Local bifurcations in the Kawahara-Kuramoto-Sivashinsky equation

Kulikov A.N., Kulikov D.A.

Yaroslavl state university, Russia

Consider the partial differential equation

$$v_t = av_x - bv_{xx} - v_{xxxx} + 2cvv_x. \tag{1}$$

Here $v = v(t, x)$ is a function of two variables t and x, $a, b, c \in R$.

This equation is one of simplest versions of the Kuramoto-Syvashinsky (KS) equation.

The premier versions of the KS equation were proposed by:

(i) Syvashinsky for the study of the weak turbulence [1]

[1] Syvashinsky G.I. Weak turbulence in periodic flow // Physica 17 D. 1985. P.243-255.

(ii) Kuramoto Y. For the study of the chemical oscillations [2]

[2] Kuramoto Y. Chemical oscillations, waves and turbulence. Berlin: Springer Verlag.1984. 156 pp.

(iii) Bradley and Harper for study of ripple topography on the surface induced by ion bombardment [3]

[3] Bradley R.M. and J.M. Harper. Theory of ripple topography induced by ion bombardment // J.Vac. Sci. Technol. 16. 1988. P. 2390-2395.

In the majority of the papers, KS equation is studied with periodic boundary condition

$$v(t, x + l) = v(t, x), l > 0.$$

Without loss generality we can assume that $l = 2\pi$. Therefore we can take periodic boundary conditions in the following form

$$v(t, x + 2\pi) = v(t, x). \tag{2}$$

Always, boundary value problem (1),(2) admits solutions

$$v(t, x) = \alpha.$$

Here, α is any arbitrary real constant.

Sometimes the equilibriums $v = const$ of problem (1),(2) are stable in the sense of the Lyapunov defenition. Hence the set of these solutions forms a local attractor. We can show that there exists sufficient condition for the coefficients of equation (1), when problem (1), (2) has another local attractor. This attractor is formed by a set of t – periodic solution, i.e.

$$v(t+T,x) = v(t,x).$$

But every solution belonging to this attractor is unstable.

Therefore, the problem (1),(2) exhibits chaos and we can demonstrate this property by the mathematical method without numerical calculations

These solutions have the different period $T(\alpha)$ which depend from the value of the spatial average

$$\alpha = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(t,x)dx.$$

Remark: *We must notice that the right part of equation (1) has the zero spacial average.*

In the remainder of this communication we wish expose the principal part of our results and basic part of the applying methods.

Reduction of our nonlinear boundary value problem

All solutions of problem (1), (2) admit the representation in the form of the Fourier series

$$v(t,x) = v_0(t) + u(t,x)$$

where

$$v_0(t) = M_0(v) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} v(t,x)\,dx$$

$$u(t,x) = \sum_{n\neq 0} u_n(t)\exp(inx).$$

It particular we obtain that $M_0(u) = 0$.

We can rewrite problem (1), (2) in convenient form. Therefore we obtain the following problem

$$\dot{v}_0 = 0 \tag{3}$$

$$u_t = au_x - bu_{xx} - u_{xxxx} + 2c(v_0 + u)u_x \tag{4}$$

$$u(t, x + 2\pi) = u(t, x), M_0(u) = 0. \tag{5}$$

It follows from equation (3) that $v_0(t) = \alpha$, where α is any arbitrary real constant.

Finally we can rewrite the boundary value problem (4), (5) for the function u in the closed form

$$u_t = Au + 2cuu_x, \tag{6}$$

$$u(t, x + 2\pi) = u(t, x), M_0(u) = 0. \tag{7}$$

Here linear operator $A = A(\alpha)$ is defined by the following equality

$$Au = A(\alpha)u = a(\alpha)u_x - bu_{xx} - u_{xxxx}, a(\alpha) = a + 2\alpha c.$$

The constant α plays the role of a parameter.

