Positive solutions of a singular Minkowski-curvature equation

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Workshop in Nonlinear PDEs

7 September 2015

We discuss the existence of radially symmetric positive solutions for the quasilinear equation

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right)+a(|x|)u=b(|x|)g(u) \quad \text{in } B,$$

where B is the open ball of radius R centered at 0 in $\mathbb{R}^N$, and the functions $a(r)$, $b(r)$ and $g(s)$ are smooth.

Boudary conditions:

- Dirichlet $u=0$ on ∂B
- Neumann $\partial u=0$ on ∂B

- Coelho, Corsato, Rivetti, *Positive radial solutions of the Dirichlet problem for the Minkowski-curvature equation in a ball*, Topol. Methods Nonlinear Anal. **44** (2014), no. 1, 23–39.
- Bonheure, Coelho, De Coster, in preparation.

Minkowski curvature operator

Newton's Second Law of Motion:

$$F = ma = (mv)' = \big(mu'\big)'$$

Special Theory of Relativity: the mass of a body increases with velocity

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where $m_0 =$ rest mass and $c = 3 \times 10^8 m/s =$ the speed of light.
With the normalization $m_0 = c = 1$, we recover the equation

$$F = \left(\frac{u'}{\sqrt{1 - |u'|^2}} \right)'$$

[The Feynman Lectures on Physics 1964]

Riemannian Geometry: Represents the local mean curvature of hypersurfaces in the Lorentz-Minkowski space $\mathbb{L}^{N+1}$ with coordinates $(x_1, \ldots, x_N, t)$ and the metric $\sum_{j=1}^{N} (dx_j)^2 - (dt)^2$.

[Bartnik Simon 1982]

Positive solutions of the Dirichlet problem

We discuss existence and multiplicity of positive radial solutions of

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = f(|x|,u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where we may assume that

$$f(r,s) = \lambda\, m(r) s^p + \mu\, n(r) s^q,$$

with $0 < p < 1 \leq q$, parameters $\lambda, \mu \geq 0$ and $m, n \colon [0,R] \to \mathbb{R}$ continuous and positive somewhere, in particular they may change sign.

The radially symmetric solutions of this problem satisfy

$$\begin{cases} -\left(\dfrac{r^{N-1}u'}{\sqrt{1-|u'|^2}}\right)' = r^{N-1}\Big(\lambda\, m(r) s^p + \mu\, n(r) s^q\Big) & \text{in }]0,R[, \\ u'(0) = 0, \quad u(R) = 0. \end{cases} \tag{D}$$

Main result for the Dirichlet problem

Theorem (existence and multiplicity of positive radial solutions)

Under our assumptions, consider problem (D).

Then we have

(sublinear problem) *if* $\mu = 0$ *and* $0 < p < 1$, *then for all* $\lambda > 0$ *problem* (D) *has at least one positive solution.*

(linear problem) *if* $\lambda = 0$ *and* $q = 1$, *there exists* $\mu^* > 0$ *such that for all* $\mu > \mu^*$ *problem* (D) *has at least one positive solution.*

(superlinear problem) *if* $\lambda = 0$ *and* $q > 1$, *there exists* $\mu^* > 0$ *such that for all* $\mu > \mu^*$ *problem* (D) *has at least two positive solutions.*

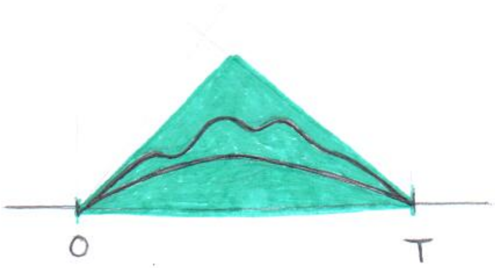
(sub-superlinear problem) *there exist* $\mu^* > 0$ *and a function* $\lambda\colon]\mu^*, +\infty[\to]0, +\infty[$ *such that, for all* $\mu > \mu^*$ *and all* $\lambda \in]0, \lambda(\mu)[$, *then* (D) *has at least three positive solutions.*

Idea of the proof

STEP 1: Global a priori estimates

Any solution of the Dirichlet problem (D) satisfies

$$\|u\|_\infty < R \quad \text{and} \quad \|u'\|_\infty \leq 1 - \varepsilon.$$

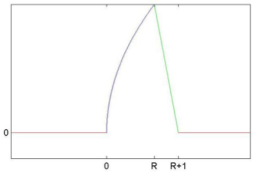


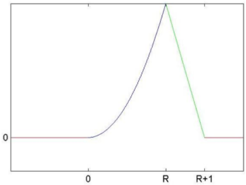
Idea of the proof (cont.)

