Let $a(t) = 1 + t^{\frac{q-p}{p}} + \frac{1}{(1+t)^{\frac{p-2}{p}}}$ and $b(t) = 1 + t^{\frac{q-p}{p}}$. In this case, a satisfies $(a_1) - (a_4)$ and b satisfies $(b_1) - (b_4)$ with $\xi_0 = \xi_1 = \xi_3 = 1, \xi_2 = 2$ and $\sigma_0 + \sigma_1 = \sigma_2 + \sigma_3 = 1$. Hence, the above Theorem can be also used for the problem

$$-\Delta_p u - \Delta_q u - \text{div}(\frac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}}) + V(x)(|u|^{p-2}u + |u|^{q-2}u) = K(x)f(u) \text{ in } \mathbb{R}^N.$$

Other interesting elliptic problems from mathematical point of view can be obtained.

Problems of $p\&q$ type have been recently studied in different cases by **C. He, G. Li**, Ann. Acad. Sci. Fenn.Math. (2008), **C.O. Alves, G.M. Figueiredo**, Adv. Nonlin. Stud. (2011), **G.M. Figueiredo**, J. Math. Anal. Appl. (2011), **S. Marano, N.S. Papageorgiou,** Nonlinear Anal. (2013), **D. Mugnai, N.S. Papageorgiou**, Trans. AMS (2014), **S. Barile, G.M. Figueiredo**, Adv. Nonlin. Stud. (2014), **S. Barile, G.M. Figueiredo**, Nonlinear Anal. (2015),...

Variational Framework

First, for every $\zeta \in \mathbb{R}$, $\zeta \geq 1$, let us consider the space

$$\mathcal{D}^{1,\zeta}(\mathbb{R}^N) := \left\{ u \in L^{\zeta^*}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^\zeta \, dx < +\infty \right\} \hookrightarrow L^{\zeta^*}(\mathbb{R}^N)$$

where $\zeta^* = \frac{\zeta N}{N-\zeta}$ if $\zeta < N$ and $\zeta^* = +\infty$ if $\zeta \geq N$. In the following, we denote by $\|\cdot\|_\zeta$ the classical norm on the space $L^\zeta(\mathbb{R}^N)$. In order to prove that problem $(\mathcal{P})$ has a variational structure, let us consider the space

$$X = \left\{ u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \cap \mathcal{D}^{1,q}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)(|u|^p + |u|^q) \, dx < +\infty \right\}$$

endowed with the norm $\|u\| = \|u\|_{V,p} + \|u\|_{V,q}$ where

$$\|u\|_{V,\zeta}^\zeta = \int_{\mathbb{R}^N} |\nabla u|^\zeta \, dx + \int_{\mathbb{R}^N} V(x)|u|^\zeta \, dx, \quad \text{for } \zeta \geq 1.$$

Let X' the dual space of X endowed with the norm $\|\cdot\|_{X'}$.
Recall that a weak solution of problem $(\mathcal{P})$ is a function $u \in X$ such that

$$\int_{\mathbb{R}^N} a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^N} V(x) b(|u|^p)|u|^{p-2} u \, v \, dx - \int_{\mathbb{R}^N} K(x) f(u) \, v \, dx = 0, \text{ for all } v \in X.$$

Now, assumed (VK_0), (VK_2) and (VK_3), $(a_1)-(b_1)$ and (f_1) or $(\widetilde{f_1})$ and (f_2), the weak solutions of $(\mathcal{P})$ are the critical points of the energy functional defined on X by

$$J(u) := \frac{1}{p}\int_{\mathbb{R}^N} A(|\nabla u|^p)\,dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)B(|u|^p)\,dx - \int_{\mathbb{R}^N} K(x)F(u)\,dx.$$

More precisely, $J \in C^1(X,\mathbb{R})$ and its differential $J' : X \to X'$ is defined as

$$\langle J'(u), v\rangle = \int_{\mathbb{R}^N} a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx + \int_{\mathbb{R}^N} V(x)b(|u|^p)|u|^{p-2}u\,v\,dx - \int_{\mathbb{R}^N} K(x)f(u)\,v\,dx,$$

for every $u, v \in X$. Let us define, for every $\zeta \in \mathbb{R}$, $\zeta \geq 1$,

$$L^{\zeta}_K(\mathbb{R}^N) = \left\{u : \mathbb{R}^N \to \mathbb{R} : \quad u \text{ is measurable and } \int_{\mathbb{R}^N} K(x)|u|^{\zeta}\,dx < +\infty\right\}.$$

By means of Hardy-type inequality, we can recover compactness.

