Positive solutions of a singular Minkowski-curvature equation

Isabel Coelho

Instituto Superior de Engenharia de Lisboa

Workshop in Nonlinear PDEs

7 September 2015
We discuss the existence of radially symmetric positive solutions for the quasilinear equation

\[-\text{div}\left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}}\right) + a(|x|)u = b(|x|)g(u) \quad \text{in } B,\]

where $B$ is the open ball of radius $R$ centered at 0 in $\mathbb{R}^N$, and the functions $a(r)$, $b(r)$ and $g(s)$ are smooth.

Boundary conditions:

- **Dirichlet** $u = 0$ on $\partial B$
- **Neumann** $\partial u = 0$ on $\partial B$

- Bonheure, Coelho, De Coster, in preparation.
Minkowski curvature operator

Newton’s Second Law of Motion:

\[ F = ma = (mv)' = (mu')' \]

Special Theory of Relativity: the mass of a body increases with velocity

\[ m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \]

where \( m_0 = \) rest mass and \( c = 3 \times 10^8 \text{m/s} = \) the speed of light.

With the normalization \( m_0 = c = 1 \), we recover the equation

\[ F = \left( \frac{u'}{\sqrt{1 - |u'|^2}} \right)' \]

[The Feynman Lectures on Physics 1964]

Riemannian Geometry: Represents the local mean curvature of hypersurfaces in the Lorentz-Minkowski space \( \mathbb{L}^{N+1} \) with coordinates \((x_1, \ldots, x_N, t)\) and the metric \( \sum_{j=1}^{N} (dx_j)^2 - (dt)^2 \).

[Bartnik Simon 1982]
Positive solutions of the Dirichlet problem

We discuss existence and multiplicity of positive radial solutions of

\[
\begin{aligned}
-\text{div}\left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}}\right) &= f(|x|, u) \quad \text{in } B, \\
u &= 0 \quad \text{on } \partial B,
\end{aligned}
\]

where we may assume that

\[
f(r, s) = \lambda m(r)s^p + \mu n(r)s^q,
\]

with \(0 < p < 1 \leq q\), parameters \(\lambda, \mu \geq 0\) and \(m, n: [0, R] \rightarrow \mathbb{R}\) continuous and positive somewhere, in particular they may change sign.

The radially symmetric solutions of this problem satisfy

\[
\begin{aligned}
-\left(\frac{r^{N-1}u'}{\sqrt{1 - |u'|^2}}\right)' &= r^{N-1}\left(\lambda m(r)s^p + \mu n(r)s^q\right) \quad \text{in } ]0, R[, \\
u'(0) &= 0, \quad u(R) = 0.
\end{aligned}
\]
Main result for the Dirichlet problem

Theorem (existence and multiplicity of positive radial solutions)

Under our assumptions, consider problem (D).

Then we have

(sublinear problem) if \( \mu = 0 \) and \( 0 < p < 1 \), then for all \( \lambda > 0 \) problem (D) has at least one positive solution.

(linear problem) if \( \lambda = 0 \) and \( q = 1 \), there exists \( \mu^* > 0 \) such that for all \( \mu > \mu^* \) problem (D) has at least one positive solution.

(superlinear problem) if \( \lambda = 0 \) and \( q > 1 \), there exists \( \mu^* > 0 \) such that for all \( \mu > \mu^* \) problem (D) has at least two positive solutions.

(sub-superlinear problem) there exist \( \mu^* > 0 \) and a function

\[
\lambda: ]\mu^*, +\infty[ \rightarrow ]0, +\infty[\]

such that, for all \( \mu > \mu^* \) and all \( \lambda \in ]0, \lambda(\mu)[ \), then (D) has at least three positive solutions.
Idea of the proof

**Step 1: Global a priori estimates**

Any solution of the Dirichlet problem (D) satisfies

\[ \|u\|_\infty < R \quad \text{and} \quad \|u'\|_\infty \leq 1 - \varepsilon. \]
Idea of the proof (cont.)

