Fast Diffusion Equation via Diffusive Scaling of Generalized Carleman Kinetic Equation

Beomjun Choi

NIMS/ Columbia University

cbj5212@gmail.com

Overview

Discrete velocity kinetic equation of a fictitious gas proposed by [Carleman '33]. For $|\alpha| \leq 1$,

$$\begin{cases} \partial_t u_i^\varepsilon + \dfrac{1}{\varepsilon}\nabla_i u_i^\varepsilon = \dfrac{1}{2n\varepsilon^2}\displaystyle\sum_{j=1}^{2n}(u_j^\varepsilon + u_i^\varepsilon)^\alpha (u_j^\varepsilon - u_i^\varepsilon) & \text{in } Q \\ \partial_t u_{i+n}^\varepsilon - \dfrac{1}{\varepsilon}\nabla_i u_{i+n}^\varepsilon = \dfrac{1}{2n\varepsilon^2}\displaystyle\sum_{j=1}^{2n}(u_j^\varepsilon + u_{i+n}^\varepsilon)^\alpha (u_j^\varepsilon - u_{i+n}^\varepsilon) & \text{in } Q \end{cases} \tag{1.1}$$

with initial condition $u_i^\varepsilon = g_i \geq 0$ at $t = 0$ $(i = 1, \cdots, 2n)$.

Introduction-Model

$$\begin{cases} \partial_t u_i^\varepsilon + \frac{1}{\varepsilon}\nabla_i u_i^\varepsilon = \frac{1}{2n\varepsilon^2}\sum_{j=1}^{2n}(u_j^\varepsilon + u_i^\varepsilon)^\alpha(u_j^\varepsilon - u_i^\varepsilon) & \text{in } Q \\ \partial_t u_{i+n}^\varepsilon - \frac{1}{\varepsilon}\nabla_i u_{i+n}^\varepsilon = \frac{1}{2n\varepsilon^2}\sum_{j=1}^{2n}(u_j^\varepsilon + u_{i+n}^\varepsilon)^\alpha(u_j^\varepsilon - u_{i+n}^\varepsilon) & \text{in } Q \end{cases}$$

- For $n = 1$ and $\alpha = 1$, proposed by Carleman
- When $\varepsilon = 1$, u_i and u_{i+n} represents density of particles moving in x_i and $-x_i$ direction with speed 1, respectively
- $(u_i + u_j)^\alpha(u_j - u_i)$ describes the interaction between two group of particles, i and j at a position x

Introduction-Model

- Eq (1.1) comes from $(x, t) \to (\varepsilon x, \varepsilon^2 t)$, diffusive limit
- Physically, $\varepsilon > 0$ represents Knudsen number and Mach number
- In $\varepsilon \to 0$ limit, this mesoscopic kinetic equation becomes a certain continuum equation

Introduction-Model

- GOAL

  Prove convergence of $\rho^\varepsilon := \sum_{i=1}^{2n} u_i^\varepsilon$ to ρ in rigorous way and show the limit ρ satisfies

$$\partial_t \rho - \nabla \cdot \left(\frac{1}{n^{1-\alpha} \cdot \rho^\alpha} \nabla \rho \right) = 0 \text{ with } \rho(0) = \sum_{i=1}^{2n} g_i \qquad (1.2)$$

- Fast Diffusion Eq if $\alpha > 0$, and Porous Medium Eq if $\alpha < 0$

Regarding Diffusive Scaling Limit

- For $n = 1$ and $\alpha = 1$, [Kurtz '73] and [McKean '75]
- For $n = 1$ and $\alpha < 1$, [Pulvirenti&Toscani '96], [Toscani&Lions '97]
- For $n \geq 1$ and $\alpha < 1$, [Toscani&Lions '97] slightly different model
- For $n = 1$ and $1 < \alpha < \frac{4}{3}$, [Salvarani&Toscani '09]
- For $n = 1$ and $|\alpha| \leq 1$, [Salvarani&Vazquez '05] with general $L^1 + L^\infty$ initial data g_i without weight or flux assumption

Introduction-History

Previous results use entropy method

- Uniform estimates (lower bound) on functionals such as $\int \sum_{i=1}^{2n} u_i^\varepsilon \log u_i^\varepsilon dx$ or $\int \sum_{i=1}^{2n} \phi(u_i^\varepsilon) dx$ allow us to pass the limit
- To obtain such an estimate, uniform estimate of a kind of weighted L^1 norm is required
- This is undesirable if target fast diffusion equation is not mass preserving
- In higher dimension, some fast diffusion is no longer mass preserving and a new method is needed.

