On the classification of nonnegative solutions to nonlinear equations in half-spaces

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Aims and motivations

- To study and to classify solutions of semilinear equations on the Euclidean *half-space* :

  $\mathbb{R}^N_+ := \{x = (x', x_N) \in \mathbb{R}^N \,:\, x_N > 0\}$, with $N \geq 2$.

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- *A priori estimates and regularity results for solutions of nonlinear second order PDE's on smooth bounded domains.*

- *Semilinear problems with small diffusion on smooth bounded domains.*

The model problem

Consider the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^N_+, \\ u > 0 & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{1}$$

where $N \geq 2$ and

$f : [0, +\infty) \to \mathbb{R}$, f is *locally Lipschitz continuous*.

Monotonicity

$f \in C^1$, with $f(0) > 0$ or both $f(0) = 0$ and $f'(0) \geq 0$, then every bounded *solution u is* monotone, *i.e. it satisfies $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$.*

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f globally Lipschitz continuous and, either $N = 2$ or $N \geq 3$ and $f(0) \geq 0$, then every positive *solution u (not necessarily bounded) is* monotone.

(Berestycki, Caffarelli and Nirenberg, 1996-1997)

One-dimensional symmetry

f globally Lipschitz continuous and let u be a bounded *solution of* (1)*.*

i) If $N = 2$, u is one-dimensional*, i.e., $u = u(x_N)$.*

ii) If $N = 3$ the same conclusion holds, if one assumes in addition that $f(0) \geq 0$ and $f \in C^1$.

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- In the previous results, it is always assumed the *boundedness of u* and/or the *global Lipschitz character* of f.
- Both the assumptions are crucial for the methods used.

Open problem

Open problem (Berestycki, Caffarelli, Nirenberg, 1996)

We have always assumed that f is globally Lipschitz*. Does the conclusion of the above theorems still hold for* unbounded *solutions of* (1) *in the case where f is* merely locally Lipschitz ?

Question

In 1997, *Berestycki, Caffarelli, Nirenberg,* wrote :

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This clearly suggests to study the general case $N \geq 3$, *even* under the (more restrictive) assumptions : f *globally Lipschitz and* u *bounded.*

Conjecture

Conjecture (Berestycki, Caffarelli, Nirenberg, 1997)

The problem

$$\begin{cases} -\Delta u = u - 1 & \text{in } \mathbb{R}^N_+, \\ u > 0 & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{2}$$

has no bounded *solutions.*

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Berestycki, Caffarelli, Nirenberg (1997) proved the above conjecture for $N = 2$ and 3.

Main difficulties

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  A priori bound on u and/or the global Lipschitz character of f ensure the possibilty to use elliptic estimates to study the asymptotic behaviour u, by means of the translation invariance of the considered problem and/or to use some comparison principles on unbounded cylindrical domains having small cross section.

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- This is the case when $f(u) = u - 1$, for instance.

  In this case f is smooth and globally Lipschitz, with $f(0) = -1 < 0$ and $u(x) = 1 - \cos(x_N)$ is a smooth, bounded and nonnegative solution.

The two-dimensional case

Theorem (F., Sciunzi, 2014)

Let $u \in C^2(\overline{\mathbb{R}^2_+})$ be a nonnegative solution to

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^2_+, \\ u \geq 0 & \text{in } \mathbb{R}^2_+, \\ u = 0 & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{3}$$

with f locally Lipschitz continuous on $[0\,,+\infty)$. Then

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(*i*) if $f(0) < 0$, either u is positive on $\mathbb{R}^2_+$, with $\frac{\partial u}{\partial x_2} > 0$ in $\mathbb{R}^2_+$, or u is one-dimensional and periodic (and unique).

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- (ii) if $f(0) \geq 0$, either u vanishes identically, or u is positive on $\mathbb{R}^2_+$ with $\frac{\partial u}{\partial x_2} > 0$ in $\mathbb{R}^2_+$.

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Method : Rotating plane/line method + unique continuation principle.

