Bound and Ground states to systems of coupled NLS-KdV equations

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Scheme of the talk

- Systems of coupled NLS-KdV equations.
  1. Application in gravity water-waves.

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  5. Our results: Existence and multiplicity of Bound-Ground states (by variational and perturbation techniques).

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  5. Our results: Existence and multiplicity of Bound-Ground states (by variational and perturbation techniques).
  6. Sketch of the proofs.

Main references

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[C1] E.C.; *Existence of Bound and Ground States for a System of Coupled Nonlinear Schrödinger-KdV Equations.* C. R. Math. Acad. Sci. Paris (CRMAS) 2015.
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Related works on systems of NLS Eq.

[AC1] A. Ambrosetti, E.C.; CRMAS 2006.

[AC2] A. Ambrosetti, E.C.; J. Lond. Math. Soc. 2007.

[ACR] A. Ambrosetti, E.C., D. Ruiz; Calc. Var. Partial Differential Equations 2007.

[BW] T. Bartsch, Z.-Q. Wang; J. Partial Differential Equations 2006.

[LW] T-C. Lin, J. Wei; Comm. Math. Phys. 2005.

[MMP] L. Maia, E. Montefusco, B. Pellacci; Jour. Diff. Eqns. 2006.

[S] B. Sirakov; Comm. Math. Phys. 2007.

System of coupled NLS-KdV equations

$$\begin{cases} if_t + f_{xx} + |f|^2 f + \beta f g & = & 0 \\ g_t + g_{xxx} + g g_x + \frac{1}{2}\beta(|f|^2)_x & = & 0 \end{cases}$$

where $f = f(x,t) \in \mathbb{C}$ while $g = g(x,t) \in \mathbb{R}$, and β is the real coupling coefficient. Moreover, $|f|$, $|g| \to 0$ as $|x| \to \infty$.

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Capillary-gravity water waves.

This system appears in phenomena of interactions between short and long dispersive waves, arising in fluid mechanics, such as the interactions of capillary - gravity water waves. Indeed, f represents the short-wave, while g stands for the long-wave; see **[FO]** for more details.

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We look for solitary “traveling-waves”, namely solutions of the form

$$(f(x,t), g(x,t)) = \left(e^{i\omega t} e^{i\frac{c}{2}x} u(x-ct), v(x-ct)\right) \quad \text{with} \quad u,\, v \in W^{1,2}(\mathbb{R}).$$

Choosing $\lambda_1 = \omega + \frac{c^2}{4}$, $\lambda_2 = c$, we get that u, v are solutions of the following stationary system:

$$(S) \equiv \begin{cases} -u'' + \lambda_1 u & = & u^3 + \beta u v \\ -v'' + \lambda_2 v & = & \frac{1}{2}v^2 + \frac{1}{2}\beta u^2. \end{cases}$$

Functional Setting

The Sobolev space $E = W^{1,2}(\mathbb{R})$ is endowed with the classical norm and we also denote the equivalent norms

$$\|u\|_j^2 = \int_{\mathbb{R}} (|\nabla u|^2 + \lambda_j u^2) dx; \quad j = 1, 2.$$

We define $\mathbb{E} = E \times E$, $H = \{u \in E : u \text{ is radially symmetric}\}$, $\mathbb{H} = H \times H$.

Energy functional

The energy functional is given by

$$\Phi(\mathbf{u}) = I_1(u) + I_2(v) - \tfrac{1}{2}\beta \int_{\mathbb{R}} u^2 v \, dx, \qquad \mathbf{u} = (u, v) \in \mathbb{E},$$

where

$$I_1(u) = \tfrac{1}{2}\|u\|_1^2 - \tfrac{1}{4} \int_{\mathbb{R}} u^4 dx, \qquad I_2(v) = \tfrac{1}{2}\|v\|_2^2 - \tfrac{1}{6} \int_{\mathbb{R}} v^3 dx, \qquad u,\, v \in E.$$

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We also write

$$G_\beta(\mathbf{u}) = \tfrac{1}{4} \int_{\mathbb{R}} u^4 dx + \tfrac{1}{6} \int_{\mathbb{R}} v^3 dx + \tfrac{1}{2}\beta \int_{\mathbb{R}} u^2 v \, dx, \ \|\mathbf{u}\|^2 = \|u\|_1^2 + \|v\|_2^2, \quad \mathbf{u} \in \mathbb{E}.$$