Introduction-Notations

These will be some notations that appear frequently.

- $Q = \mathbb{R}^n \times (0,\infty)$, $Q_T = \mathbb{R}^n \times (0,T)$, $T>0$, $Q_{R,T} = B_R(0)\times(0,T)$
- $\rho^\varepsilon = \sum_{i=1}^{2n} u_i^\varepsilon$, $\rho_i^\varepsilon = u_i^\varepsilon + u_{i+n}^\varepsilon$, $\rho_{i,j}^\varepsilon = u_i^\varepsilon + u_j^\varepsilon$
- $J^\varepsilon = (J_1^\varepsilon, J_2^\varepsilon, \dots, J_n^\varepsilon)$, $J_i^\varepsilon = \frac{u_i^\varepsilon - u_{i+n}^\varepsilon}{\varepsilon}$, $J_{i,j}^\varepsilon = \frac{u_i^\varepsilon - u_j^\varepsilon}{\varepsilon}$

J^ε will behave as $-C\frac{\nabla\rho}{\rho^\alpha}$ in $\varepsilon \to 0$ in limit

Introduction-Formal Computation in 1D

In $1D$ case, our equation could be written

$$\begin{cases} \partial_t u_1^\varepsilon + \frac{1}{\varepsilon}\nabla_x u_1^\varepsilon = \frac{1}{2\varepsilon^2}(u_1^\varepsilon + u_2^\varepsilon)^\alpha (u_2^\varepsilon - u_1^\varepsilon) \\ \partial_t u_2^\varepsilon - \frac{1}{\varepsilon}\nabla_x u_2^\varepsilon = \frac{1}{2\varepsilon^2}(u_1^\varepsilon + u_2^\varepsilon)^\alpha (u_1^\varepsilon - u_2^\varepsilon) \end{cases} \tag{1.3}$$

By adding and subtracting equations above for each i,

$$\begin{cases} \partial_t \rho^\varepsilon + \nabla_x J^\varepsilon = 0 \\ \varepsilon^2 \partial_t J^\varepsilon + \nabla_x \rho^\varepsilon = -(\rho^\varepsilon)^\alpha J^\varepsilon. \end{cases} \tag{1.4}$$

If ρ^ε, J^ε are ε-uniformly L^2_{loc} bounded in space-time (x,t), we obtain $\rho^\varepsilon \rightharpoonup \rho$, $J^\varepsilon \rightharpoonup J$ in L^2_{loc}.

$$\begin{cases} \partial_t \rho + \nabla_x J = 0 \\ \nabla_x \rho = -(\rho)^\alpha J \end{cases} \tag{1.5}$$

$$\therefore \text{ if } \rho \text{ is nonzero in certain sense, } \partial_t \rho = \nabla_x \left(\frac{\nabla_x \rho}{\rho^\alpha} \right).$$

In fact, Divergence-Curl Lemma says convergence of ρ^ε is strong in L^2_{loc}, which reminds us compactness theorem in Sobolev spaces.

L^2 bounds for ρ^ε and J^ε

A bound for ρ^ε is easy. One can show $\sum_i ||u_i^\varepsilon||_p^p$ is decreasing in time, or use comparison principle. To control J^ε, we use entropy formula. Multify $(u_i \log u_i)'$ on the equation and integrate in x we get,

$$\begin{aligned}\partial_t \int u_1^\varepsilon \log u_1^\varepsilon + u_2^\varepsilon \log u_2^\varepsilon dx &= -\int \frac{1}{\varepsilon^2}(\rho^\varepsilon)^\alpha (u_1^\varepsilon - u_2^\varepsilon)(\log u_1^\varepsilon - \log u_2^\varepsilon) dx \\ &\le -\int (J^\varepsilon)^2 (\rho^\varepsilon)^{\alpha-1} dx \\ &\le -\int (J^\varepsilon)^2 ||\rho^\varepsilon||_\infty^{\alpha-1} dx.\end{aligned}$$

This implies...