The two-dimensional case : symmetry

Theorem (F., Sciunzi, 2014)

Assume that f is locally Lipschitz continuous on $[0\,,+\infty)$. Let $u \in C^2(\overline{\mathbb{R}^2_+})$ be a nonnegative solution to (3) with $|\nabla u| \in L^\infty(\mathbb{R}^2_+)$. Then u is one-dimensional, i.e.

$$u(x) \,=\, u_0(x_2) \qquad \forall\, x \in \mathbb{R}^2_+.$$

for some smooth 1D-profile u_0.

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  since :
- elliptic estimates imply that any *bounded solution* of the considered equation has *bounded gradient*,
- the linear function $u(x) = x_2$ is an *unbounded, monotone, one-dimensional* harmonic function with *bounded gradient*.
- (necessity) : $u(x_1, x_2) = x_2 e^{x_1}$ is a positive monotone solution of $-\Delta u = -u$, with $\nabla u \notin L^\infty$ and which is not 1D.

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If $\frac{\partial u}{\partial x_N} > 0$ in $\mathbb{R}^N_+$, by the implicit function theorem, any connected component of each level set $\{ u = \alpha \}$ is a $N - 1$-dimensional smooth manifold, hence we can introduce the principal curvatures $\kappa_1, \dots, \kappa_{N-1}$ at any point of such manifold.

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We set

$$\mathcal{K}^2 := \kappa_1^2 + \cdots + \kappa_{N-1}^2$$

$$\nabla_T = \text{tangential gradient along level sets}$$

(*the orthogonal projection of the gradient on the tangent space to level sets*).

Sketch of proof : A geometric Poincaré-type formula in any dimension

Theorem (F., Valdinoci, 2010)

Assume $N \geq 2$ and suppose that $u \in C^2(\overline{\mathbb{R}^N_+})$ satisfies

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^N_+ \\ \frac{\partial u}{\partial x_N} > 0 & \text{in } \mathbb{R}^N_+ \\ u = const. & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{4}$$

with f locally Lipschitz continuous.

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with f locally Lipschitz continuous.
Then for any $\varphi \in C^{0,1}_c(\underline{\mathbb{R}^N})$ we have

$$\int_{\mathbb{R}^N_+} \left(|\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2 \right) \varphi^2 \leq \int_{\mathbb{R}^N_+} |\nabla u|^2 |\nabla \varphi|^2 \tag{5}$$

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- $\forall\, R > 1, \qquad \varphi_R(x) := \chi_{B_{\sqrt{R}}}(x) + \frac{2\ln(R/|x|)}{\ln R}\chi_{B_R \setminus B_{\sqrt{R}}}(x).$

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- $\forall\, R > 1, \qquad \varphi_R(x) := \chi_{B_{\sqrt{R}}}(x) + \frac{2\ln(R/|x|)}{\ln R}\chi_{B_R \setminus B_{\sqrt{R}}}(x).$
- Plug φ_R inside (5) to obtain

$$\int_{\mathbb{R}^2_+ \cap B_{\sqrt{R}}} \left(|\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2\right) \leq \frac{C}{(\ln R)^2} \int_{\mathbb{R}^2_+ \cap (B_R \setminus B_{\sqrt{R}})} \frac{1}{|x|^2} \leq \frac{C'}{\log R}$$

for appropriate $C,\ C' > 0$, since $|\nabla u|$ is bounded.

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u is 1D.

The two-dimensional case : the conjecture and the complete classification

Theorem (F., Sciunzi, 2014)

Let $u \in C^2(\overline{\mathbb{R}^2_+})$ be a nonnegative solution to

$$\begin{cases} -\Delta u = u - 1 & \text{in } \mathbb{R}^2_+ \\ u \geqslant 0 & \text{in } \mathbb{R}^2_+ \\ u = 0 & \text{on } \partial\mathbb{R}^2_+ . \end{cases} \tag{6}$$

Then

$$u(x) \; = \; 1 - \cos x_2 \, .$$

The 3-dimensional case : the conjecture

Theorem (F., Soave 2013)

Let $u \in C^2(\overline{\mathbb{R}^3_+})$ be a nonnegative solution to

$$\begin{cases} -\Delta u = u - 1 & \text{in } \mathbb{R}^3_+ \\ u \geqslant 0 & \text{in } \mathbb{R}^3_+ \\ u = 0 & \text{on } \partial\mathbb{R}^3_+, \end{cases} \tag{7}$$

with $|\nabla u| \in L^\infty(\mathbb{R}^3_+)$.
Then

$$u(x) = 1 - \cos x_3 \, .$$

Different method : Fourier series + Liouville theorems (in low dimension).