Compact Embeddings. *Assume that $(V,K) \in \mathcal{K}$ holds. Then, if (VK_2) holds, E is compactly embedded in $L^{\zeta}_K(\mathbb{R}^N)$ for every $\zeta \in (q, q^*)$. If (VK_3) holds, E is compactly embedded in $L^m_K(\mathbb{R}^N)$.*

Assume $(f_1)-(f_2)$ or $(\widetilde{f}_1)-(f_2)$ and $(V,K)\in\mathcal{K}$. If $u_n \rightharpoonup u$ in X, then

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} K(x)F(u_n)\,dx = \int_{\mathbb{R}^N} K(x)F(u)\,dx$$

and

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} K(x)f(u_n)u_n\,dx = \int_{\mathbb{R}^N} K(x)f(u)u\,dx.$$

In order to find a positive solution, we assume that $f(t)=0$ for all $t\in(-\infty,0]$ while for a negative solution we suppose $f(t)=0$ for all $t\in[0,+\infty)$.

Definition (Palais-Smale Condition) We say that J satisfies (PS) if any sequence $(u_n)_n\subset X$ such that

$$(J(u_n))_n \text{ is bounded} \quad \text{and} \quad dJ(u_n)\ \to\ 0 \quad \text{as } n\to+\infty,$$

(i.e. (u_n) is a sequence of Palais-Smale) converges in X up to subsequences.

If $(V,K)\in\mathcal{K}$, $(a_1)-(a_2)-(a_4)$, $(b_1)-(b_2)-(b_4)$ and $(f_1)-(f_3)$ hold, then J satisfies (PS). In particular, by (a_1) and (a_4) we exploit

$$C|u-v|^p \le \big\langle a(|u|^p)|u|^{p-2}u - a(|v|^p)|v|^{p-2}v, u-v\big\rangle$$
$$C'|u-v|^q \le \big\langle a(|u|^p)|u|^{p-2}u - a(|v|^p)|v|^{p-2}v, u-v\big\rangle$$

for all $u,v\in\mathbb{R}^N$, $N\ge 1$ and $\langle\cdot,\cdot\rangle$ the scalar product in $\mathbb{R}^N$. Similar inequalities follow for b by (b_1) and (b_4).

Theorem (Mountain Pass)

Let $J \in C^1(X, \mathbb{R})$, X *a real Banach space. If* J *satisfies (PS) and*

(J_0) $J(0) = 0$,

(J_α) *there exist* $\alpha, \delta > 0$ *such that* $J_{|\partial B_\delta} \geq \alpha$,

(J_e) *there exists* $e \in X \setminus B_\delta$ *such that* $J(e) \leq 0$,

then, set $\Gamma = \{g \in C([0,1], X) : g(0) = 0,\ g(1) = e\}$, J *has a critical value* $c \geq \alpha$ *where*

$$c = \inf_{g \in \Gamma} \max_{u \in g[0,1]} J(u).$$

A. Ambrosetti, P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. 14 (1973).

Therefore, there exists $u \in X$ such that $J(u) = c$, $J'(u) = 0$ and $u \geq 0$. By adapting some arguments from **G. Li**, Annales Acad. Sci. Fennicae (1989), we conclude that $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}(\mathbb{R}^N)$ for some $0 < \alpha < 1$. From Harnack's inequality in **N.S. Trudinger**, Comm. Pure and Appl. Math. (1967) we get $u(x) > 0$ for all $x \in \mathbb{R}^N$.
If $f = 0$ in $[0, +\infty)$, by the same arguments we prove the existence of $u \in X$ such that $J(u) = c$, $J'(u) = 0$ with $u(x) < 0$ for all $x \in \mathbb{R}^N$.

Least Energy nodal solution: Minimization and Deformation Lemma

We also show the existence of a $w \in \mathcal{M}$ such that $J(w) = \min_{v \in \mathcal{M}} J(v)$ with $\mathcal{M} = \left\{ v \in \mathcal{N} : v^{\pm} \neq 0, \langle J'(v^{\pm}), v^{\pm} \rangle = 0 \right\}$ the subset of the Nehari manifold $\mathcal{N}$ containing all changing-sign solutions of (P).