- $\int u_1^\varepsilon \log u_1^\varepsilon + u_2^\varepsilon \log u_2^\varepsilon dx$ is decreasing in time
- For $t_1 < t_2$, if an upper bound of this entropy (above) at t_1 and a lower bound at t_2 are given, we obtain an uniform $L^2([t_1, t_2] \times \mathbb{R})$ bound of J^ε
- Same probem happens in higher dimension case
- [Salvarani&Vazquez '05] use local(in space) entropy which gives motivation of our work

Flux Estimate via Local Entropy

Proposition

Suppose u_i^ε i=1,. . . ,2n, are uniformly bounded above and below, say $0 < \frac{1}{M} \leq u_i^\varepsilon \leq M < \infty$ on $K \times [0.T]$. Then for all $\phi \in C_0^\infty(\mathbb{R}^n)$ with $supp(\phi) \subset K$,

$$\int_0^T \int \sum_{i,j} (u_i^\varepsilon + u_j^\varepsilon)^\alpha J_{i,j}^{\varepsilon 2} \phi^2 dxdt < C$$

for some $C = C(\phi, M, \alpha, T, n) > 0$, but independent of ε.

Proof.

By multiplying $\frac{1}{u_i^\varepsilon}\phi^2$ both sides of (1.1) and integrating w.r.t. x, we obtain

$$\begin{aligned} &\partial_t \int \sum_{i=1}^{2n} \log(u_i^\varepsilon)\phi^2 dx + \frac{1}{\varepsilon}\int \sum_{i=1}^{n} \nabla_i \log(u_i^\varepsilon) - \nabla_i \log(u_{i+n}^\varepsilon)\phi^2 dx \\ &= \frac{1}{4n\varepsilon^2}\int \sum_{i\neq j, 1\leq i,j\leq 2n} (u_i^\varepsilon + u_j^\varepsilon)^\alpha (u_j^\varepsilon - u_i^\varepsilon)(\frac{1}{u_i^\varepsilon} - \frac{1}{u_j^\varepsilon})\phi^2 dx. \end{aligned} \tag{2.1}$$

and integration by parts gives us

$$\begin{aligned} \partial_t \int \sum_{i=1}^{2n} \log(u_i^\varepsilon)\phi^2 dx &= \frac{2}{\varepsilon}\int \sum_{i=1}^{n} (\log(u_i^\varepsilon) - \log(u_{i+n}^\varepsilon))\phi\nabla_i\phi dx \\ &- \frac{1}{4n}\int \sum_{i\neq j, 1\leq i,j\leq 2n} \frac{(u_i^\varepsilon + u_j^\varepsilon)^\alpha (J_{j,i}^\varepsilon)^2}{(u_i^\varepsilon u_j^\varepsilon)}\phi^2 dx. \end{aligned} \tag{2.2}$$

...

Diffusive Limit in Easy Cases

Theorem

Suppose initial data $\{g_i\}$ *has uniform lower and upper bounds* $0 < N \leq g_i \leq M$ *for* $i = 1, \dots, 2n$*, then the unique weak solution of* (1.1) u_i^ε *converge to the same limit* $\frac{\rho}{2n}$ *in* $L^2_{loc}(Q)$ *as* $\varepsilon \to 0$*. Here* ρ *is the unique weak solution of the Cauchy problem*

$$\partial_t \rho - \nabla \cdot \left(\frac{1}{n^{1-\alpha} \cdot \rho^\alpha} \nabla \rho \right) = 0$$

with initial data $\sum_{i=1}^n g_i$.