Using this notation (useful for some computations) we can rewrite the energy functional

$$\Phi(\mathbf{u}) = \frac{1}{2}\|\mathbf{u}\|^2 - G_\beta(\mathbf{u}), \quad \mathbf{u} \in \mathbb{E}.$$

Nehari manifold

We work mainly in $\mathbb{H}$. Setting

$$\Psi(\mathbf{u}) = (\nabla\Phi(\mathbf{u})|\mathbf{u}) = (I_1'(u)|u) + (I_2'(v)|v) - \frac{3}{2}\beta\int_{\mathbb{R}} u^2 v\,dx,$$

we define the corresponding Nehari manifold

$$\mathcal{N} = \{\mathbf{u}\in\mathbb{H}\setminus\{\mathbf{0}\} : \Psi(\mathbf{u}) = 0\}.$$

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$$(\nabla\Psi(\mathbf{u}) \mid \mathbf{u}) = -\|\mathbf{u}\|^2 - \int_{\mathbb{R}} u^4\,dx < 0, \quad \forall\,\mathbf{u} \in \mathcal{N},$$

thus $\mathcal{N}$ is a smooth manifold locally near any point $\mathbf{u} \neq \mathbf{0}$ with $\Psi(\mathbf{u}) = 0$.
Moreover, it is not difficult to show that is a natural restriction and the functional is bounded below on $\mathcal{N}$.

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PS condition does not hold (in general)
We remember that in the one dimensional case, one cannot expects a compact embedding of E into $L^q(\mathbb{R})$ for $2 < q < \infty$.
Indeed, working on H, it is not true too. However, we will show that for a PS sequence we can find a subsequence for which the weak limit is a solution.
With some extra work one could also consider the non-negative radially decreasing functions, where one has compactness thanks to Beresticky-Lions (ARMA 1989).

The semi-trivial solution

Let V denotes the unique positive solution of $-v'' + v = v^2$, in H; Kwong, ARMA 89. Setting

$$V_2(x) = 2\lambda_2\, V(\sqrt{\lambda_2}\, x) = \frac{3\lambda_2}{\cosh^2\left(\frac{\sqrt{\lambda_2}}{2}x\right)},$$

one has that V_2 is the unique positive solution of $-v'' + \lambda_2 v = \frac{1}{2}v^2 \quad$ in H. Hence $\mathbf{v}_2 := (0, V_2)$ is a particular solution of our problem for any $\beta \in \mathbb{R}$. Furtheremore, it is the unique non-negative semi-trivial critical point of Φ.

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Proposition **[E.C.]**. There exists $\Lambda > 0$ such that:

(i) If $\beta < \Lambda$, then $\mathbf{v}_2$ is a strict local minimum of Φ constrained on $\mathcal{N}$.

(ii) For any $\beta > \Lambda$, then $\mathbf{v}_2$ is a saddle point of Φ constrained on $\mathcal{N}$. Moreover, $\inf_{\mathcal{N}} \Phi < \Phi(\mathbf{v}_2)$.

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Idea of the proof. Defining

$$\Lambda = \inf_{\varphi \in H \setminus \{0\}} \frac{\|\varphi\|_1^2}{\int_{\mathbb{R}} V_2 \varphi^2 dx}.$$

(i) We show that if $\beta < \Lambda$, $\forall\, \mathbf{h} \in T_{\mathbf{v}_2}\mathcal{N}$ then $D^2\Phi_{\mathcal{N}}(\mathbf{v}_2)[\mathbf{h}]^2 > c\|\mathbf{h}\|^2$, for some $c > 0$.

(ii) On the contrary, if $\beta > \Lambda$, we find $\mathbf{h}_0 \in T_{\mathbf{v}_2}\mathcal{N}$ so that $D^2\Phi_{\mathcal{N}}(\mathbf{v}_2)[\mathbf{h}_0]^2 < 0$, proving that the morse index of $\mathbf{v}_2$ as critical point of Φ on $\mathcal{N}$ is at least 1. Plainly, we infer that $\inf_{\mathcal{N}} \Phi < \Phi(\mathbf{v}_2)$.

Previous mathematical results

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Remarks. 1. Its solution could coincide with the semi-trivial solution $\mathbf{v}_2 = (0, V)$.
2. They do not study the range β small or negative.

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Theorem **[LZ]**. In the dimensional case $n = 2,\ 3$. $\exists\, \gamma > 0$ such that (S) has a radial bound state with both components nonzero provided $\beta > \gamma$.