Let

$$u(0,x) = f(x) \in H_0. \tag{8}$$

Notice that $H = H_2^4$ is Sobolev space of the 2π periodic functions of x with square integrable distributional partial derivatives up to the fourth order, i.e.

$$\int_{-\pi}^{\pi} (f^{(j)}(x))^2 dx < \infty, j = 0, 1, 2, 3, 4$$

Finally, H_0 contains the function belonging to H, which have zero average, i.e.

$$\int_{0}^{2\pi} f(x) dx = 0.$$

According to result of Sobolevsky [4] we can affirm that mix problem (6),(7),(8) is locally resolvable.

[4] P.E. Sobolevsky. On Equations of Parabolic type in Banach space. Proceeding. Moscow. 1961. Math. Sci. 10. P.297-350.

Recall that boundary value problem (6), (7) has the zero equilibrium

$$u(t,x) = 0.$$

Stability of the trivial equilibrium for the problem (6),(7)

To study the stability of this solution we consider the following linearized problem

$$u_t = A(\alpha)u, \tag{9}$$

$$u(t, x + 2\pi) = u(t, x), M_0(u) = 0. \tag{10}$$

Here

$$A(\alpha)u = -u_{xxx} - bu_{xx} + a(\alpha)u_x.$$

Let λ be an eigenvalue of the linear differential operator $A(\alpha)$. It is defined on smooth functions satisfying the periodic condition.

The following proposition is valid.

Lemma 1. *This operator $A(\alpha)$ has the eigenvalues*

$$\lambda_n = -n^4 + bn^2 + a(\alpha)in, n \neq 0$$

corresponding to the eigenfunctions

$$e_n(x) = \exp(inx).$$

In our case the following propositions are well-known:

(i) if for all $n \in Z \ \ n \neq 0$, $Re\lambda_n < 0$ **than the solution** $u = 0$ **is asymptotically stable for problem (9), (10) and for the problem (6), (7);**

(ii) if for any $n = n_0$ **we have** $Re\lambda_{n_0} > 0$ **that it is unstable.**

If the following conditions are valid:

$$1. Re\lambda_n \leq 0, n \in Z \ n \neq 0;$$

2. for any $n_1, n_2, \ldots, n_p$ we have $Re\lambda_{n_j} = 0, j = 1, \ldots, p.$

Than we have a critical case for the stability of the trivial solution. Here the critical case is realized if $b = 1$.

Bifurcation Analysis

To study local bifurcations (i.e. in the neighbourhood of the zero equilibrium).

We put in this section

$$b = 1 + \gamma\varepsilon.$$

Here ε is a small positive parameter, $\gamma = \pm 1$.

It means that the next conditions are valid

$$b < 1 \text{ if } \gamma = -1 \textit{ or } b > 1 \text{ if } \gamma = 1.$$

We must define the operator $A(\alpha, \varepsilon)$ by the equality

$$A(\alpha, \varepsilon)w(x) = -w^{IV}(x) - (1 + \gamma\varepsilon)w''(x) + aw'(x),$$

$$w(x + 2\pi) = w(x).$$

This operator has two eigenvalues

$$\lambda_{1,-1} = \tau(\varepsilon) \pm i\sigma(\varepsilon)$$

where

$$\tau(\varepsilon) = \gamma\varepsilon, \sigma(\varepsilon) = a(\alpha).$$

Notice that the real part $\tau(\varepsilon)$ don't depend from α and in particular $\tau(0) = 0$.

The remaining eigenvalues λ_n $(n = \pm 2, \pm 3, \ldots)$ have the negative real part

$$Re\lambda_n \leq -\gamma_0 < 0.$$

Note than $A(\alpha) = A(\alpha, 0)$. It possesses one pair pure imaginary complex conjugated eigenvalues.