Diffusive Limit with General Initial Data

- Theorem requires uniform initial bounds $N \leq g_i \leq M$
- Local uniform bounds $N \leq u_i^\varepsilon \leq M$ on $K \subset\subset Q$ on each K is enough
- IDEA. Assume $\Psi(x) \leq g_i \leq M$ and prove u_i keep this shape of lower bound shape by making explicit barrier having initial shape like Ψ
- To construct explicit (sub or super) solution, we find u_i^ε of the form $u_i^\varepsilon = \frac{1}{2n}(\rho + A_i\varepsilon + B_i\varepsilon^2)$ and find A_i and B_i using formal asymptotic expansion.

Comparison Principle

Definition

$(u_i)_{i=1}^{2n} \in C([0,T], L^\infty_{loc}(\mathbb{R}^n)) \cap L^1_{loc}(Q_T)$ is a weak subsolution of (1.1) if

$$\begin{cases} \partial_t u_i^\varepsilon + \frac{1}{\varepsilon}\nabla_i u_i^\varepsilon - \frac{1}{2n\varepsilon^2}\sum_{j=1}^{2n}(u_j^\varepsilon + u_i^\varepsilon)^\alpha(u_j^\varepsilon - u_i^\varepsilon) \le 0 \\ \partial_t u_{i+n}^\varepsilon - \frac{1}{\varepsilon}\nabla_i u_{i+n}^\varepsilon - \frac{1}{2n\varepsilon^2}\sum_{j=1}^{2n}(u_j^\varepsilon + u_{i+n}^\varepsilon)^\alpha(u_j^\varepsilon - u_{i+n}^\varepsilon) \le 0 \end{cases} \tag{4.1}$$

in distribution sense.

Similarly we may define supersolution of (1.1).

Comparison Principle(2)

Lemma (L^1-contraction)

If u_i and v_i are subsolution and supersolution of (1.1), respectively, then for $t_2 > t_1 \geq 0$

$$\left[\int_{B(0,R)} \sum_{i=1}^{2n} (u_i - v_i)^+ dx\right](t_2) \leq \left[\int_{B(0,R+\frac{t_2-t_1}{\varepsilon})} \sum_{i=1}^{2n} (u_i - v_i)^+ dx\right](t_1) \tag{4.2}$$

$$\left[\int_{\mathbb{R}^n} \sum_{i=1}^{2n} (u_i - v_i)^+ dx\right](t_2) \leq \left[\int_{\mathbb{R}^n} \sum_{i=1}^{2n} (u_i - v_i)^+ dx\right](t_1). \tag{4.3}$$

- $u_i \leq v_i$ on $B(0, R + \frac{t_2-t_1}{\varepsilon})$ at $t = t_1$ for all $i = 1, \ldots, 2n$ implies that $u_i \leq v_i$ on $B(0,R)$ $t \in (t_1, t_2)$ for all $i = 1, \ldots, 2n$.
- $u_i \leq v_i$ on $\mathbb{R}^n$ at $t = t_1$ for all $i = 1, \ldots, 2n$ implies that $u_i \leq v_i$ on $\mathbb{R}^n$ $t \in (t_1, t_2)$ for all $i = 1, \ldots, 2n$.

After a long computation with symmetry assumption

$$A_i = -\left(\frac{n}{\rho}\right)^{\alpha} D_{v_i}\rho = \begin{cases} -\left(\frac{n}{\rho}\right)^{\alpha}\partial_i\rho, & 1 \le i \le n \\ \left(\frac{n}{\rho}\right)^{\alpha}\partial_{i-n}\rho, & n+1 \le i \le 2n \end{cases} \quad \text{and}$$

$$B_i = B_{i+n} = \left(\frac{n}{\rho}\right)^{\alpha}\partial_i\left(\left(\frac{n}{\rho}\right)^{\alpha}\partial_i\rho\right) - \frac{\alpha}{2\rho}\left(\left(\frac{n}{\rho}\right)^{\alpha}\partial_i\rho\right)^2 \quad 1 \le i \le n. \tag{4.4}$$

For our purpose, $A_i\varepsilon$ and $B_i\varepsilon^2$ need to be comparable to ρ, but usually $|B_i\varepsilon^2|\&|A_i\varepsilon| >> \rho$ as $x \to \infty$. Roughly, our "key observation" is that we need $|B_i\varepsilon^2|\&|A_i\varepsilon| < c\rho$ only on the region $|x| = O(\frac{1}{\varepsilon})$ since we need a local bound and equation has finite speed $(= \frac{1}{\varepsilon})$ of propagation.