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Remarks. 1. It is not considered the case β small or negative too.
2. Even more, in **[DFO,LZ]** have not been analyzed the existence of ground states.

Our main results

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Theorem 2 (Bound States).

(i) There exists $\varepsilon_0 > 0$ so that for any $0 < \varepsilon < \varepsilon_0$ and $\beta = \varepsilon\widetilde{\beta} > 0$, there exists an even bound state $\mathbf{u}_\varepsilon > \mathbf{0}$ with $\mathbf{u}_\varepsilon \to \mathbf{u}_0 = (U_1, V_2)$ as $\varepsilon \to 0$.

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(ii) If $\lambda_2 > \Lambda_2$ and $0 < \beta < \Lambda$, there exists an even bound state $\mathbf{u}^* > \mathbf{0}$ with $\Phi(\mathbf{u}^*) > \Phi(\mathbf{v}_2)$.

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(ii) If $\lambda_2 > \Lambda_2$ and $0 < \beta < \Lambda$, there exists an even bound state $\mathbf{u}^* > \mathbf{0}$ with $\Phi(\mathbf{u}^*) > \Phi(\mathbf{v}_2)$.

Remark. The coexistence of positive bound and ground states for $0 < \beta < \Lambda$ and λ_2 large is a great novelty and difference with the more studied (in the last years) systems of NLS equations, for which there is uniqueness of positive solution under appropriate conditions on the parameters (Ikoma, NODEA '09; Wei-Yao, CPAA '12), including the case $\beta > 0$ small.

Sketch of the proofs

Theorem 1 (Ground States).

1. Suppose that $\beta > \Lambda$, then (S) has a positive even ground state $\widetilde{\mathbf{u}}$.
2. There exists $\Lambda_2 > 0$ so that if $\lambda_2 > \Lambda_2$, for every $\beta > 0$, (S) has a positive even ground state $\widetilde{\mathbf{u}}$.

Proof of Theorem 1 (main ideas).

1. The previous Proposition gives that $\mathbf{v}_2$ is a saddle point of Φ on $\mathcal{N}$. By the Ekeland's Variational Principle, $\exists\, \{\mathbf{u}_k\}_{k\in\mathbb{N}} \subset \mathcal{N}$, a PS minimizing sequence, i.e.,

$$\Phi(\mathbf{u}_k) \to c = \inf_{\mathcal{N}} \Phi, \qquad \nabla_{\mathcal{N}} \Phi(\mathbf{u}_k) \to 0.$$

Using the properties of $\mathcal{N}$ we show that $\Phi'(\mathbf{u}_k) \to 0$ an also the sequence is bounded. By some measure arguments, (for a subsequence without changing notation) and after a translation, we do not loss the mass at infinity, i.e.,

$$\liminf_{k\to\infty} \int_{B_R(0)} (u_k^2 + v_k^2)dx \geq C > 0,$$

proving that the weak limit $\widetilde{\mathbf{u}} \not\equiv \mathbf{0}$. Even more, we show that $\widetilde{\mathbf{u}}$ is a critical point of Φ satisfying $\Phi(\widetilde{\mathbf{u}}) = \inf_{\mathcal{N}} \Phi$.

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Proof of Theorem 1 (main ideas).

1. To finish, we use the classical properties of the Schwarz symmetrization combined with variational arguments to show that indeed

$$\Phi(\widetilde{\mathbf{u}}) = \inf\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathbb{E}, \Phi'(\mathbf{u}) = 0\}.$$

To do so, we suppose for a contradiction that $\exists\, \mathbf{w}_0 \in \mathbb{E}$ a non-trivial critical point of Φ such that

$$\Phi(\mathbf{w}_0) < \Phi(\widetilde{\mathbf{u}}) = \min\{\Phi(\mathbf{u}) \, : \, \mathbf{u} \in \mathcal{N}\}.$$

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Proof of Theorem 1 (main ideas).

