Structural instability of nonlinear plates modelling suspension bridges

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Workshop in Nonlinear PDEs, Brussels September 7-11, 2015

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References

- E. B., A. Ferrero, F. Gazzola, *Structural instability of nonlinear plates modelling suspension bridges: mathematical answers to some long-standing questions*, arXiv:1502.05851

- E. B., F. Gazzola, C. Zanini, *Which residual mode captures the energy of the dominating mode in second order Hamiltonian systems?*, arXiv:1410.2374

Stability in suspension bridge models

The spectacular collapse of the Tacoma Narrows Bridge has attracted the attention of engineers, physicists, and mathematicians in the last 74 years.

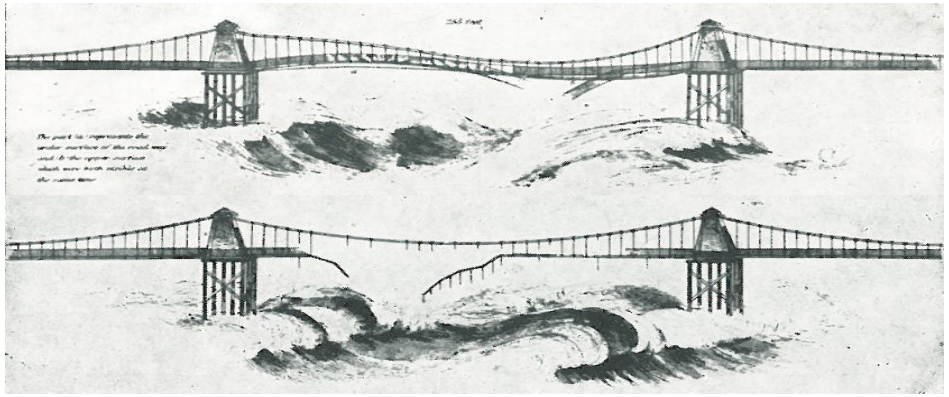
The collapsed Tacoma Narrows Bridge (1940).

The crucial event in the collapse was the sudden change from a **vertical** to a **torsional** mode of oscillation.

- O.H. Ammann, T. von Kármán, G.B. Woodruff, *The failure of the Tacoma Narrows Bridge*, Federal Works Agency (1941)

Stability in suspension bridge models

Many others bridges manifested similar uncontrolled torsional oscillations.



Brighton Chain Pier (1836).

Why do torsional oscillations appear suddenly?

There have been many attempts to answer but up to nowadays an *"opinion on the exact cause of the Tacoma Narrows Bridge collapse is even today **not unanimously shared**."*

- R. Scott, *In the wake of Tacoma. Suspension bridges and the quest for aerodynamic stability*, ASCE Press (2001)

Full theoretical answers to the above questions are not available.

- P.J. McKenna, *Oscillations in suspension bridges, vertical and torsional*, DCDS (2014)

Facts:

The only torsional mode which developed under wind action on the bridge or on the model is that with a single node at the center of the main span.

- F.C. Smith, G.S. Vincent, *Aerodynamic stability of suspension bridges: with special reference to the TNB*, Univ. Washington Press (1950)

...the motions, which a moment before had involved a number of waves (nine or ten) had shifted almost instantly to two.

- O.H. Ammann, T. von Kármán, G.B. Woodruff, *The failure of the Tacoma Narrows Bridge*, Federal Works Agency (1941)

