Plane-like minimizers for a non-local Ginzburg-Landau-type energy in a periodic medium

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One of the most interesting themes in the study of the qualitative properties shared by the solutions of semilinear equations is **one-dimensional symmetry**.

In particular, an important problem in this direction is the determination of conditions ensuring that the solutions of, say,

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$$-\Delta u = u(1-u^2) \quad \text{in } \mathbb{R}^n,$$

such that $\partial_{x_n} u > 0$ in $\mathbb{R}^n$, then u must be one-dimensional, at least if $n \leq 8$.

Positive answers have been provided by Ghoussoub & Gui for $n = 2$ and Ambrosio & Cabré for $n = 3$. Savin proved its validity for $4 \leq n \leq 8$, under the additional assumption

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In recent years, a similar question involving the **fractional Laplace operator**

$$(-\Delta)^s u(x) = \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy,$$

with $s \in (0, 1)$, has been proposed. Partial answers can be found in works authored by Cabré, Cinti, Sire, Solà-Morales and Valdinoci, essentially proving the statement for $n = 2$ and $n = 3$, $s \geq 1/2$.

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The existence of one-dimensional solution of

$$-(-\Delta)^s u = W'(u) \quad \text{in } \mathbb{R}^n,$$

where W is a **double-well potential**, was investigated in papers by Cabré, Palatucci, Savin, Sire, Solà-Morales and Valdinoci.

A more general existence result is contained in a work in preparation with Tommaso Passalacqua, which deals with an equation obtained by replacing the fractional Laplacian with the integral operator

$$\text{P.V.} \int_{\mathbb{R}^n} (u(y) - u(x)) \, K(y - x) \, dy,$$

where $K(z)$ is an even, measurable kernel comparable to $|z|^{-n-2s}$.

In the work that I present here - which is written in collaboration with Enrico Valdinoci - we address a slightly different problem. We consider the energy functional formally defined by

$$\mathcal{E}(u) := \iint_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 \, K(x,y) \, dx dy + \int_{\mathbb{R}^n} W(x, u(x)) \, dx,$$

and the associated Euler-Lagrange equation

(E) $$L_K u = W_u(\cdot, u) \quad \text{in } \mathbb{R}^n,$$

with

$$L_K u(x) := \text{P.V.} \int_{\mathbb{R}^n} (u(y) - u(x)) \, K(y,x) \, dy.$$

Here, $K : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$ is a measurable function such that

$$K(x, y) = K(y, x) \quad \text{for a.a. } x, y \in \mathbb{R}^n,$$

and

$$\frac{\lambda \chi_{B_1}(x - y)}{|x - y|^{n+2s}} \le K(x, y) \le \frac{\Lambda}{|x - y|^{n+2s}} \quad \text{for a.a. } x, y \in \mathbb{R}^n.$$

On the other hand, $W = W(x, u) : \mathbb{R}^n \times [0, +\infty)$ is measurable in x, differentiable in u and satisfies

$$W(x, u) \ge \gamma(\theta) \quad \text{for a.a. } x \in \mathbb{R}^n \text{ and any } u \in (-\theta, \theta), \theta \in [0, 1),$$

for some positive non-increasing function γ, and

$$W(x, -1) = W(x, 1) = 0 \quad \text{for a.a. } x \in \mathbb{R}^n.$$

Notice that equation (E) is not translation-invariant, due to the fact that both L_K and W are space-dependent. Consequently, there is no hope that one-dimensional solutions (and one-dimensional minimizers) may exist.

However, if we impose a periodic condition on the "coefficients", which is encoded in the requirement

$$K(x+k, y+k) = K(x, y) \quad \text{and} \quad W(x+k, u) = W(x, u),$$

for a.a. $x, y \in \mathbb{R}^n$, any $u \in \mathbb{R}$ and any $k \in \mathbb{Z}^n$, it is reasonable to question the existence of **plane-like minimizers**, that is minimizers u whose "interface" $\{|u| < 9/10\}$ lies in some slab.

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More in detail, we can prove the following

Theorem (C.-Valdinoci, '15)

There is a constant $M_0 > 0$, depending only on universal quantities, such that, given any $\omega \in \mathbb{R}^n \setminus \{0\}$, there exists a minimizer u_ω of the energy $\mathcal{E}$ in $\mathbb{R}^n$ for which

$$\left\{ x \in \mathbb{R}^n : |u_\omega(x)| < \frac{9}{10} \right\} \subseteq \left\{ x \in \mathbb{R}^n : \frac{\omega}{|\omega|} \cdot x \in [0, M_0] \right\}.$$

Furthermore,

- *if $\omega \in \mathbb{Q}^n \setminus \{0\}$, then u_ω is periodic with respect to $\sim_\omega$;*
- *if $\omega \in \mathbb{R}^n \setminus \mathbb{Q}^n$, then u_ω is the locally uniform limit of a sequence of periodic class A minimizers.*

Here, $\sim_\omega$ is the equivalence relation defined for any $x, y \in \mathbb{R}^n$ by

$$x \sim_\omega y \quad \text{iff} \quad y - x = k, \ \text{for some } k \in \mathbb{Z}^n \text{ s.t. } \omega \cdot x = 0.$$

A theorem like this was first obtained by Valdinoci for a "local" energy functional of the form

$$\int_{\mathbb{R}^n} \langle A(x)\nabla u(x), \nabla u(x)\rangle + W(x, u(x))\, dx,$$

with A bounded, elliptic and Hölder continuous.

