$$-\Delta u = f(u) - kuv \qquad \text{in } \Omega \qquad (1)$$
$$-\Delta v = g(v) - kuv$$
$$u, v \geq 0 \text{ in } \Omega$$
$$u = v = 0 \text{ on } \partial\Omega$$

Here $f(0) = 0$, f is sublinear and is negative for large y

and g has similar properties

<u>population model</u>

$$-\Delta u = f(u) - kv^2 u \qquad \text{in } \Omega \qquad (2)$$
$$-\Delta v = g(v) - ku^2 v$$
$$u, v \geq 0 \text{ in } \Omega$$
$$u = v = 0 \text{ on } \partial\Omega$$

f and g have similar properties to above

Bose Einstein condensates.

LIMIT PROBLEMS FOR LARGE k

$$-\Delta w = f(w^+) - g(-w^-) \text{ on } \Omega \qquad (3)$$
$$w = 0 \text{ on } \Omega$$

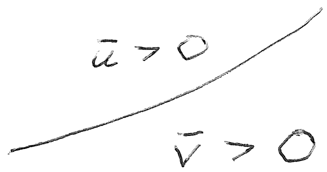
(where $w \sim u - v$, w^+ is the positive part of w
$w = w^+ + w^-$)

The idea for (1) is to use the equation for $u - v$ and bound u, v in $W^{1,2}(\Omega) \cap L^\infty(\Omega)$ and prove kuv is bounded in $L^1(\Omega)$ and $uv \to 0$ in Ω as $k \to \infty$ and hence in the limit $uv = 0$

The argument in the second case is rather more involved. One first shows that

$$-\Delta \bar{u} = f(\bar{u}) - m_1$$
$$-\Delta \bar{v} = g(\bar{v}) - m_2$$

where m_i is supported on $\bar{u} = \bar{v} = 0$, m_i are positive measures and m_1 is the weak* limit of $k v^2 u$ and $\bar{u}\bar{v} = 0$. One gets the same limit equation as before if ~~we~~ we prove $m_1 = m_2$. To prove this, we use a monotonicity formula to prove that $u = v = 0$ is mostly a Lipschitz surface with $u \neq 0$ on one side and $v \neq 0$ on the other



and also that the measures which both are the normal derivatives across the free boundary match up.

This argument uses the variational structure but this is not completely necessary for the lower order terms.

If we look for solutions with u and v both small we have limiting equations for the second problem

$$\begin{aligned} -\Delta u &= au - v^2 u \quad \text{in } \Omega \\ -\Delta v &= dv - u^2 v \\ u &= v = 0 \quad \text{on } \partial\Omega \\ u, v &\geq 0 \end{aligned} \qquad (4)$$

where $a = f'(0)$, $d = g'(0)$.

For the first problem, the correspong equations are

$$\begin{aligned} -\Delta u &= au - uv \\ -\Delta v &= dv - uv \\ u, v &\geq 0 \\ u = v &= 0 \text{ on } \partial\Omega \end{aligned} \quad \text{on } \Omega \qquad (\cancel{4}\ 5)$$

Now the limiting problem for (3) ~~no~~ near zero is

$$\begin{aligned} -\Delta w &= aw^+ + dw^- \quad \text{in } \Omega \\ w &= 0 \quad \text{on } \partial\Omega \end{aligned} \qquad (6)$$

(where $w^- \leq 0$).

This is also the limiting problem for (4) or (5) at infinity in the sense that if we have large solutions of (4) or (5) then u normalized must converge to w^+ and v normalized ~~—~~ converges to $-w^-$ where w is a changing sign solution of (6)