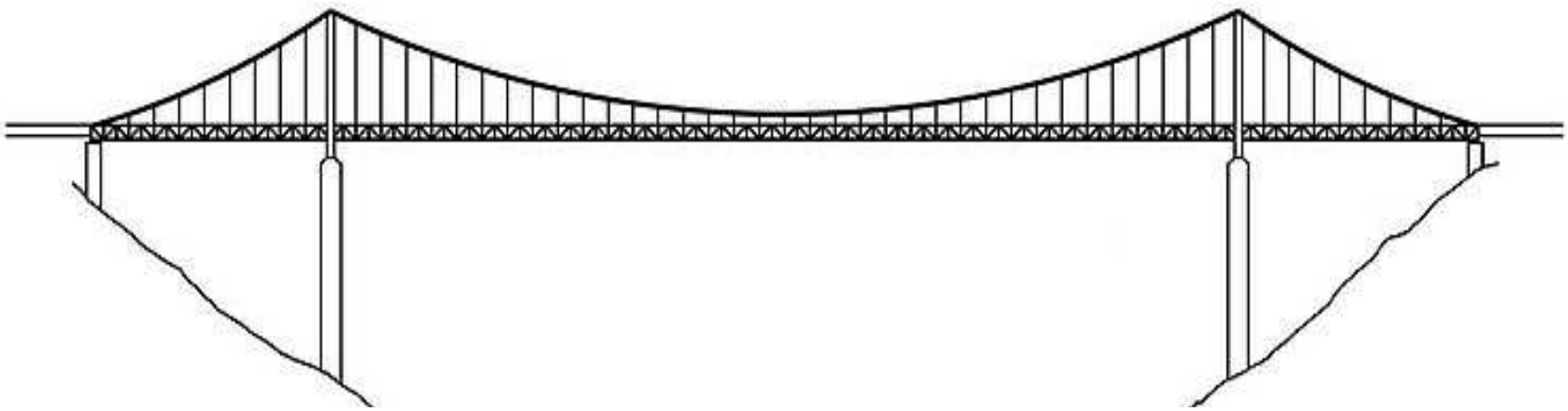
Further questions:

Why do torsional oscillations appear with a node at midspan?

Are there longitudinal oscillations which are more prone to generate torsional oscillations?

PDE's approach: a continuous model for suspension bridges

We view the bridge as a plate $\Omega = (0, L) \times (-\ell, \ell)$ ($\ell << L$) which is hinged on the small edges and free on the large edges:



$\Downarrow$

The nonlinear hyperbolic problem

$$\begin{cases} mu_{tt}+\delta u_t+\frac{E\,d^3}{12(1-\sigma^2)}\Delta^2 u+h(y,u)=f & (x,y,t)\in\Omega\times(0,T) \\ u(0,y,t)=u_{xx}(0,y,t)=u(L,y,t)=u_{xx}(L,y,t)=0 & (y,t)\in(-\ell,\ell)\times(0,T) \\ u_{yy}(x,\pm\ell,t)+\sigma u_{xx}(x,\pm\ell,t)=0 & (x,t)\in(0,L)\times(0,T) \\ u_{yyy}(x,\pm\ell,t)+(2-\sigma)u_{xxy}(x,\pm\ell,t)=0 & (x,t)\in(0,L)\times(0,T) \\ u(x,y,0)=u_0(x,y)\,,\quad u_t(x,y,0)=u_1(x,y) & (x,y)\in\Omega \end{cases}$$

Physical constants: m mass density, $\delta>0$ damping, h restoring force due to the cables, f external forces, d thickness of the plate, E Young modulus and σ Poisson ratio

Our assumptions

- ✓ bridge $=$ isolated system (no dissipation, no interactions);
- ✓ The suspended structure of the TNB consisted of a "mixture" of concrete and metal $\Longrightarrow \sigma = 0.2$;
- ✓ TNB data: $L = 2800\,\text{ft.} \approx 853.44\,m\,, \quad 2\ell = 39\,\text{ft.} \approx 11.89\,m$ $\Longrightarrow \frac{2\ell}{L} = \frac{39}{2800} \approx \frac{1}{75} = \frac{\frac{2\pi}{150}}{\pi} \Longrightarrow$ by scaling $L = \pi$ and $\ell = \frac{\pi}{150}$,
- ✓
$$h(y,u) = \Upsilon(y)\left(k_1\,u + k_2\,u^3\right),$$
Υ characteristic function of $(-\frac{\pi}{150}, -\frac{3\pi}{500}) \cup (\frac{3\pi}{500}, \frac{\pi}{150})$
  - Plaut-Davis J. Sound and Vibrations 2007
- ✓ change of variable (not renamed)

$$u(x,y,t) = \sqrt{\frac{k_1}{k_2}}\,u\left(\frac{\pi x}{L}, \frac{\pi y}{L}, \sqrt{\frac{k_1}{m}}\,t\right), \quad \gamma = \frac{E\,d^3}{12k_1(1-\sigma^2)}\,\frac{\pi^4}{L^4}$$

Dimensionless and isolated problem.