Hence to study bifurcations in question we must consider already two parameter – dependent boundary value problem

$$u_t = A(\alpha, \varepsilon)u + 2cuu_x, \tag{11}$$

$$u(t, x + 2\pi) = u(t, x),\ M_0(u) = 0. \tag{12}$$

According to Hopf-Andronov theorem (see, for example, [4]) for the parabolic equations the bifurcation problem in question can be reduce to the study any ordinary differential equation for the complex-value function $z = z(t)$

$$\dot{z} = \varepsilon[(\gamma_1 + i\gamma_2)z + (l_1 + il_2)z|z|^2] + o(\varepsilon). \tag{13}$$

To obtain in the terminal conclusion we must calculate the coefficients of equation (13). The last equation (namely (13)) is called normal form in the Poincare-Dulac sense.

According to the papers [5,6] we can restore the right part of equation (13) if we apply an algorithm. It is the modified the Krylov-Bogolubov algorithm.

[4] Mariana Haragus, Gerard Iooss. Local bifurcations, center manifolds and normal forms in infinite-dimenstional dynamical systems. Springer. 2011.

[5] A. N. Kulikov and D. A. Kulikov Formation of wavy nanostructures on the surface of flat substrates by ion bombardment // Comput. Math. Math. Phys. 2012. Vol. 52. No. 5. P. 800–814.

[6] A.N. Kulikov, D.A. Kulikov. Bifurcations in a Boundary-Value Problem of Nanoelectronics // Journal of mathematical sciences. 2015. Vol. 208. No. 5. P. 211-221.

Solutions of problem (11), (12) on the center manifold M_2 are sought in the form

$$u(t,x,\varepsilon) = \varepsilon^{1/2}u_1(t,x,z,\overline{z}) + \varepsilon u_2(t,x,z,\overline{z}) + \\ + \varepsilon^{3/2}u_3(t,x,z,\overline{z}) + O(\varepsilon), \tag{14}$$

where $z = z(t)$ are solutions of normal form (13), which described the solution dynamics on the 2-dimentional invariant manifold M_2 (center manifold).

Here we get in the formula (14)

$$u_1 = z\exp(i\sigma t + ix) + \overline{z}\exp(-i\sigma t - ix),$$

the smooth functions u_2, u_3 satisfy periodic boundary condition, belongs H_0 for any fixed t and $2\pi/\sigma$ periodic in t. At last the functions u_2, u_3 satisfy the following equality

$$M_0(u_j) = M_{\pm 1}(u_j) = 0, \ \ j = 1, 2,$$

$$M_{\pm 1}(u_j) = (1/(2\pi)) \int\limits_{-\pi/\sigma}^{-\pi/\sigma} \int\limits_{\pi}^{\pi} u_j \exp(\pm i\sigma x \pm ix)\,dx\,dt.$$

To determine u_2 and u_3 we substitute sum (14) into (11), (12) and equate the coefficients of ε and $\varepsilon^{3/2}$.

As a result for u_2, u_3 we obtain two nongomogeneous boundary value problem

$$u_{2t} = A(\alpha, 0)u + 2u_1 u_{1x}, \tag{15}$$

$$u_2(t, x + 2\pi) = u_2(t, x), \tag{16}$$

$$\begin{gathered} u_{3t} = A(\alpha, 0)u + 2u_1 u_{2x} + 2u_2 u_{2x} + \gamma u_{1xx} - \\ -(z' \exp(i\sigma t + ix) + \overline{z'} \exp(-i\sigma t - ix)), \end{gathered} \tag{17}$$

$$u_3(t, x + 2\pi) = u_3(t, x). \tag{18}$$

Problem (15), (16) has unique suitable solution

$$u_2 = q_2 z^2 \exp(2i\sigma t + 2ix) + \overline{q}_2 \overline{z}^2 \exp(-2i\sigma t - 2ix),$$

$$q_2 = i\frac{c}{6}, \overline{q}_2 = -i\frac{c}{6}.$$

Applying the solvability condition to problem (17), (18) gives that

$$\gamma_1 = \gamma,\ \gamma_2 = 0, l_1 = -\frac{c^2}{6} < 0, l_2 = 0.$$

Remark:(solvability condition in our case)*If we have the problem*

$$u_t = A(\alpha, 0)u + F(t, x),$$

$$u(t, x + 2\pi) = u(t, x), M_0(u) = 0, M_{\pm 1}(u) = 0.$$

This problem has the unique solution if $M_0(F) = M_{\pm 1}(F) = 0.$

Notice that 3 last equalities are solvability condition.