Sketch of proof

Consider the strip $\bar{\Sigma} = \mathbb{R}^{N-1} \times [0, 2\pi]$, $N \geq 2$.
For every $x' \in \mathbb{R}^{N-1}$, we denote by $\tilde{u}(x', \cdot)$ the 2π-periodic extension of $x_N \mapsto u(x', x_N)$. In view of the smoothness of u, it follows that the Fourier expansion of $x_N \mapsto \tilde{u}(x', x_N)$, i.e.

$$\frac{a_0(x')}{2} + \sum_{m=1}^{+\infty} \left(a_m(x') \cos(m x_N) + b_m(x') \sin(m x_N)\right), \tag{8}$$

where

$$\begin{aligned} a_m(x') &:= \frac{1}{\pi} \int_0^{2\pi} u(x', x_N) \cos(m x_N)\, dx_N \qquad \forall m \geq 0, \\ b_m(x') &:= \frac{1}{\pi} \int_0^{2\pi} u(x', x_N) \sin(m x_N)\, dx_N \qquad \forall m \geq 1, \end{aligned} \tag{9}$$

is convergent.

Sketch of proof

Lemma

Let $N \geq 2$. For any $m \geq 1$ we have

$$\Delta' a_m(x') = (m^2 - 1)a_m(x') + \frac{1}{\pi}\left(u_N(x',0) - u_N(x',2\pi)\right) \tag{10}$$

$$\Delta' b_m(x') = (m^2 - 1)b_m(x') + \frac{m}{\pi}u(x',2\pi). \tag{11}$$

Also,

$$\Delta' a_0(x') = 2 - a_0(x') + \frac{1}{\pi}\left(u_N(x',0) - u_N(x',2\pi)\right).$$

For any $m \geq 1$ we have

$$\begin{aligned}
\Delta' a_m(x') &= \frac{1}{\pi}\int_0^{2\pi} \Delta' u(x', x_N) \cos(m x_N)\, dx_N \\
&= \frac{1}{\pi}\int_0^{2\pi} (1 - u(x', x_N) - u_{NN}(x', x_N)) \cos(m x_N)\, dx_N \\
&= -\frac{1}{\pi}\int_0^{2\pi} (u(x', x_N) + u_{NN}(x', x_N)) \cos(m x_N)\, dx_N .
\end{aligned}$$

For any $m \geq 1$ we have

$$\begin{aligned}\Delta' a_m(x') &= \frac{1}{\pi}\int_0^{2\pi} \Delta' u(x', x_N)\cos(mx_N)\,dx_N \\ &= \frac{1}{\pi}\int_0^{2\pi} (1 - u(x', x_N) - u_{NN}(x', x_N))\cos(mx_N)\,dx_N \\ &= -\frac{1}{\pi}\int_0^{2\pi} (u(x', x_N) + u_{NN}(x', x_N))\cos(mx_N)\,dx_N.\end{aligned}$$

Integrating by parts twice the last term we obtain

$$\begin{aligned}\Delta' a_m(x') = &-\frac{1}{\pi}\int_0^{2\pi} (u(x', x_N)\cos(mx_N) + mu_N(x', x_N)\sin(mx_N))\ dx_N \\ &-\frac{1}{\pi}\left[u_N(x', x_N)\cos(mx_N)\right]_{x_N=0}^{2\pi} \\ =&\frac{m^2-1}{\pi}\int_0^{2\pi} u(x', x_N)\cos(mx_N)\,dx_N \\ &-\frac{1}{\pi}(u_N(x', 2\pi) - u_N(x', 0))\end{aligned}$$

Sketch of proof

Lemma

For $N \leq 3$. Both b_1 and a_1 are constant; moreover,

$$u(x', 2\pi) = 0, \quad u_N(x', 0) = 0 \quad \text{and} \quad u_N(x', 2\pi) = 0 \quad \forall x' \in \mathbb{R}^{N-1}.$$

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An important consequence of the previous Lemma is that the equations for a_m and b_m simplify as

$$\Delta' a_m(x') = (m^2 - 1) a_m(x') \qquad \forall m \geq 2 \tag{12}$$

$$\Delta' b_m(x') = (m^2 - 1) b_m(x') \qquad \forall m \geq 2. \tag{13}$$

Conclusion of the proof

The Fourier coefficients a_m and b_m are identically 0 for any $m \geq 2$.