$$\begin{cases} u_{tt}+\gamma\Delta^2 u+\Upsilon(y)(u+u^3)=0 & \text{in } \Omega\times(0,\infty) \\ u(0,y,t)=u_{xx}(0,y,t)=u(\pi,y,t)=u_{xx}(\pi,y,t)=0 & \text{for } (y,t)\in(-\frac{\pi}{150},\frac{\pi}{150})\times(0,\infty) \\ u_{yy}(x,\pm\frac{\pi}{150},t)+0.2\cdot u_{xx}(x,\pm\frac{\pi}{150},t)=0 & \text{for } (x,t)\in(0,\pi)\times(0,\infty) \\ u_{yyy}(x,\pm\frac{\pi}{150},t)+1.8\cdot u_{xxy}(x,\pm\frac{\pi}{150},t)=0 & \text{for } (x,t)\in(0,\pi)\times(0,\infty) \\ u(x,y,0)=u_0(x,y)\,, \quad u_t(x,y,0)=u_1(x,y) & \text{for } (x,y)\in\Omega \end{cases}$$

where $\Omega=(0,\pi)\times(-\frac{\pi}{150},\frac{\pi}{150})$ and $\sigma=0.2$.

The ibv problem is isolated, its energy is constant in time:

$$E(u)=\int_\Omega\left\{\frac{u_t^2}{2}+\frac{\gamma}{2}(\Delta u)^2+\frac{4\gamma}{5}(u_{xy}^2-u_{xx}u_{yy})+\Upsilon(y)\left(\frac{u^2}{2}+\frac{u^4}{4}\right)\right\}.$$

The associated eigenvalues problem

$$\begin{cases} \Delta^2 w = \lambda w & (x,y) \in \Omega \\ w(0,y) = w_{xx}(0,y) = w(\pi,y) = w_{xx}(\pi,y) = 0 & y \in (-\ell,\ell) \\ w_{yy}(x,\pm\ell) + \sigma w_{xx}(x,\pm\ell) = w_{yyy}(x,\pm\ell) + (2-\sigma) w_{xxy}(x,\pm\ell) = 0 & x \in (0,\pi) \end{cases}$$

- A. Ferrero, F. Gazzola, *A partially hinged rectangular plate as a model for suspension bridges*, Disc. Cont. Dynam. Syst. A 2015

Theorem

The set of eigenvalues may be ordered in an increasing sequence $\{\lambda_k\}$ of strictly positive numbers diverging to $+\infty$;
Any eigenfunction belongs to $C^\infty(\overline{\Omega})$ and the set of eigenfunctions is a complete system in

$$H^2_*(\Omega) := \left\{ w \in H^2(\Omega);\ w = 0 \text{ on } \{0,\pi\} \times (-\ell,\ell) \right\}.$$

Moreover, all the eigenvalues and eigenfunctions belong to one of the following families:

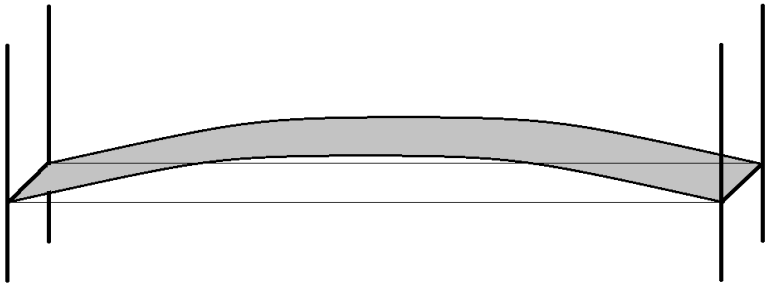
(i) for any $m \geq 1$, there exists a unique eigenvalue $\lambda = \mu_{m,1} \in ((1-\sigma)^2 m^4, m^4)$ with corresponding eigenfunction

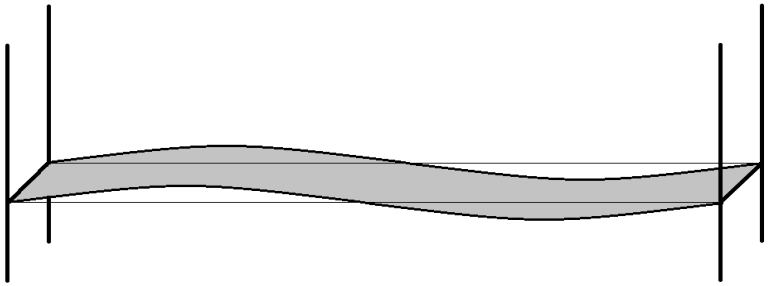
$$\left[A_m \cosh\left(y\sqrt{m^2 + \mu_{m,1}^{1/2}}\right) + B_m \cosh\left(y\sqrt{m^2 - \mu_{m,1}^{1/2}}\right)\right] \sin(mx)\,;$$