Return to study normal form (13) which rewrite in truncated form

$$\dot{z} = \varepsilon\left[\gamma z - \frac{c^2}{6} z\left|z\right|^2\right] \tag{13a}$$

Lemma. *Normal form (13) has the set of equilibrium* $V_1(\varphi)$

$$z = \eta \exp(i\varphi),\ \varphi \in R,\ \eta = \sqrt{-\frac{\gamma}{l_1}},$$

if $\gamma l_1 > 0$.

In our case $l_1 < 0$. Hence, $\gamma > 0 (\gamma = 1)$. These solutions are stable. Hence, we have attractor.

Using the results of the articles [5,6] we obtain.

Theorem 1. There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all α problem (11),(12) has the cycle L given by the asymptotical formula

$$u_*(t,x,\alpha,\varepsilon) = \varepsilon^{1/2}\eta[\exp(i\sigma t + i\alpha + i\varphi) + \exp(-i\sigma t - i\alpha - i\varphi)] + \\ +\varepsilon\eta^2[q_2\exp(2i\sigma t + 2ix + 2i\varphi) + \bar{q}_2\exp(-2i\sigma t - 2ix - 2i\varphi)] + o(\varepsilon), \quad (19)$$

$\sigma = a(\alpha)$.

Main theorem. *If* $b = 1 + \gamma\varepsilon$, *value boundary problem (1),(2) has 2-dimentional invariant manifold* $V_2(\varepsilon,\alpha)$ *formed by the 2 parametric set of periodic functions*

$$v(t,x,\varepsilon,\alpha) = \alpha + u_*(t,x,\alpha,\varepsilon) \tag{20}$$

where the function u_* *was determined by formula (19).*

Notice that $V_2(\varepsilon,\alpha)$ is cylindrical surface of the 2 dimension.

This manifold is attractor but it can be shown in the standard way that every solution of the set (20) is unstable.

Hence, we have any attractor for the solution of problem in question which possess two property:

1. All solutions belonging to $V_2(\varepsilon, \alpha)$ (to this manifold) are periodic. In general, these solutions have different periods.

2. All solutions are unstable.

To demonstrate the instability of the solutions belonging to our attractor it is sufficient to prove the following property.

Let $v_1(t,x,\varepsilon), v_2(t,x,\varepsilon)$ be two solutions $\in V_2(\alpha,\varepsilon)$ than we have 2 conditions

- $\max d(t)_{t\geq 0} \geq \sqrt{(\alpha_1-\alpha_2)^2+4\varepsilon\eta^2\pi}, \ \min d(t)_{t\geq 0} = |\alpha_1-\alpha_2|.$

Here

$$d(t) = |v_1 - v_2|_H, \ \alpha_j = M_0(v_j),$$

η is the amplitude of the periodic solution of normal form (13a).

The obtained results may be used for explanation of weak turbulence as the attractor in question is formed by the unstable periodic solutions.

We verified 2 properties from the Devaney definition [6] containing 3 properties.