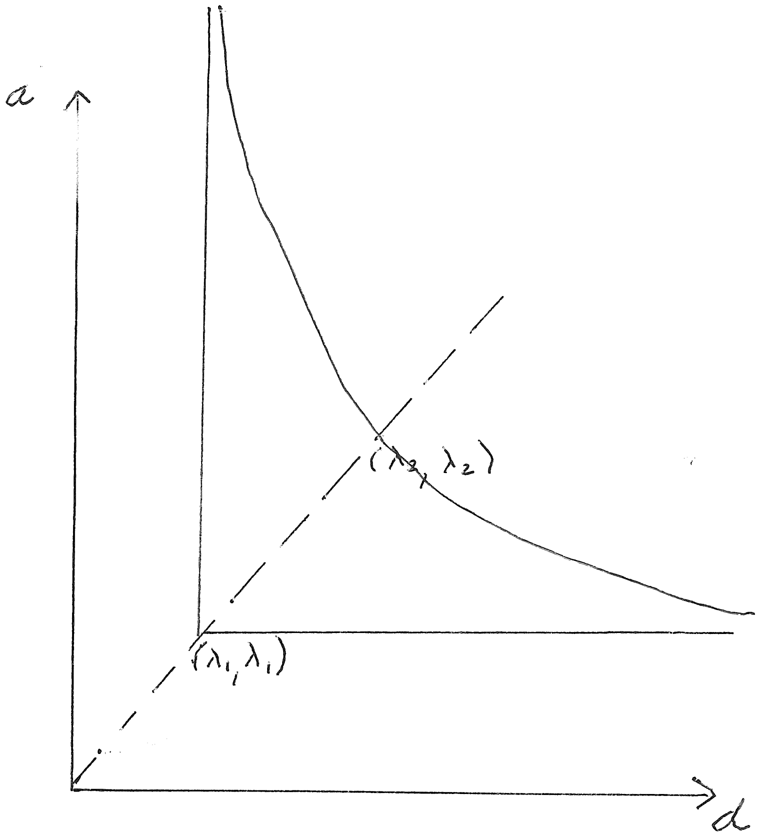
The set of $(a,d) \in (\lambda_1, \infty) \times (\lambda_1, \infty)$ such that (6) has a non-trivial solution plays an important role in our problems. It is called the Fucik spectrum or sometimes the D-Fucik spectrum. Unfortunately it is incompletely understood. Moreover, if (a,d) is not in this spectrum we do not completely understand the degree of the map

$w \to w - (-\Delta)^{-1}(aw^+ + dw^-)$ in appropriate spaces

(We sometimes abuse notation and talk about the degree of (6)).

(Note that the degree can be a large integer.)

It was essentially proved by D'Aujourdhui and improved by Gossez and de Figueiredo (who gave a variational characterization) that there is a first non-trivial curve of the Fucik spectrum bifurcating from (λ_2, λ_2).



For equation (5), if (a, d) is not in the Fucik spectrum, the degree of (5) on large balls (in the cone of non-negative functions) is the degree of (6) on the unit ball. In particular, this implies that (1) has a positive solution for large k if (6) has non-zero degree.

We conjecture that this is also true for (4) ~~a~~ but the proof seems not so easy.

Similarly, if w is an isolated solution of (3) which changes sign and has non-zero degree then there exists a solution (u, v) of (1) for large k with u close to w^+ and v close to $-w^-$. This solution is unique if w is a non-degenerate solution of (3) and is stable if w is a non-~~dne~~ degenerate solution of (3) which is stable.

The idea of the proof is to work with $u, u-v$

We conjecture similar results hold for (2). In this case, we may need a better ~~approxim~~ approximate solution.

There is a different way of proceeding for (2) by using the variational structure. This holds if (a, d) lies above the Aysardhi curve. In this case there are ~~non-~~ non-degenerate positive solutions $(\bar{u}, 0)$, $(0, \bar{v})$.
We can find a positive solution by looking for a solution in the order interval between $(\bar{u}, 0)$ and $(0, \bar{v})$.
Our assumptions ensure $(0,0)$ is not a mountain pass.
This works very well if (3) has no stable changing sign solution but gets a ~~lit~~ little more complicated otherwise.

$$\dot{u} = \Delta u + f(u) - kuv$$
$$\dot{v} = \mu \Delta v + g(v) - kuv$$
$$u = v = 0 \quad \text{on } \partial\Omega$$
$$\dot{w} = \Delta(w^+ + \mu w^-) + f(w^+) - g(-w^-)$$

Question If $\mu \neq 1$, is $\int_\Omega |\nabla w|^2$ continuous in t ?

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