(ii) for any $m \geq 1$, there exist infinitely many eigenvalues $\lambda = \mu_{m,k} > m^4$ $(k \geq 2)$ with corresponding eigenfunctions

$$\left[A_m \cosh\left(y\sqrt{\mu_{m,k}^{1/2} + m^2}\right) + B_m \cos\left(y\sqrt{\mu_{m,k}^{1/2} - m^2}\right)\right] \sin(mx)\,;$$

These are the **longitudinal eigenfunctions**, of the kind $c_m \sin(mx)$.





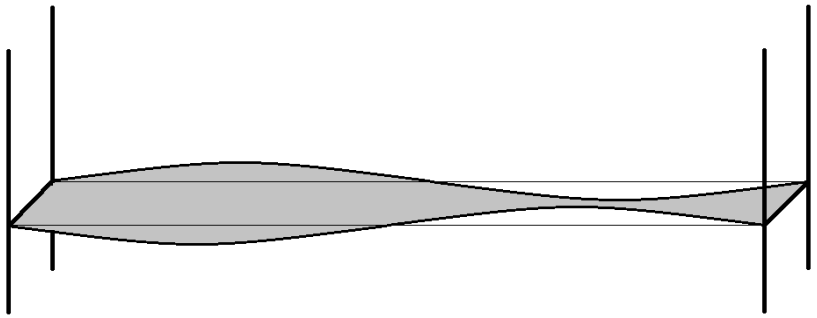
(*iii*) for any $m \geq 1$, there exist infinitely many eigenvalues $\lambda = \nu_{m,k} > m^4$ ($k \geq 2$) with corresponding eigenfunctions

$$\left[A_m \sinh\left(y\sqrt{\nu_{m,k}^{1/2} + m^2}\right) + B_m \sin\left(y\sqrt{\nu_{m,k}^{1/2} - m^2}\right)\right] \sin(mx)\,;$$

(*iv*) for any $m \geq 1$ satisfying $\ell m\sqrt{2}\coth(\ell m\sqrt{2}) > \left(\frac{2-\sigma}{\sigma}\right)^2$ there exists an eigenvalue $\lambda = \nu_{m,1} \in (\mu_{m,1}, m^4)$ with corresponding eigenfunction

$$\left[A_m \sinh\left(y\sqrt{m^2 + \nu_{m,1}^{1/2}}\right) + B_m \sinh\left(y\sqrt{m^2 - \nu_{m,1}^{1/2}}\right)\right] \sin(mx)\,.$$

These are the **torsional eigenfunctions**, of the kind $c_m y \sin(mx)$.



the destructive oscillations observed in several suspension bridges are of the kind: y sin(2x) (**one node at midspan**).

The eigenvalues solve explicit equations, we numerically obtained the following values:

eigenvalue	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
kind	$\mu_{1,1}$	$\mu_{2,1}$	$\mu_{3,1}$	$\mu_{4,1}$	$\mu_{5,1}$	$\mu_{6,1}$	$\mu_{7,1}$	$\mu_{8,1}$
$\sqrt{\text{eigenvalue}} \approx$	0.98	3.92	8.82	15.68	24.5	35.28	48.02	62.73

eigenvalue	λ_9	λ_{10}	λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{16}
kind	$\mu_{9,1}$	$\mu_{10,1}$	$\nu_{1,2}$	$\mu_{11,1}$	$\mu_{12,1}$	$\mu_{13,1}$	$\mu_{14,1}$	$\nu_{2,2}$
$\sqrt{\text{eigenvalue}} \approx$	79.39	98.03	104.61	118.62	141.19	165.72	192.21	209.25

Approximate value of the least 16 eigenvalues.