Step 2: An equivalent problem

- We replace s^p and s^q with $\tilde{p}(s)$ and $\tilde{q}(s)$: continuous potentials with compact support

$$\tilde{p}(s) = \begin{cases} s^p & \text{if } 0 \le s \le R, \\ \text{linear} & \\ 0 & \text{if } |s| \ge R+1, \end{cases} \qquad \tilde{q}(s) = \begin{cases} s^q & \text{if } 0 \le s \le R, \\ \text{linear} & \\ 0 & \text{if } |s| \ge R+1, \end{cases}$$





- We replace the operator with $\tilde{\psi}$: odd, increasing, asymptotically linear diffeomorphism of $\mathbb{R}$ into $\mathbb{R}$

$$\tilde{\psi}(y) = \begin{cases} \dfrac{y}{\sqrt{1-y^2}} & \text{if } |y| \le 1-\varepsilon, \\ \text{linear} & \text{if } |y| > 1-\varepsilon. \end{cases}$$

Idea of the proof (cont.)

Proposition

A positive function $u \in C^1([0,R])$ *is a solution of problem*

$$\begin{cases} -\left(\dfrac{r^{N-1}u'}{\sqrt{1-|u'|^2}}\right)' = r^{N-1}\Big(\lambda\, m(r) s^p + \mu\, n(r) s^q\Big) \quad \textit{in }]0,R[, \\ u'(0)=0, \quad u(R)=0, \end{cases} \tag{D}$$

if and only if it is a solution of the modified problem

$$\begin{cases} -\big(r^{N-1}\tilde{\psi}(u')\big)' = r^{N-1}\Big(\lambda\, m(r)\tilde{p}(u) + \mu\, n(r)\tilde{q}(u)\Big) \quad \textit{in }]0,R[, \\ u'(0)=0, \quad u(R)=0. \end{cases} \tag{D$_{mod}$}$$

Goal: Find positive solutions of (D$_{mod}$).

Idea of the proof (conclusion)

STEP 3: Variational setting

We look for critical points of the action functional associated with the modified problem (D_{mod})

$$\tilde{\mathcal{I}}(u) = \int_0^R r^{N-1}\tilde{\Psi}(u')\,dr - \lambda\int_0^R r^{N-1}m(r)\tilde{P}(u)\,dr - \mu\int_0^R r^{N-1}m(r)\tilde{Q}(u)\,dr,$$

in the functional space

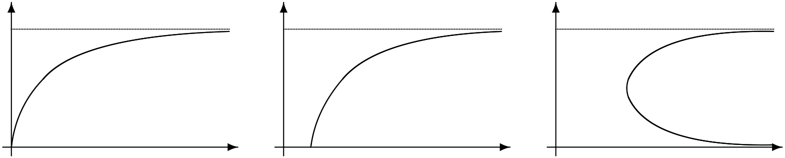
$$\mathcal{H}_{N-1}(0,R) = \Big\{ w \in W^{1,1}_{\text{loc}}(]0,R]) : \int_0^R r^{N-1}\left|w'\right|^2\,dr < +\infty \quad \text{and} \quad w(R) = 0 \Big\},$$

(see [Bonheure, Gomes, Sanchez 2005])

where

$$\tilde{\Psi}(y) = \int_0^y \psi(\xi)\,d\xi, \quad \tilde{P}(s) = \int_0^s \tilde{p}(\xi)\,d\xi \quad \text{and} \quad \tilde{Q}(s) = \int_0^s \tilde{q}(\xi)\,d\xi.$$

Bifurcation diagrams



$p < 1$ $p = 1$ $p > 1$

for the Minkowski-curvature equation

$p < 1$ $p = 1$ $p > 1$

for the classical semi-linear equation

Positive decreasing solutions of the Neumann problem

We discuss existence of radial positive decreasing solutions of

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) + a(|x|)u = b(|x|)u^q & \text{in } B, \\ \partial_\nu u = 0 & \text{on } \partial B, \end{cases} \tag{N}$$

where the functions $a(r)$ and $b(r)$ are continuous and positive, a is increasing and b is decreasing and $q > 1$.

Our main result:

Theorem (existence of positive decreasing radial solutions)

Under our assumptions, problem (N) *has at least one positive radially decreasing solution.*

Idea of the proof

STEP 1: Global a priori estimates

Any solution of the Neumann problem (N) satisfies

$$\|u\|_\infty < c_\infty \quad \text{and} \quad \|u'\|_\infty \leq 1-\varepsilon.$$

STEP 2: An equivalent problem

We replace the operator with ψ_β : odd, increasing diffeomorphism of $\mathbb{R}$ into $\mathbb{R}$ with superlinear growth at infinity

$$\psi_\beta(y) \begin{cases} \dfrac{y}{\sqrt{1-y^2}} & \text{if } |y| \leq 1-\varepsilon, \\ a_\beta y^{p-1} + b_\beta y & \text{if } |y| > 1-\varepsilon, \end{cases}$$

with $p > N \geq 2$.

Idea of the proof (cont.)

Proposition

A positive function $u \in C^1([0,R])$ is a solution of problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) + a(|x|)u = b(|x|)u^q & \textit{in } B, \\ \partial_\nu u = 0 & \textit{on } \partial B, \end{cases} \tag{N}$$

if and only if it is a solution of the modified problem

$$\begin{cases} -\mathrm{div}\big(\psi_\beta(\nabla u)\big) + a(|x|)u = b(|x|)u^q & \textit{in } B, \\ \partial_\nu u = 0 & \textit{on } \partial B. \end{cases} \tag{N$_{mod}$}$$

The statement is a consequence of our choice of $p > N$ and the Sobolev Embedding Theorem

$$W^{1,p}(B) \subset L^\infty(B).$$

Idea of the proof (cont.)