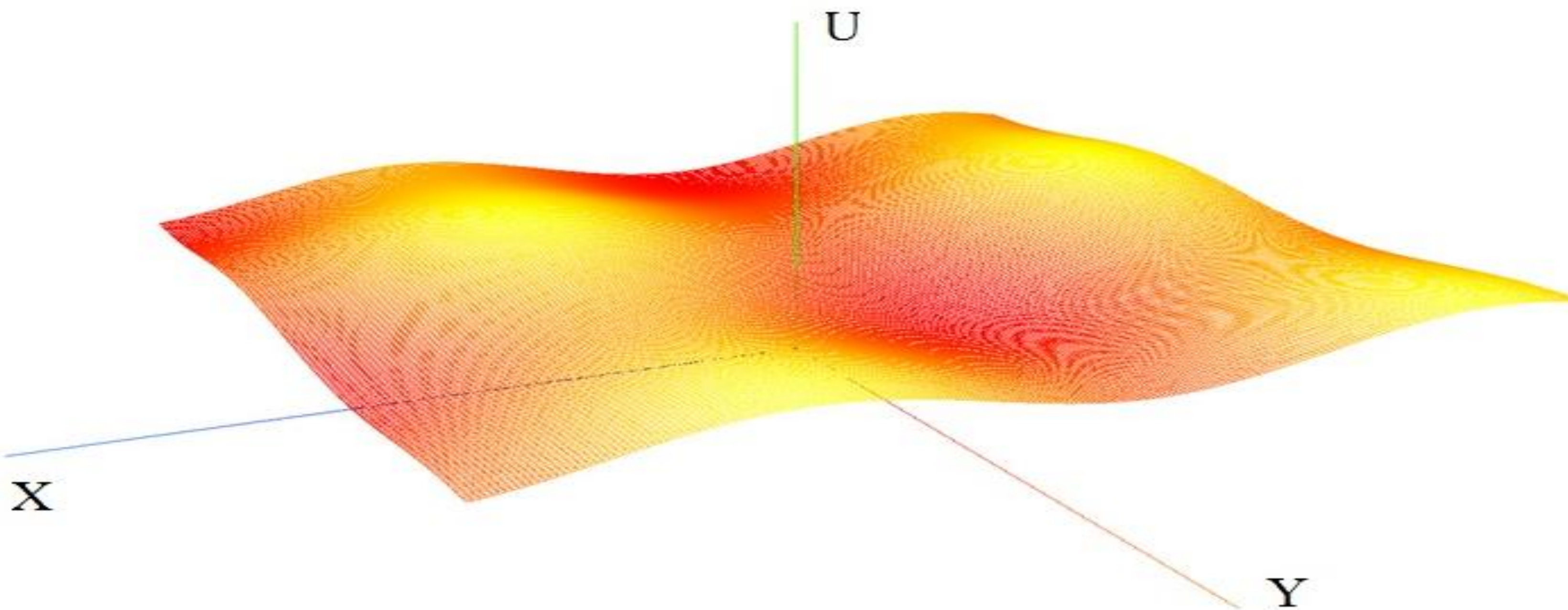
Third property is transitivity. It is desirable that 3 properties be valid. In this case we shall get the chaotic dynamic in the sense of Devaney definition. But now we have the attractor which is quasichaotic.

We can say that the solutions of problem (1), (2) demonstrate the weak turbulence [1].

The part of these results was published in the papers [4,5] (also see the references in these articles).

[6] Devaney R.L. An Introduction to chaotic Dynamical Systems. A.-W.P.C. 1989. 333 p.

To describe the formation of nongomogeneous relief under ion bombardment we considered the generalized KS equation in which the function depends from 3 variables t, x, y [5]. In this case we studied the similar problem and obtained the nongomogeneous relief in similar form (see, [19] and (20)). You can see on the poster the picture of that relief



Here t is fixed. The similar relief was obtained in the experiences.

Any other works about this investigation:

[7] Kulikov A.N., Kulikov D.A. Bifurcations of Spatially Inhomogeneous Solutions in Two boundary value Problems for the Generalized Kuramoto-Syvashinsky Equation // Vestnik Mephi(Вестник Национального исследовательского ядерного университета "МИФИ"). 2014. Vol. 3. No 4. P. 408-415(in Russian).

[8] Kulikov A.N., Metlitskaya A.V., Rudyi A.S. Formation of the Nanorelief at a Surface Erosion by Ion Bombardment within the Bradley-Harper Model // Russian Microelectronics. 2013. Vol. 42. No 4. P. 238-245.

Thank you for attention