- ✓ the 10 lowest eigenfunctions are all longitudinal
- ✓ the first torsional eigenfunction is the 11th which corresponds to the eigenvalue $\nu_{1,2}$ and to the eigenfunction $\approx Cy \sin x$
- ✓ the second torsional eigenfunction is the 16th which corresponds to the eigenvalue $\nu_{2,2}$ and to the eigenfunction $\approx Cy \sin 2x$
- ✓ Federal Report of 1941: *in the TNB the motions, which a moment before had involved ten waves had shifted almost instantly to two* (having one node at midspan)

$$\text{Why } \sin(10x) \to y \sin(2x) \text{ ?}$$

Back to the evolution equation

$$\begin{cases} u_{tt}+\gamma\Delta^2 u+\Upsilon(y)(u+u^3)=0 & \text{in } (0,\pi)\times(-\frac{\pi}{150},\frac{\pi}{150})\times(0,\infty) \\ u(0,y,t)=u_{xx}(0,y,t)=u(\pi,y,t)=u_{xx}(\pi,y,t)=0 & \text{for } (y,t)\in(-\frac{\pi}{150},\frac{\pi}{150})\times(0,\infty) \\ u_{yy}(x,\pm\frac{\pi}{150},t)+0.2\cdot u_{xx}(x,\pm\frac{\pi}{150},t)=0 & \text{for } (x,t)\in(0,\pi)\times(0,\infty) \\ u_{yyy}(x,\pm\frac{\pi}{150},t)+1.8\cdot u_{xxy}(x,\pm\frac{\pi}{150},t)=0 & \text{for } (x,t)\in(0,\pi)\times(0,\infty) \\ u(x,y,0)=u_0(x,y)\,,\quad u_t(x,y,0)=u_1(x,y) & \text{for } (x,y)\in(0,\pi)\times(-\frac{\pi}{150},\frac{\pi}{150}) \end{cases}$$

$$u\in C^0(\mathbb{R}_+;H^2_*(\Omega))\cap C^1(\mathbb{R}_+;L^2(\Omega))\cap C^2(\mathbb{R}_+;\mathcal{H}(\Omega))$$

is a **solution** if it satisfies the initial conditions and if

$$\langle u''(t),v\rangle+\gamma(u(t),v)_{H^2_*}+(h(y,u(t)),v)_{L^2}=0 \qquad \forall v\in H^2_*(\Omega) \text{ and } \forall t\in(0,T)\,,$$

$H^2_*(\Omega):=\left\{w\in H^2(\Omega);\ w=0 \text{ on } \{0,L\}\times(-\ell,\ell)\right\}$ with $\mathcal{H}(\Omega)$ its dual space.

Theorem

For all $u_0\in H^2_*(\Omega)$ and $u_1\in L^2(\Omega)$ there exists a unique global solution u.

Finite dimensional analysis: Galerkin procedure

$$u_m(t) = \sum_{i=1}^{m} g_i^m(t) w_i \qquad \text{and} \qquad g^m(t) := (g_1^m(t), \dots, g_m^m(t))^T$$

(w_i eigenfunctions) as $m \to \infty$ we have

$$u_m \to u \text{ in } C^0([0,T]; H^2_*(\Omega)) \cap C^1([0,T]; L^2(\Omega))$$

where

$$\begin{cases} (g^m(t))'' + \gamma \Lambda_m g^m(t) + \Phi_m(g^m(t)) = 0 \qquad \forall t \in (0,T) \\ \\ g^m(0) = ((u_0, w_1)_{L^2}, \dots, (u_0, w_m)_{L^2})^T, \ (g^m)'(0) = ((u_1, w_1)_{L^2}, \dots, (u_1, w_m)_{L^2})^T \end{cases}$$

$\Lambda_m := \mathrm{diag}(\lambda_1, \dots, \lambda_m)$ and $\Phi_m : \mathbb{R}^m \to \mathbb{R}^m$

$$\Phi_m(\xi_1, \dots, \xi_m) := \left(\left(h\Big(y, \sum_{j=1}^{m} \xi_j w_j\Big), w_1 \right)_{L^2}, \dots, \left(h\Big(y, \sum_{j=1}^{m} \xi_j w_j\Big), w_m \right)_{L^2} \right)^T .$$

Stability analysis

The Federal Report and the fact that among the least 16 modes, there are 14 longitudinal modes and 2 torsional modes, suggest to fix $m = 16$.