Theorem (Weierstrass Theorem)

Let $(X, \|\cdot\|)$ be a reflexive Banach (o Hilbert) space and $M \subseteq X$ be a weakly closed subset of X, i.e.,

$$\text{for every } \{u_n\} \subset M,\ u_n \rightharpoonup u \text{ it results } u \in M.$$

Suppose that the functional $J : M \to \mathbb{R}$ is a weakly sequentially lower semi-continuous on M, namely

$$\text{for every } \{u_n\} \subset M,\ u_n \rightharpoonup u \text{ it is } J(u) \leq \liminf_{n \to +\infty} J(u_n),$$

and weakly coercive on M, i.e.,

$$\text{for every } \{u_n\} \subset M,\ \|u_n\| \to +\infty\text{: } J(u_n) \to +\infty.$$

Then, J is bounded below on M and

$$\text{there exists } u_0 \in M \text{ such that } J(u_0) = \min_{u \in M} J(u).$$

Let us observe that minimizers or maximizers of J on $\mathcal{M}$ are not automatically (weak) solutions of problem $(\mathcal{P})$. Indeed, in **T. Bartsch, T. Weth,** Topol. Methods Nonlinear Anal. (2003), it has been proved that $\mathcal{M}$ is not a submanifold of X with $X = W_0^{1,2}(\Omega)$ since the map $u \to u^{\pm}$ looses differentiability and $\mathcal{M}$ is a codimension 2 submanifold of $W^{2,2}(\Omega)$. So one cannot talk about vector fields on $\mathcal{M}$ and deformations cannot be easily construct on $\mathcal{M}$. A similar situation occurs in our case.

However, we can define a suitable function and its gradient vector field which are related to functional J; thanks to some monotonicity conditions which follow from (a_3), (b_3) and (f_4), these functions will be involved in particular in the application of the deformation lemma.
Indeed, for each $v \in X$ with $v^{\pm} \neq 0$, let us consider $h^v : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ given by

$$h^v(t, s) = J(tv^+ + sv^-), \text{ for every } (t, s) \in [0, +\infty) \times [0, +\infty),$$

and its gradient $\Phi^v : [0, +\infty) \times [0, +\infty) \to \mathbb{R}^2$ defined by

$$\begin{aligned} \Phi^v(t, s) &= \Big(\Phi_1^v(t, s), \Phi_2^v(t, s)\Big) = \Big(\frac{\partial h^v}{\partial t}(t, s), \frac{\partial h^v}{\partial s}(t, s)\Big) \\ &= \Big(\big\langle J'(tv^+ + sv^-), v^+\big\rangle, \big\langle J'(tv^+ + sv^-), v^-\big\rangle\Big) \end{aligned}$$

for every $(t, s) \in [0, +\infty) \times [0, +\infty)$.

By adapting some arguments in

T. Bartsch, T. Weth, M. Willem, *Partial symmetry of least energy nodal solutions to some variational problems*, J. Anal. Math. 96 (2005)

we are able to prove that every minimizer on $\mathcal{M}$ of $J|_{\mathcal{M}}$ is a critical point of J by the following

Lemma (Quantitative Deformation Lemma)

Let X be a Banach space, $J \in C^1(X, \mathbb{R})$, $B(u, \delta)$ the open ball of center u and radius δ, $c_0 \in \mathbb{R}$, and $\varepsilon, \delta > 0$ such that

$$\text{for every} \quad u \in J^{-1}([c_0 - 2\varepsilon, c_0 + 2\varepsilon]) \cap B(u, \delta) : \|J'(u)\| \geq C(\varepsilon, \delta) > 0.$$

Then, there exists a deformation $\eta \in C([0,1] \times X, X)$ such that

(i) $\eta(t, u) = u$ if $t = 0$ or if $u \notin J^{-1}([c_0 - 2\varepsilon, c_0 + 2\varepsilon]) \cap B(u, \delta)$,

(ii) $\eta(1, J^{c_0+\varepsilon} \cap B(u, \delta)) \subset J^{c_0-\varepsilon}$,

(iii) $J(\eta(\cdot, u))$ is non-increasing for every $u \in X$,

(iv) $\eta(t, \cdot)$ is an homeomorphism on X, for every $t \in [0, 1]$.

M. Willem, *Minimax Theorems*, PNLDE and their Appl., Birkhauser (1996).