1. Setting $\mathbf{w} = |\mathbf{w}_0|$ there holds $\Phi(\mathbf{w}) = \Phi(\mathbf{w}_0)$, $\Psi(\mathbf{w}) = \Psi(\mathbf{w}_0)$.
For $\mathbf{w} = (w_1, w_2)$, we set $\mathbf{w}^\star = (w_1^\star, w_2^\star)$, where $w_j^\star$ is the Schwarz symmetric function associated to $w_j \geq 0$. Then, by the classical properties of the Schwarz symmetrization,

$$\|\mathbf{w}^\star\|^2 \leq \|\mathbf{w}\|^2, \qquad G_\beta(\mathbf{w}^\star) \geq G_\beta(\mathbf{w}),$$

thus, in particular, $\Psi(\mathbf{w}^\star) \leq \Psi(\mathbf{w})$. It allow us to prove that $\exists\,!\, 0 < t(\mathbf{w}^\star) < 1$ so that $t(\mathbf{w}^\star)\mathbf{w}^\star \in \mathcal{N}$. Thus,

$$\Phi(t(\mathbf{w}^\star)\mathbf{w}^\star) = \frac{1}{6}t^2\|\mathbf{w}^\star\|^2 + \frac{1}{12}t^4\int_{\mathbb{R}}(w_1^\star)^4\,dx \leq \frac{1}{6}\|\mathbf{w}\|^2 + \frac{1}{12}\int_{\mathbb{R}} w_1^4\,dx$$

$$= \Phi(\mathbf{w}) < \Phi(\widetilde{\mathbf{u}}) = \min\{\Phi(\mathbf{u}) \,:\, \mathbf{u} \in \mathcal{N}\}.$$

A contradiction which demonstrates the result.

Sketch of the proofs

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Proof of Theorem 1 (main ideas).

2. Since by previous proposition $\mathbf{v}_2$ is a strict local minimum for $\beta < \Lambda$, a priori, $\widetilde{\mathbf{u}}$ could coincide with $\mathbf{v}_2$.

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Proof of Theorem 1 (main ideas).

2. Since by previous proposition $\mathbf{v}_2$ is a strict local minimum for $\beta < \Lambda$, a priori, $\widetilde{\mathbf{u}}$ could coincide with $\mathbf{v}_2$.
The idea to prove that $\widetilde{\mathbf{u}} \not\equiv \mathbf{v}_2$, is to find $\mathbf{u}_1 \in \mathcal{N}$ so that $\Phi(\mathbf{u}_1) < \Phi(\mathbf{v}_2)$.
Indeed, we look for $\mathbf{u}_1 = t(V_2, V_2)$ where $t > 0$ is the unique value such that $\mathbf{u}_1 \in \mathcal{N}$.

Sketch of the proofs

Theorem 1 (Ground States).

1. Suppose that $\beta > \Lambda$, then (S) has a positive even ground state $\widetilde{\mathbf{u}}$.
2. There exists $\Lambda_2 > 0$ so that if $\lambda_2 > \Lambda_2$, for every $\beta > 0$, (S) has a positive even ground state $\widetilde{\mathbf{u}}$.

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The positivity follows by the maximum principle applied to each single equation.

Sketch of the proofs

Theorem 2 (Bound States).

1. There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and $\beta = \varepsilon\widetilde{\beta} > 0$, there exists an even bound state $\mathbf{u}_\varepsilon > \mathbf{0}$ with $\mathbf{u}_\varepsilon \to \mathbf{u}_0 = (U_1, V_2)$ as $\varepsilon \to 0$.
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1. Roughly speaking, by the non-degeneracy of U_1 and V_2 as critical points of their corresponding energy functionals I_1 and I_2 on the radial space H, clearly $\mathbf{u}_0 = (U_1, V_2)$ is a non-degenerate critical point of Φ on $\mathbb{H}$.

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In order to show the positivity of $\mathbf{u}_\varepsilon$, we assume the contrary, $\|\mathbf{u}_\varepsilon^-\| \neq 0$, then estimating (by variational techniques) the energies of $\mathbf{u}_\varepsilon^\pm$ we obtain a contradiction.

Sketch of the proofs

Theorem 2 (Bound States).

1. There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and $\beta = \varepsilon\widetilde{\beta} > 0$, there exists an even bound state $\mathbf{u}_\varepsilon > \mathbf{0}$ with $\mathbf{u}_\varepsilon \to \mathbf{u}_0 = (U_1, V_2)$ as $\varepsilon \to 0$.
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Proof of Theorem 2 (main ideas).