The system becomes

$$\begin{cases} \varphi_k''(t) + \gamma\, \mu_{k,1}\varphi_k(t) + \Phi_k\big(\varphi_1(t), ..., \varphi_{14}(t), \tau_1(t), \tau_2(t)\big) = 0 \\ \tau_k''(t) + \gamma\, \nu_{k,2}\tau_k(t) + \Gamma_k\big(\varphi_1(t), ..., \varphi_{14}(t), \tau_1(t), \tau_2(t)\big) = 0 \\ \varphi_k(0) = \phi_0^k\,, \quad \varphi_k'(0) = \phi_1^k\,, \quad \tau_k(0) = \eta_0^k\,, \quad \tau_k'(0) = \eta_1^k \end{cases}$$

with

φ_k (coefficients of the **longitudinal** eigenfunctions) for $k = 1, ..., 14$,

τ_k (coefficients of the **torsional** eigenfunctions) for $k = 1, 2$.

We isolate each longitudinal mode φ_k by setting to 0 all the remaining 15 components and we solve

$$\begin{cases} \varphi_k''(t) + (\gamma\mu_{k,1} + a_k)\varphi_k(t) + b_k\varphi_k^3(t) = 0 \qquad \forall t > 0 \\ \varphi_k(0) = \phi_0^k\,, \quad \varphi_k'(0) = \phi_1^k\,. \end{cases}$$

For $1 \leq k \leq 14$, we call **k-th longitudinal mode** at energy $E(\phi_0^k, \phi_1^k) > 0$ the unique (periodic) solution $\overline{\varphi}_k$ of the above Cauchy problem.
Since the problem is nonlinear, the period of $\overline{\varphi}_k$ <u>depends</u> on the energy.

The torsional part of the linearized system around $(0, ..., \overline{\varphi}_k, ..., 0)$ reads

$$\xi''(t) + \Big(\gamma\nu_{l,2} + \bar{a}_l + d_{l,k}\overline{\varphi}_k^2(t)\Big)\xi(t) = 0\,, \qquad (l = 1, 2)\,.$$

Fix $1 \leq k \leq 14$ and $l = 1, 2$. We say that the k-th longitudinal mode $\overline{\varphi}_k$ at energy $E(\phi_0^k, \phi_1^k)$ is **stable with respect to the l-th torsional mode** if the trivial solution of the above Hill equation is stable.

Theorem

Fix $1 \leq k \leq 14$, $l \in \{1, 2\}$. Then there exists $E_k^l > 0$ and a strictly increasing function Λ such that $\Lambda(0) = 0$ and such that the k-th longitudinal mode $\overline{\varphi}_k$ at energy $E(\phi_0^k, \phi_1^k)$ is **stable with respect to the l-th torsional mode** provided that