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The Fourier coefficients a_m and b_m are identically 0 for any $m \geq 2$.
Hence, the Fourier series reduces to

$$\frac{a_0(x')}{2} + a_1 \cos x_N + b_1 \sin x_N \tag{14}$$

and, for every $x \in \bar{\Sigma}$, it is equal to $u(x)$.
The initial condition $u(x', 0) = 0$ reads

$$\frac{a_0(x')}{2} + a_1 = 0 \Rightarrow a_0 \text{ is constant, equal to } -2a_1.$$

We also proved that $u_N(x', 0) = 0$, which implies $b_1 = 0$.
Plugging the expression of u inside the equation $-\Delta u = u - 1$ we obtain

$$-a_1 \cos x_N + \frac{a_0}{2} + a_1 \cos x_N - 1 = 0 \Rightarrow a_0 = 2,$$

and hence $a_1 = -1$.
We proved that $u(x', x_N) = 1 - \cos x_N$ in $\bar{\Sigma} = \mathbb{R}^{N-1} \times [0, 2\pi]$.

Monotonicity

Theorem

Assume $N \geq 3$ and f locally Lipschitz continuous with $f(0) \geq 0$.
Assume that u is bounded on the strips $\mathbb{R}^{N-1} \times [0, t]$, for every $t > 0$.
Then u is monotone.

The case $N \leq 5$

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For $N \leq 5$, every *bounded and positive* solution u in $\mathbb{R}^N_+$ is *one-dimensional and monotone*,

if one of the two following conditions occurs :

- $f \in C^1$ and $f \geq 0$ on $[0, +\infty)$

  or

- $f \in C^1$ and f changes sign "only once".

Sketch of proof

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- u is monotone.
- $$v(x_1, \dots, x_{N-1}) := \lim_{x_N \to +\infty} u(x) \tag{15}$$
  belongs to $\mathcal{C}^2(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$ and satisfies
  $$\begin{cases} -\Delta v = f(v) & \text{in } \mathbb{R}^{N-1}, \\ v > 0 & \text{in } \mathbb{R}^{N-1}. \end{cases} \tag{16}$$

Sketch of proof

- v is a *stable* solution of

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Theorem (F., Dupaigne,2010)

Assume $f \in \mathcal{C}^1$, $f \geq 0$, $M \leq 4$. Assume $v \in \mathcal{C}^2(\mathbb{R}^M)$ is a bounded, stable solution of

$$-\Delta v = f(v) \qquad in \quad \mathbb{R}^M$$

Then v is constant.

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Then v is constant.

- $\Longrightarrow \quad f(\sup u) = f(\sup v) = 0 \quad \Longrightarrow \quad u$ is 1D.

Thank you for your attention!

The talk is based on

- *Flattening results for elliptic PDEs in unbounded domains with applications to overdetermined problems*, Archive for Rational Mechanics and Analysis, 195 (2010), 1025 - 1058. (with E. Valdinoci).
- *Symmetry and uniqueness of non-negative solutions of some problems in the half-space,* Journal of Mathematical Analysis and Applications, 403, (2013),1, 215-233 (with N. Soave).
- *Qualitative properties and classification of nonnegative solutions to $-\Delta u = f(u)$ in unbounded domains when $f(0) < 0$*, (2014) To appear in Revista Matemática Iberoamericana (with B. Sciunzi).
- *Splitting theorems, symmetry results and overdetermined problems for Riemannian manifolds*, Comm. in PDEs, Vol. 38, issue 10, 2013 (with L. Mari and E. Valdinoci).