2. By the proposition above and Theorem 1, we have that $\widetilde{\mathbf{u}}$, $\mathbf{v}_2$ are strict local minima of Φ on $\mathcal{N}$. Thus, by the Mountain Pass Theorem (Ambrosetti-Rabinowitz, JFA 1973) applied on the set of paths

$$\Gamma = \{\gamma \in \mathcal{C}([0,1];\mathcal{N}) : \gamma(0) = \widetilde{\mathbf{u}}, \gamma(1) = \mathbf{u}_2\},$$

we are able to find a critical point $\mathbf{u}^*$ with

$$\Phi(\mathbf{u}^*) = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \Phi(\gamma(t)) > \max\{\Phi(\widetilde{\mathbf{u}}), \Phi(\mathbf{v}_2)\} = \Phi(\mathbf{v}_2).$$

Note. The lack of compactness can be circumvent in a similar way as in previous Theorem 1.

Sketch of the proofs

Theorem 2 (Bound States).

1. There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and $\beta = \varepsilon\widetilde{\beta} > 0$, there exists an even bound state $\mathbf{u}_\varepsilon > \mathbf{0}$ with $\mathbf{u}_\varepsilon \to \mathbf{u}_0 = (U_1, V_2)$ as $\varepsilon \to 0$.
2. If $\lambda_2 > \Lambda_2$ and $0 < \beta < \Lambda$, there exists an even bound state $\mathbf{u}^* > \mathbf{0}$ with $\Phi(\mathbf{u}^*) > \Phi(\mathbf{v}_2)$.

Proof of Theorem 2 (main ideas).

2. In order to prove $\mathbf{u}^* > \mathbf{0}$, we consider the new system:

$$(S^+) \equiv \begin{cases} -u'' + \lambda_1 u &= (u^+)^3 + \beta u^+ v \\ -v'' + \lambda_2 v &= \frac{1}{2}v^2 + \frac{1}{2}\beta(u^+)^2. \end{cases}$$

The corresponding associated functional Φ^+, and Nehari Manifold $\mathcal{N}^+$ have similar properties, but Φ^+ is not $\mathcal{C}^2$.

To circumvent this difficulty, we show directly (without using the second derivative of Φ on $\mathcal{N}$ as in the proposition) that $\mathbf{v}_2$ is a strict local minimum of Φ^+ on $\mathcal{N}^+$.

To do so, if $\mathbf{h} \in T_{\mathbf{v}_2}\mathcal{N}^+$ with $\|\mathbf{h}\| = 1$, we consider $\mathbf{v}_\varepsilon = (\varepsilon h_1, V_2 + \varepsilon h_2)$. Plainly, there exists a unique $t_\varepsilon > 0$ so that $t_\varepsilon \mathbf{v}_\varepsilon \in \mathcal{N}^+$. Thus, we have to prove there exists $\varepsilon_1 > 0$ so that

$$\Phi^+(t_\varepsilon \mathbf{v}_\varepsilon) > \Phi^+(\mathbf{v}_2) \qquad \forall\, 0 < \varepsilon < \varepsilon_1.$$

Sketch of the proofs

Theorem 2 (Bound States).

1. There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and $\beta = \varepsilon\widetilde{\beta} > 0$, there exists an even bound state $\mathbf{u}_\varepsilon > \mathbf{0}$ with $\mathbf{u}_\varepsilon \to \mathbf{u}_0 = (U_1, V_2)$ as $\varepsilon \to 0$.
2. If $\lambda_2 > \Lambda_2$ and $0 < \beta < \Lambda$, there exists an even bound state $\mathbf{u}^* > \mathbf{0}$ with $\Phi(\mathbf{u}^*) > \Phi(\mathbf{v}_2)$.

Proof of Theorem 2 (main ideas).

2. There holds

$$\Phi^+(t_\varepsilon \mathbf{v}_\varepsilon) = I_2(t_\varepsilon(V_2 + \varepsilon h_2)) + I_1^+(t_\varepsilon \varepsilon h_1) - \frac{1}{2}\beta\varepsilon^2 t_\varepsilon^2 \int_{\mathbb{R}} (h_1^+)^2 (V_2 + \varepsilon h_2)\, dx.$$

Since V_2 is the positive ground state of I_2,

$$\Phi^+(t_\varepsilon \mathbf{v}_\varepsilon) > \Phi^+(\mathbf{v}_2) + I_1^+(t_\varepsilon \varepsilon h_1) - \frac{1}{2}\beta\varepsilon^2 t_\varepsilon^2 \int_{\mathbb{R}} (h_1^+)^2 (V_2 + \varepsilon h_2)\, dx.$$

To do so, it is sufficient to show that

$$\mathcal{J}(t_\varepsilon \mathbf{v}_\varepsilon) := I_1^+(t_\varepsilon \varepsilon h_1) - \frac{1}{2}\beta\varepsilon^2 t_\varepsilon^2 \int_{\mathbb{R}} (h_1^+)^2 (V_2 + \varepsilon h_2)\, dx \geq 0 \qquad \forall\, 0 < \varepsilon < \varepsilon_1.$$

Sketch of the proofs

Theorem 2 (Bound States).