STEP 3: A completely continuous map in a cone

We will look for a solution of the Neumann problem in the cone of positive, nonincreasing, radial continuous functions:

$$\mathcal{C}_- = \big\{u \in C(\overline{B}) \colon u \text{ is radial}, \ \ u \geq 0 \quad \text{and} \ \ u(r) \geq u(s) \ \ \text{for every } 0 \leq r \leq s \leq R\big\}.$$

To prove our existence result, we will apply a suitable fixed point theorem to the operator $T : \mathcal{C}_- \to C^1(\overline{B})$ defined as

$$T(u) = v$$

with

$$\begin{cases} -\operatorname{div}\big(\psi_\beta(\nabla v)\big) + a(|x|)v = b(|x|)u^q & \text{in } B, \\ \partial_\nu v = 0 & \text{on } \partial B. \end{cases}$$

Idea of the proof (cont.)

Lemma (Auxiliary)

Let $w \in \mathcal{C}_-$. Then problem

$$\begin{cases} -\mathrm{div}\big(\psi_\beta(\nabla u)\big) + a(|x|)u = w(|x|) & \text{in } B, \\ \partial_\nu u = 0 & \text{on } \partial B, \end{cases} \qquad (\mathsf{N}_{aux})$$

has a unique solution u_β that belongs to $\mathcal{C}_-$.
Moreover, the map $K\colon \mathcal{C}_- \to \mathcal{C}_-$ defined by

$$K(w) = u_\beta \qquad \text{where } u_\beta \text{ is the solution of } (\mathsf{N}_{aux})$$

is completely continuous.

Corollary

The operator T is completely continuous and satisfies $T(\mathcal{C}_-) \subseteq \mathcal{C}_-$.

Idea of the proof (conclusion)

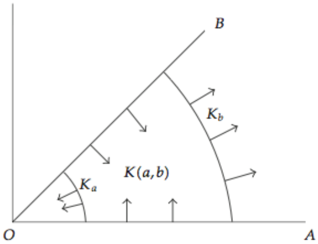
Step 4: A fixed point theorem in a cone

Theorem (expansive form [Benjamin 1971])

Let $\mathcal{C}_-$ be a cone in X and $T : \mathcal{C}_- \to \mathcal{C}_-$ a completely continuous operator. If there exist positive constants $a < b$ such that

- $Tu \neq \lambda u$ *for any* $\lambda > 1$ *for all* $u \in \mathcal{C}_-$ *with* $\|u\| = a$
- $\exists\, p \in \mathcal{C}_- \setminus \{0\}$ *such that* $u - Tu \neq \lambda p$ *for any* $\lambda \geq 0$ *for all* $u \in \mathcal{C}_-$ *with* $\|u\| = b$

then T has a fixed point u in $\mathcal{C}_-$ satisfying $a \leq \|u\| \leq b$.



Conclusion

- We proved the existence of multiple radial positive solutions of the Dirichlet problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = \mu\, f(|x|,u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{D}$$

according to the value of the parameter μ and the behaviour of $f(r,s)$ near $s=0$.

- We proved the existence of a positive radial decreasing solution of the Neumann problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) + a(|x|)u = b(|x|)u^q & \text{in } B, \\ \partial_\nu u = 0 & \text{on } \partial B, \end{cases} \tag{N}$$

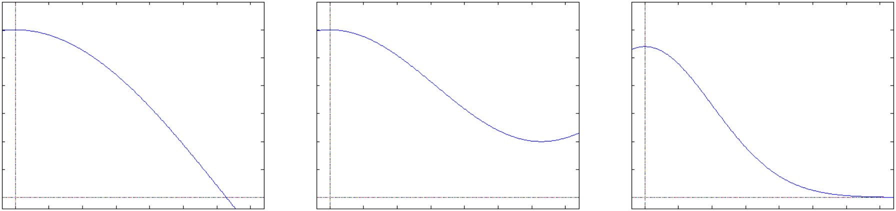
for $a(r)$ is increasing, $b(r)$ is decreasing and $q>1$.

For here to the whole space...

We could use our results to study the existence of entire solutions of

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) + a(|x|)u = b(|x|)u^q & \text{in } \mathbb{R}^N, \\ u(x) \longrightarrow 0 & \text{as } |x| \to +\infty. \end{cases} \tag{E}$$

What can happen:



Theorem

Problem (E) has at least one positive radially decreasing solution.

THANK YOU FOR YOUR ATTENTION !

- R. Feynman, *The Feynman Lectures on Physics*, 1964.
- R. Bartnik, L. Simon, *Spacelike Hypersurfaces with Prescribed Boundary Values and Mean Curvature*, Commun. Math. Phys., 1982.
- D. Bonheure, J.M. Gomes, L. Sanchez, *Positive solutions of a second-order singular ordinary differential equation*, 2005.
- T. B. Benjamin, *A unified theory of conjugate flows*, 1971.