**Step 2: An equivalent problem**

- We replace $s^p$ and $s^q$ with $	ilde{p}(s)$ and $	ilde{q}(s)$: continuous potentials with compact support

$$
\tilde{p}(s) = \begin{cases} 
  s^p & \text{if } 0 \leq s \leq R, \\
  \text{linear} & \text{if } |s| \geq R + 1,
\end{cases}
\tilde{q}(s) = \begin{cases} 
  s^q & \text{if } 0 \leq s \leq R, \\
  \text{linear} & \text{if } |s| \geq R + 1,
\end{cases}
$$

- We replace the operator with $\tilde{\psi}$: odd, increasing, asymptotically linear diffeomorphism of $\mathbb{R}$ into $\mathbb{R}$

$$
\tilde{\psi}(y) = \begin{cases} 
  \frac{y}{\sqrt{1 - y^2}} & \text{if } |y| \leq 1 - \varepsilon, \\
  \text{linear} & \text{if } |y| > 1 - \varepsilon.
\end{cases}
$$
Idea of the proof (cont.)

**Proposition**

A positive function $u \in C^1([0, R])$ is a solution of problem

\[
\begin{align*}
- \left( \frac{r^{N-1}u'}{\sqrt{1 - |u'|^2}} \right)' &= r^{N-1} \left( \lambda m(r)s^p + \mu n(r)s^q \right) \quad \text{in } ]0, R[, \\
\left( \begin{array}{c}
u'(0) = 0, \\
u(R) = 0,
\end{array} \right)
\end{align*}
\]

if and only if it is a solution of the modified problem

\[
\begin{align*}
- \left( r^{N-1}\tilde{\psi}(u') \right)' &= r^{N-1} \left( \lambda m(r)\tilde{p}(u) + \mu n(r)\tilde{q}(u) \right) \quad \text{in } ]0, R[, \\
\left( \begin{array}{c}
u'(0) = 0, \\
u(R) = 0.
\end{array} \right)
\end{align*}
\]

(D)

Goal: Find positive solutions of $(D_{mod})$. 

Isabel Coelho (ISEL)
Idea of the proof (conclusion)

**Step 3: Variational setting**

We look for critical points of the action functional associated with the modified problem \((D_{mod})\)

\[\tilde{I}(u) = \int_0^R r^{N-1} \tilde{\Psi}(u') \, dr - \lambda \int_0^R r^{N-1} m(r) \tilde{P}(u) \, dr - \mu \int_0^R r^{N-1} m(r) \tilde{Q}(u) \, dr,\]

in the functional space

\[\mathcal{H}_{N-1}(0, R) = \left\{ w \in W^{1,1}_{loc}([0, R]) : \int_0^R r^{N-1} |w'|^2 \, dr < +\infty \right\},\]

where

\[\tilde{\Psi}(y) = \int_0^y \psi(\xi) \, d\xi, \quad \tilde{P}(s) = \int_0^s \tilde{p}(\xi) \, d\xi \quad \text{and} \quad \tilde{Q}(s) = \int_0^s \tilde{q}(\xi) \, d\xi.\]

(see [Bonheure, Gomes, Sanchez 2005])
Bifurcation diagrams

$p < 1$

$p = 1$

$p > 1$

for the Minkowski-curvature equation

$p < 1$

$p = 1$

$p > 1$

for the classical semi-linear equation
Positive decreasing solutions of the Neumann problem

We discuss existence of radial positive decreasing solutions of

\[
\begin{cases}
- \text{div} \left( \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) + a(|x|)u = b(|x|)u^q & \text{in } B, \\
\partial_{\nu} u = 0 & \text{on } \partial B,
\end{cases}
\]

(N)

where the functions \(a(r)\) and \(b(r)\) are continuous and positive, \(a\) is increasing and \(b\) is decreasing and \(q > 1\).

Our main result:

**Theorem (existence of positive decreasing radial solutions)**

*Under our assumptions, problem (N) has at least one positive radially decreasing solution.*
Idea of the proof

**Step 1: Global a priori estimates**
Any solution of the Neumann problem (N) satisfies

\[ \|u\|_{\infty} < c_{\infty} \quad \text{and} \quad \|u'\|_{\infty} \leq 1 - \varepsilon. \]

**Step 2: An equivalent problem**
We replace the operator with \( \psi_{\beta} : \) odd, increasing diffeomorphism of \( \mathbb{R} \) into \( \mathbb{R} \) with superlinear growth at infinity

\[
\psi_{\beta}(y) = \begin{cases} 
\frac{y}{\sqrt{1 - y^2}} & \text{if } |y| \leq 1 - \varepsilon, \\
\alpha_{\beta} y^{p-1} + b_{\beta} y & \text{if } |y| > 1 - \varepsilon,
\end{cases}
\]

with \( p > N \geq 2 \).
Idea of the proof (cont.)