- This requires profile of ρ should not decay so fast as $|x| \to \infty$
- Simple calculation says $|x|^{-\frac{2}{\alpha}}$ optimal rate in this regards
- Tail profile of Barrenblat solution of equation $\partial_t \rho = \Delta(\rho^{1-\alpha})$

For $n \geq 3$, $0 \leq 1-\alpha < \frac{n-2}{n}$, and $R > 0$, define

$$\Psi_{\alpha,n,R,T}(x,t) = C_\alpha \left(\frac{T-ct}{|x|^2+R^2} \right)^{\frac{1}{\alpha}}, \quad c-1 > \frac{2}{\alpha} \frac{1}{n-\frac{2}{\alpha}} \tag{4.5}$$

Then we may check

$$\begin{aligned} |A_i| &\leq c_0 \frac{\Psi}{T-ct}(|x|^2+R^2)^{\frac{1}{2}}, & |\partial_t A_i| &\leq c_0 \frac{\Psi}{(T-ct)^2}(|x|^2+R^2)^{\frac{1}{2}} \\ |B_i| &\leq c_0 \frac{\Psi}{(T-ct)^2}(|x|^2+R^2), & |D_{v_i} B_i| &\leq c_0 \frac{\Psi}{(T-ct)^2}(|x|^2+R^2)^{\frac{1}{2}} \\ |\partial_t B_i| &\leq c_0 \frac{\Psi}{(T-ct)^3}(|x|^2+R^2), & |\Psi^\alpha A_i| &\leq c_0 \frac{\Psi}{(|x|^2+R^2)^{\frac{1}{2}}} \\ |\Psi^\alpha B_i| &\leq c_0 \frac{\Psi}{(T-ct)} & &\text{and so on.} \end{aligned} \tag{4.6}$$

Proposition

In above setting, if $\overline{u_i}$ *are defined explicitly using* Ψ *in place of* ρ, *there exists* $c_1 = c_1(c, n, \alpha) > 0$ *s.t.*

$$\frac{1}{4n}\Psi \leq \overline{u_i^\varepsilon} \leq \frac{3}{4n}\Psi \quad \text{for all } i = 1, \dots, 2n$$

on $(|x|^2 + R^2)^{\frac{1}{2}} < \frac{c_1}{\varepsilon}(T - tc)$ *and* $t \in [0, \frac{T}{c})$.

Proposition

For $\Psi(x, t)$, *there is* $c_2 = c_2(c, n, \alpha) > 0$ *such that* $\overline{u_i}$ *is subsolution of* (1.1) *on* $(|x|^2 + R^2)^{\frac{1}{2}} < \frac{c_2}{\varepsilon}(T - tc)$ *and* $t \in [0, \frac{T}{c})$.

Proposition

For $n \geq 3$, $0 \leq 1-\alpha < \frac{n-2}{n}$, *suppose mild solution,* u_i^ε, *of* (1.1) *has initial data* $g_i \in L^1_{loc}(\mathbb{R}^n)$ *with lower bound* $g_i \geq \frac{3}{4n}\Psi_{R,T}(0)$ *for some* R *and* $T > 0$. *Then there exists universal constants* $C = C(n,\alpha) > 0$ *such that for every space-time compact set* $K \subset \mathbb{R}^n \times [0, CT)$, *there exist* $\varepsilon_K > 0$ *satisfying* $u_i^\varepsilon \geq \frac{1}{4n}\Psi_{R,T}$ *on* K *for* $0 < \varepsilon < \varepsilon_K$.

