A new Steklov-type problem for the biharmonic operator

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$$\begin{cases} \Delta^2 u - \tau \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ \tau \frac{\partial u}{\partial \nu} - \mathrm{div}_{\partial\Omega}\left(D^2 u.\nu\right) - \frac{\partial \Delta u}{\partial \nu} = \lambda u, & \text{on } \partial\Omega. \end{cases}$$

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Stekloff, *Sur les problémes fondamentaux de la physique mathèmatique (suite et fin)*, Ann. Sci. École Norm. Sup., 3, no. 19 (1902), 455-490.

Another Steklov-type problem

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Bucur, Ferrero, Gazzola, *On the first eigenvalue of a fourth order Steklov problem*, Calc. Var. Partial Differential Equations, 35 (2009), 103-131.

Weak formulation

The weak formulation of the problem is

$$\int_{\Omega} D^2u : D^2\phi + \tau \nabla u \cdot \nabla \phi dx = \lambda \int_{\partial\Omega} u\phi d\sigma, \ \forall \phi \in H^2(\Omega),$$

where $D^2u : D^2\phi = \sum_{i,j=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j}$.

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$$0 = \lambda_1[\Omega] < \lambda_2[\Omega] \leq \cdots \leq \lambda_j[\Omega] \leq \cdots$$

The Neumann problem

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Chasman, *An isoperimetric inequality for fundamental tones of free plates*, Comm. Math. Phys., 303, no. 2 (2011), 421-449.

Neumann vs Steklov

Let us consider

$$\begin{cases} \Delta^2 u - \tau \Delta u = \lambda(\varepsilon)\rho_\varepsilon u, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ \tau \frac{\partial u}{\partial \nu} - \mathrm{div}_{\partial\Omega}\left(D^2 u.\nu\right) - \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

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where

$$\rho_\varepsilon = \begin{cases} \varepsilon, & \text{in } \Omega \setminus \overline{\Omega}_\varepsilon, \\ C_\varepsilon, & \text{in } \Omega_\varepsilon, \end{cases}$$

$\Omega_\varepsilon = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \varepsilon\}$ and $\int_\Omega \rho_\varepsilon = M$ for all $\varepsilon \in]0, \varepsilon_0[$.

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Lamberti, Provenzano, *Viewing the Steklov eigenvalues of the Laplace operator as critical Neumann eigenvalues*, Current Trends in Analysis and Its Applications, Proceedings of the 9th ISAAC Congress, Kraków 2013, Birkhäuser, Basel (2015), 171-178.

Shape differentiability

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$$\Phi(\Omega) = \left\{ \phi \in \left(C^2(\overline{\Omega}) \right)^N, \text{ injective : } \inf_{\Omega} |\det D\phi| > 0 \right\},$$

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$$\Phi(\Omega) = \left\{ \phi \in \left(C^2(\overline{\Omega}) \right)^N, \text{ injective}: \inf_{\Omega} |\mathrm{det} D\phi| > 0 \right\},$$

and consider the dependence $\phi \mapsto \lambda_j[\phi(\Omega)] \equiv \lambda_j[\phi]$.

Shape differentiability

Theorem

Let Ω be a bounded domain of $\mathbb{R}^N$ of class C^1. Let F be a finite non-empty subset of $\mathbb{N}$. Let

$$\mathcal{A}_\Omega[F] = \{\phi \in \Phi(\Omega) : \lambda_l[\phi] \notin \{\lambda_j[\phi] : j \in F\} \quad \forall l \in \mathbb{N} \setminus (F \cup \{0\})\}$$

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Then the set $\mathcal{A}_\Omega$ is open in $\Phi(\Omega)$ and the maps $\Lambda_{F,s}$ from $\mathcal{A}_\Omega$ to $\mathbb{R}$ defined by

$$\Lambda_{F,s}[\phi] = \sum_{j_1 < \cdots < j_s \in F} \lambda_{j_1}[\phi] \cdots \lambda_{j_s}[\phi]$$

for $s \in \{1, ..., |F|\}$ are real analytic.

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Let Ω be a bounded domain in $\mathbb{R}^N$. Let F a finite non-empty subset of $\mathbb{N}$. Let $\tilde{\phi} \in \mathcal{A}_\Omega[F]$ be such that all the eigenvalues with indexes in F have a commond value λ_F and moreover that $\partial\tilde{\phi}(\Omega) \in C^4$. Let $v_1, ..., v_{|F|}$ be a orthonormal basis of the eigenspace associated with the eigenvalue $\lambda_F[\tilde{\phi}]$.

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$$d|_{\phi=\tilde{\phi}}(\Lambda_{F,s})[\psi] = -\lambda_F^s[\tilde{\phi}]\binom{|F|-1}{s-1}\sum_{l=1}^{|F|}\int_{\partial\tilde{\phi}(\Omega)}\Big(\lambda_F K v_l^2 + \lambda_F\frac{\partial(v_l^2)}{\partial\nu} - \tau|\nabla v_l|^2 - |D^2 v_l|^2\Big)(\psi\circ\phi^{(-1)})\cdot\nu d\sigma,$$

where K denotes the mean curvature on $\partial\tilde{\phi}(\Omega)$.

Isovolumetric perturbations

Now we consider volume preserving perturbations, i.e., ϕ such that $|\phi(\Omega)| = |\Omega|$.

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Theorem

Under the same assumptions, the function $\tilde{\phi}$ is a critical point for $\Lambda_{F,s}$ if and only if

$$\sum_{l=1}^{|F|} \left(\lambda_F[\tilde{\phi}] \left(K v_l^2 + \frac{\partial (v_l)^2}{\partial \nu} \right) - \tau |\nabla v_l|^2 - |D^2 v_l|^2 \right)$$

is constant on $\partial\tilde{\phi}(\Omega)$.

Isovolumetric perturbations

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Let B be a ball in $\mathbb{R}^N$ centered at zero. Let $v_1, \dots, v_{|F|}$ be an orthonormal basis of the eigenspace associated with the eigenvalue λ.

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are radial functions.

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Corollary

Let $\tilde{\phi}$ be such that $\tilde{\phi}(\Omega)$ is a ball. Then $\tilde{\phi}$ is a critical point for $\Lambda_{F,s}$ under volume constraint, for any $s = 1, \dots, |F|$.

The isoperimetric inequality

Theorem

For every domain Ω in $\mathbb{R}^N$ of class C^1 the following estimate holds

$$\lambda_2(\Omega) \leq \lambda_2(\Omega^*) \left(1 - \delta_N \mathcal{A}(\Omega)^2\right),$$

where δ_N is a positive dimensional constant, $\mathcal{A}(\Omega)$ is the Fraenkel asymmetry of Ω, and Ω^* is a ball with the same measure as Ω.

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Corollary

For every domain Ω in $\mathbb{R}^N$ of class C^1,

$$\lambda_2(\Omega) \leq \lambda_2(\Omega^*),$$

with equality if and only if Ω is a ball.

Thank you for your attention