$$E \leq E_k^l$$

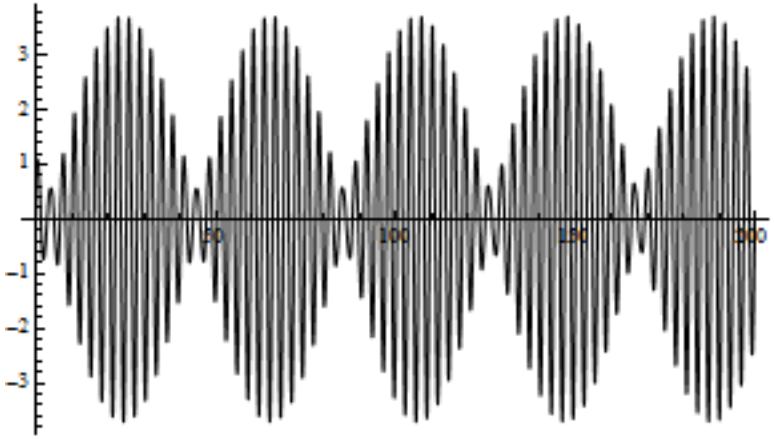
or, equivalently, provided that

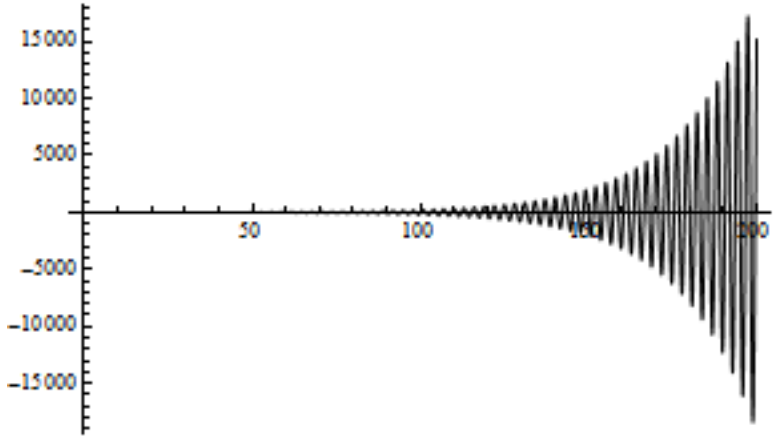
$$\|\overline{\varphi}_k\|_\infty^2 \leq \Lambda(E_k^l)\,.$$

It is not a perturbation result (Floquet theory), we have **explicit values** for E_k^l.

- N.E. Zhukovskii (1892), V.A. Yakubovich, V.M. Starzhinskii (1975)

Numerical experiments confirm the existence of an **energy threshold** where stability is lost.





Plot of the solution of the linearized equation for $k = 14$ and $l = 1$ ($\bar{\varphi}_{14}(0) = 0.79$ (left) and 0.8 (right), $\bar{\varphi}'_{14}(0) = 0$ and $\xi_1(0) = \xi'_1(0) = 1$).

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A_1^k	1.44	0.91	0.9	0.88	0.82	0.69	>10	>10	>10	>10	0.2	0.44	0.63	0.8
A_2^k	1.87	2.91	1.86	1.85	1.82	1.77	1.69	1.54	1.3	0.76	>10	>10	>10	>10

Instability thresholds for the initial data of the k-th longitudinal mode with respect to the first and the second torsional modes: A_1^k and A_2^k

Our theoretical (and numerical) results suggest the following explanations of the origin of torsional instability:

*it is due to an **internal resonance** which generates an energy transfer between different oscillation modes. When the bridge is oscillating **longitudinally** with a suitable amplitude, part of the energy is suddenly transferred to a torsional mode giving rise to wide **torsional** oscillations.*

Furthermore, from the computed energies the **tenth** longitudinal mode seems the **most prone** to generate the **second torsional** mode (on the day of the TNB collapse the motions were considerably less than had occurred many times before).

- what happens for large energies?
- which is the criterion governing the transfer of energy?

Prototype problem: μ, λ_1, $\lambda_2 > 0$, $x_0 \in \mathbb{R} \setminus \{0\}$ and $\varepsilon > 0$,

$$\begin{cases} \ddot{\varphi} + \mu^2 \varphi + U_\varphi(\varphi, \tau_1, \tau_2) = 0 & \varphi(0) = x_0,\ \dot{\varphi}(0) = 0 \\ \ddot{\tau}_1 + \lambda_1^2 \tau_1 + U_{\tau_1}(\varphi, \tau_1, \tau_2) = 0 & \tau_1(0) = \varepsilon x_0,\ \dot{\tau}_1(0) = 0 \\ \ddot{\tau}_2 + \lambda_2^2 \tau_2 + U_{\tau_2}(\varphi, \tau_1, \tau_2) = 0 & \tau_2(0) = \varepsilon x_0,\ \dot{\tau}_2(0) = 0\,, \end{cases}$$

where

$$U(\varphi, \tau_1, \tau_2) = \frac{\varphi^2 \tau_1^2 + \varphi^2 \tau_2^2 + \tau_1^2 \tau_2^2}{2}\,.$$

For $\varepsilon = 0$, unique solution $(x_0 \cos(\mu t), 0, 0)$ with conserved energy

$$E := \frac{\dot{\varphi}^2}{2} + \frac{\mu^2}{2}\varphi^2 = \frac{\mu^2}{2} x_0^2.$$