Main Theorem

Definition (Admissible Initial data)

$X_{n,\alpha}$ is a collection of nonnegative $g \in L^1_{loc}(\mathbb{R}^n)$ such that there exists nonnegative $f \in L^1_{loc}(\mathbb{R}^n)$ with $f - g \in L^1(\mathbb{R}^n)$ and f satisfies

$$\liminf_{|x|\to\infty} |x|^{\frac{2}{\alpha}} f > 0 \quad \text{if } \frac{2}{n} \leq \alpha \leq 1 \tag{5.1}$$

and

$$\begin{aligned} \limsup_{|x|\to\infty} |x|^{-2} \log(f+1) < \infty \qquad & \text{if } 0 \leq \alpha \leq 1 \\ \limsup_{|x|\to\infty} |x|^{\frac{2}{\alpha}} f < \infty \qquad & \text{if } -1 \leq \alpha < 0 \end{aligned} \tag{5.2}$$

Theorem

For the interaction rates $(a+b)^\alpha$ *with* $|\alpha| \le 1$, *suppose* $\{u_i^\varepsilon\}$ *are unique mild solutions of* (1.1) *and have initial data* $g_i \in X_{n,\alpha}$.

- *Then, there exists* $\overline{T} = \overline{T}(n, \alpha, \{g_i\}) \in (0, \infty]$ *such that*

$$u_i^\varepsilon \to \frac{\rho}{2n} \text{ in } L^1_{loc}(Q_T) \text{ for all } 0 < T < \overline{T} \text{ as } \varepsilon \to 0. \tag{5.3}$$

  In case $\alpha = 1$, *if solution of* (1.2) *have no uniqueness for the initial data* $\sum_{i=1}^{2n} g_i$, *the convergence takes along a subsequence for each given sequence* $\varepsilon_j \to 0$. *Otherwise, the convergence is arbitrary as* $\varepsilon \to 0$.

continued

- If initial data g_i has decomposition

$$g_i = l_i + h_i,\ \ h_i \in L^1(\mathbb{R}^n),\ \text{ and }\ \ l_i \in C^1(\mathbb{R}^n) \text{ with } \int |Dl_i| < \infty \quad (5.4)$$

, then

$$\rho^\varepsilon \to \rho \text{ in } C([0,T], L^1_{loc}(\mathbb{R}^n)) \text{ for all } 0 < T < \overline{T}. \quad (5.5)$$

- When $\alpha \neq 1$, $\rho \in C([0,T], L^1_{loc}(\mathbb{R}^n))$ is the unique weak solution of $\partial_t \rho - \nabla \cdot (\frac{1}{n^{1-\alpha} \cdot \rho^\alpha} \nabla \rho) = 0$ with initial data $\sum_{i=1}^{2n} g_i$. In case $\alpha = 1$, $\rho \in C([0,T], L^1_{loc}(\mathbb{R}^n))$ is some weak solution of $\partial_t \rho - \nabla \cdot (\frac{1}{\cdot \rho} \nabla \rho) = 0$ with initial data $\sum_{i=1}^{2n} g_i$.

continued

- $\overline{T}$ could be taken as $\overline{T} = C(n,\alpha) \cdot min(T_1, T_2),\ T_i \in (0,\infty]$ where

$$T_1 = \begin{cases} [min_{i=1}^{2n}(\liminf_{|x|\to\infty} |x|^{\frac{2}{\alpha}} f_i)]^{\frac{1}{\alpha}} & \text{if } \frac{2}{n} \le \alpha \le 1 \\ \infty & \text{if } -1 \le \alpha < \frac{2}{n} \end{cases}$$

and

$$T_2 = \begin{cases} [max_{i=1}^{2n} \limsup_{|x|\to\infty} |x|^{-2} \log(f_i + 1)]^{-1} & \text{if } 0 \le \alpha \le 1 \\ [max_{i=1}^{2n}(\limsup_{|x|\to\infty} |x|^{\frac{2}{\alpha}} f_i)]^{\frac{1}{\alpha}} & \text{if } -1 \le \alpha < 0. \end{cases}$$

Here, negative power of 0 is defined to be ∞ and f_i is a function with $f_i - g_i \in L^1$ given from definition of admissible data.

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