Such propositions are in the same spirit of a work by Caffarelli & de la Llave, where plane-like minimal surfaces for periodic area functionals are constructed. More generally speaking, they are deeply connected to other classical results such as Aubry-Mather theory for dynamical systems and the construction of line-like geodesics on periodic Riemannian surfaces, done by Bliss, Morse, Hedlund and others.

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Sketch of the proof:

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Fix $M > 0$ and consider the regions

$$\mathcal{S}_\omega^M := \Big\{ \omega \cdot x \in (0, M) \Big\} \quad \text{and} \quad \widetilde{\mathcal{S}}_\omega^M := \mathcal{S}_\omega^M / \sim_\omega .$$

Via the direct method (applied to a suitable auxiliary functional), one shows the existence of a $\sim_\omega$-periodic function u such that $u(x) \geq 9/10$ if $\omega \cdot x \leq 0$, $u(x) \leq -9/10$ if $\omega \cdot x \geq M$ and

$$\mathcal{E}(u; \widetilde{\mathcal{S}}_\omega^M) \leq \mathcal{E}(u + \varphi; \widetilde{\mathcal{S}}_\omega^M) \quad \text{for any } \varphi \text{ supported in } \widetilde{\mathcal{S}}_\omega^M.$$

Here $\mathcal{E}(u; \Omega)$ denotes the restriction of $\mathcal{E}$ to the set Ω, i.e.

$$\mathcal{E}(u; \Omega) := \iint_{\mathbb{R}^{2n} \setminus (\mathbb{R}^n \setminus \Omega)^2} |u(x) - u(y)|^2 K(x, y) \, dx dy + \int_\Omega W(x, u(x)) \, dx.$$

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- First consider $\omega \in \mathbb{Q}^n \setminus \{0\}$:
  - Obtain the existence of a local minimizer.
  - Introduce the minimal minimizer.

Define the **minimal minimizer**

$$u^{\omega}_{M}(x) := \inf u(x) \quad \text{for any } x \in \mathbb{R}^n,$$

where the $\inf$ ranges over all the periodic minimizers previously constructed.

The minimal minimizers exists, is unique and enjoys two key features:

- The doubling property.
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The **doubling property** essentially tells that u_M^ω is still a minimizer with respect to perturbations supported in $\mathcal{S}_M^\omega = \{\omega \cdot x \in (0, M)\}$ and inside a fundamental region of any equivalence relation multiple of $\sim_\omega$.

Consequently, u_M^ω is a minimizer of $\mathcal{E}$ under any compact perturbations occurring within the strip $\mathcal{S}_M^\omega$.

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  - Enlarge the walls!
  - Lift the ceiling!

The goal is to prove the existence of a universal value $M_0 > 0$ such that if $M \geq M_0|\omega|$, then

$$u_M^\omega \equiv u_{M+a}^\omega \quad \text{for any } a \geq 0.$$

To do this, it suffices to show that the interface of u_M^ω does not stick to the upper constraint, i.e.

$$\operatorname{dist}\left(\left\{|u_M^\omega| < \frac{9}{10}\right\}, \left\{\omega \cdot x = M\right\}\right) \geq 1.$$

The **Birkhoff property** ensures that it is then enough to find a ball $\widetilde{B}_{\sqrt{n}}$ of radius $\sqrt{n}$ such that

$$\widetilde{B}_{\sqrt{n}} \subset \left\{|u_M^\omega| \geq \frac{9}{10}\right\}.$$

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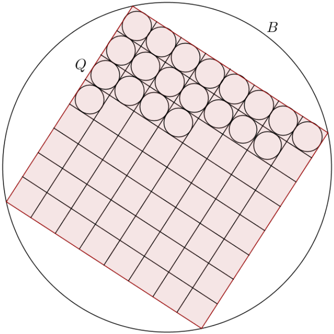
We consider a ball $B \subset\subset \mathcal{S}^\omega_M$ of radius $\simeq M$ and take a maximal cube Q concentric to B. Arguing by contradiction, we can find $\simeq M^n$ non-overlapping balls of radius $\sqrt{n}$ having non-empty intersection with the set $\{|u^\omega_M| < 9/10\}$.



- By uniform C^α bounds,

$$\mathcal{E}(u^\omega_M; B) \gtrsim M^n.$$

- By energy estimates,

$$\mathcal{E}(u^\omega_M; B) \lesssim o(M^n).$$

By taking M big enough, we are led to a contradiction.

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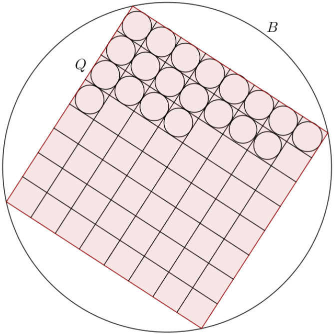

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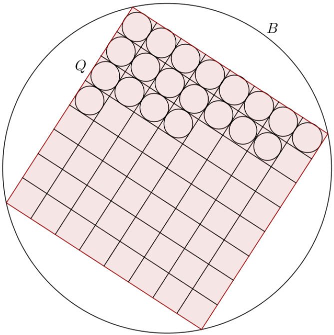

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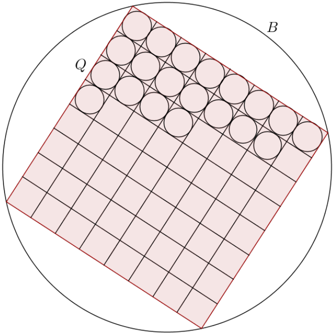

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- For $\omega \in \mathbb{R}^n \setminus \mathbb{Q}^n$ use a limiting argument.

Thank you for the attention!!