1. There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and $\beta = \varepsilon\widetilde{\beta} > 0$, there exists an even bound state $\mathbf{u}_\varepsilon > \mathbf{0}$ with $\mathbf{u}_\varepsilon \to \mathbf{u}_0 = (U_1, V_2)$ as $\varepsilon \to 0$.
2. If $\lambda_2 > \Lambda_2$ and $0 < \beta < \Lambda$, there exists an even bound state $\mathbf{u}^* > \mathbf{0}$ with $\Phi(\mathbf{u}^*) > \Phi(\mathbf{v}_2)$.

Proof of Theorem 2 (main ideas).

2. After some estimates, there exist h_1 and $\frac{\beta}{\Lambda} < \alpha < 1$ so that for ε small

$$\mathcal{J}(t_\varepsilon \mathbf{v}_\varepsilon) > \frac{1}{2} t_\varepsilon^2 \varepsilon^2 \|h_1\|_1^2 (1 - \alpha - c t_\varepsilon^2 \varepsilon^2), \quad \text{for a constant } c > 0.$$

Taking $\varepsilon > 0$ sufficiently small we infer

$$\Phi^+(t_\varepsilon \mathbf{v}_\varepsilon) > \varepsilon^2 c_0 \|h_1\|_1^2 + \Phi^+(\mathbf{v}_2) > \Phi^+(\mathbf{v}_2), \text{ for a constant } c_0 > 0,$$

proving that $\mathbf{v}_2$ is a strict local minimum for Φ^+ on $\mathcal{N}^+$.

Sketch of the proofs

Theorem 2 (Bound States).

1. There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and $\beta = \varepsilon\widetilde{\beta} > 0$, there exists an even bound state $\mathbf{u}_\varepsilon > \mathbf{0}$ with $\mathbf{u}_\varepsilon \to \mathbf{u}_0 = (U_1, V_2)$ as $\varepsilon \to 0$.
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Proof of Theorem 2 (main ideas).

2. From the preceding arguments, it follows that Φ^+ has a MP critical point $\mathbf{u}^* \in \mathcal{N}^+$, which gives rise to a solution of (S^+) with $u^*, v^* \geq 0$.

Sketch of the proofs

Theorem 2 (Bound States).

1. There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and $\beta = \varepsilon\widetilde{\beta} > 0$, there exists an even bound state $\mathbf{u}_\varepsilon > \mathbf{0}$ with $\mathbf{u}_\varepsilon \to \mathbf{u}_0 = (U_1, V_2)$ as $\varepsilon \to 0$.
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Proof of Theorem 2 (main ideas).

2. From the preceding arguments, it follows that Φ^+ has a MP critical point $\mathbf{u}^* \in \mathcal{N}^+$, which gives rise to a solution of (S^+) with $u^*, v^* \geq 0$.

Additionally, since $\mathbf{u}^*$ is a MP critical point, we have that $\Phi(\mathbf{u}^*) = \Phi^+(\mathbf{u}^*) > \Phi^+(\mathbf{v}_2) = \Phi(\mathbf{v}_2) > 0$, which implies $u^* \not\equiv 0$, and by the maximum principle applied to each single equation we get u^*, $v^* > 0$, hence $\mathbf{u}^* > \mathbf{0}$.

Further results

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3.- It is remarkable that some of these techniques have been used for the first time in systems of NLS equations in **[AC1,AC2]**.

[AC1] A. Ambrosetti, E.C.; C. R. Math. Acad. Sci. Paris (CRMAS) 2006.

[AC2] A. Ambrosetti, E.C.; J. Lond. Math. Soc. 2007.

[C1] E.C.; *Existence of Bound and Ground States for a System of Coupled Nonlinear Schrödinger-KdV Equations.* CRMAS 2015.

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Also, the first time that these techniques are applied to NLS-KdV systems is in **[C1,C2]**, giving better results than what was proved before, in which usually one consider the conservation of the L^2 norm...and the energy functional is constrained to the corresponding L^2 norm constant.

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Thank you for your attention!