Proposition

A positive function $u \in C^1([0, R])$ is a solution of problem

\[
\begin{cases}
-\text{div} \left( \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) + a(|x|)u = b(|x|)u^q & \text{in } B, \\
\partial \nu u = 0 & \text{on } \partial B,
\end{cases}
\]

if and only if it is a solution of the modified problem

\[
\begin{cases}
-\text{div} (\psi_\beta(\nabla u)) + a(|x|)u = b(|x|)u^q & \text{in } B, \\
\partial \nu u = 0 & \text{on } \partial B.
\end{cases}
\]

The statement is a consequence of our choice of $p > N$ and the Sobolev Embedding Theorem

\[W^{1,p}(B) \subset L^\infty(B).\]
Idea of the proof (cont.)

**Step 3:** A completely continuous map in a cone

We will look for a solution of the Neumann problem in the cone of positive, nonincreasing, radial continuous functions:

\[ C_- = \{ u \in C(\overline{B}) : u \text{ is radial, } u \geq 0 \text{ and } u(r) \geq u(s) \text{ for every } 0 \leq r \leq s \leq R \}. \]

To prove our existence result, we will apply a suitable fixed point theorem to the operator \( T : C_- \to C^1(\overline{B}) \) defined as

\[ T(u) = v \]

with

\[
\begin{cases}
  -\text{div} (\psi_\beta(\nabla v)) + a(|x|)v = b(|x|)u^q & \text{in } B, \\
  \partial_\nu v = 0 & \text{on } \partial B.
\end{cases}
\]
Idea of the proof (cont.)

**Lemma (Auxiliary)**

Let $w \in C_-$. Then problem

\[
\begin{cases}
-\text{div}(\psi_\beta(\nabla u)) + a(|x|)u = w(|x|) & \text{in } B, \\
\partial_\nu u = 0 & \text{on } \partial B,
\end{cases}
\]

has a unique solution $u_\beta$ that belongs to $C_-$. Moreover, the map $K : C_- \to C_- \text{ defined by}$

\[K(w) = u_\beta\quad \text{where } u_\beta \text{ is the solution of } (N_{aux})\]

is completely continuous.

**Corollary**

The operator $T$ is completely continuous and satisfies $T(C_-) \subseteq C_-$. 
Idea of the proof (conclusion)

**Step 4**: A fixed point theorem in a cone

**Theorem (expansive form [Benjamin 1971])**

Let $\mathcal{C}_-$ be a cone in $X$ and $T: \mathcal{C}_- \rightarrow \mathcal{C}_-$ a completely continuous operator. If there exist positive constants $a < b$ such that

- $Tu \neq \lambda u$ for any $\lambda > 1$ for all $u \in \mathcal{C}_-$ with $\|u\| = a$
- $\exists p \in \mathcal{C}_- \setminus \{0\}$ such that $u - Tu \neq \lambda p$ for any $\lambda \geq 0$ for all $u \in \mathcal{C}_-$ with $\|u\| = b$

then $T$ has a **fixed point** $u$ in $\mathcal{C}_-$ satisfying $a \leq \|u\| \leq b$. 

![Diagram of cones and operators](image)
We proved the existence of multiple radial positive solutions of the Dirichlet problem

\[
\begin{cases}
-\text{div} \left( \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) = \mu f(|x|, u) & \text{in } B, \\
u = 0 & \text{on } \partial B,
\end{cases}
\]

(D)

according to the value of the parameter \( \mu \) and the behaviour of \( f(r, s) \) near \( s = 0 \).

We proved the existence of a positive radial decreasing solution of the Neumann problem

\[
\begin{cases}
-\text{div} \left( \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) + a(|x|)u = b(|x|)u^q & \text{in } B, \\
\partial_{\nu} u = 0 & \text{on } \partial B,
\end{cases}
\]

(N)

for \( a(r) \) is increasing, \( b(r) \) is decreasing and \( q > 1 \).
For here to the whole space...

We could use our results to study the existence of entire solutions of

\[
\begin{aligned}
&\left\{\begin{array}{l}
-\text{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) + a(|x|)u = b(|x|)u^q \\
u(x) \rightarrow 0 \quad \text{as} \quad |x| \rightarrow +\infty.
\end{array}\right. \\
\end{aligned}
\]

\quad \text{in } \mathbb{R}^N,

(E)

What can happen:

Theorem

Problem (E) has at least one positive radially decreasing solution.
THANK YOU FOR YOUR ATTENTION!