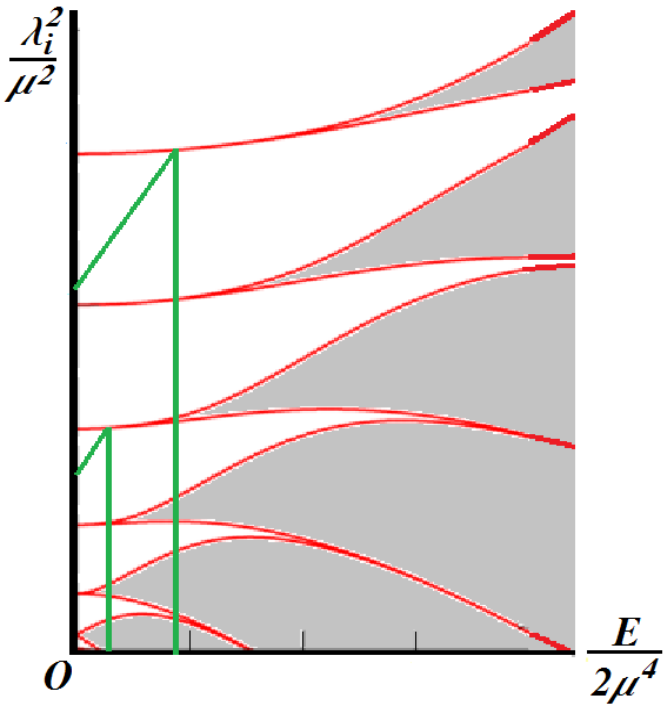
For ε small, we linearize the τ_i equations around the above solution and we obtain the following family of Mathieu equations

$$\ddot{w} + (\alpha_i + 2q\cos(2t))\, w = 0$$

where

$$\alpha_i = \alpha_i(q) = \frac{\lambda_i^2}{\mu^2} + 2q \quad \text{and} \quad q = q(x_0) = \frac{x_0^2}{4\mu^2} = \frac{E}{2\mu^4}\,.$$

Euristic idea



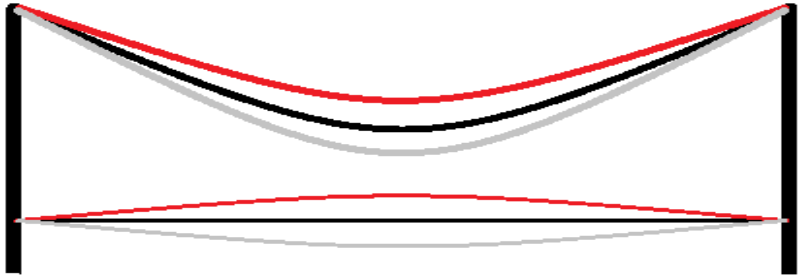
The instability regions in the Mathieu diagram become <u>more narrow</u> as <u>*a* increases</u>. Since the parametric lines associated to each torsional component take their origin when $a = \lambda_i^2/\mu^2$, it would be desirable that $\lambda_i >> \mu$.

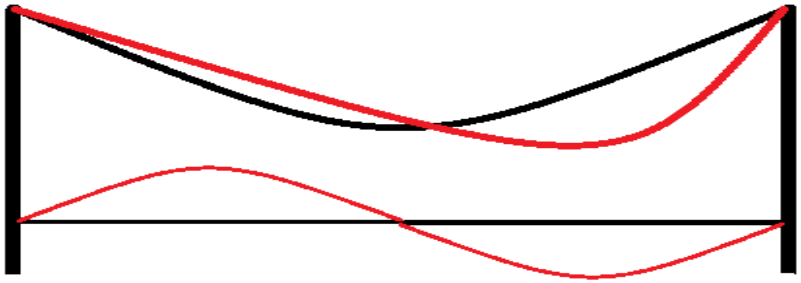
*The **torsional stability** of a suspension bridge depends on the **ratios between torsional and vertical frequencies**; the larger they are, more stable is the bridge.*

Coming back to the initial questions:

- ✓ Why do torsional oscillations appear with a node at midspan?
- ✓ Are there longitudinal oscillations which are more prone to generate torsional oscillations?

The 10th longitudinal eigenvalue minimizes the ratio $\frac{2^{nd}\text{torsional eigenvalue}}{\text{longitudinal eigenvalue}}$ and is therefore the most unstable.





In fact, the first mode does not exist, instead of moving to $y \sin x$, the cable forces the motion to move to $y \sin(2x)$.

- P. Bergot, L. Civati, *Dynamic structural instability in suspension bridges*, Master Thesis, Civil Engineering, Politecnico of Milan, Italy (2014)

To be continued.....

F. Gazzola's talk (Wednesday 14:00 p.m.)