Sharp asymptotic estimates for eigenvalues of Aharonov-Bohm operators with varying poles

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Workshop in Nonlinear PDEs

joint works with L. Abatangelo

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The Aharonov-Bohm operator

For $a = (a_1, a_2) \in \mathbb{R}^2$ and $\gamma \in \mathbb{R} \setminus \mathbb{Z}$, we consider the vector potential

$$A_a(x_1, x_2) = \gamma \left(\frac{-(x_2 - a_2)}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2} \right),$$

$(x_1, x_2) \in \mathbb{R}^2 \setminus \{a\}$. A_a generates the Aharonov-Bohm magnetic field in $\mathbb{R}^2$ with pole a and circulation γ.

The AB magnetic field is produced by an infinitely long thin solenoid intersecting perpendicularly the plane (x_1, x_2) at the point a, as the radius of the solenoid goes to zero and the magnetic flux remains constantly equal to γ. Negletting the irrelevant coordinate along the solenoid, the problem becomes 2-dimensional.

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Aharonov-Bohm effect [Y. Aharonov, D. Bohm, Phys. Rev. (1959)]

The AB magnetic field is a δ-like magnetic field: a quantum particle moving in $\mathbb{R}^2 \setminus \{a\}$ is affected by the magnetic potential, despite being confined to a region in which the magnetic field is zero.

The magnetic eigenvalue problem

Let us consider the Schrödinger operator with AB vector potential

$$(i\nabla + A_a)^2 u = -\Delta u + 2iA_a \cdot \nabla u + |A_a|^2 u.$$

In a bounded, open and simply connected domain $\Omega \subset \mathbb{R}^2$, $\forall\, a \in \Omega$, we consider the eigenvalue problem

$$\begin{cases} (i\nabla + A_a)^2 u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \qquad (E_a)$$

in a weak sense.

The functional setting

For every $a \in \Omega$, $H^{1,a}(\Omega,\mathbb{C})$ is the completion of $\{u \in H^1(\Omega,\mathbb{C}) \cap C^\infty(\Omega,\mathbb{C}) : u$ vanishes in a neighborhood of $a\}$ w.r.t.

$$\|u\|_{H^{1,a}(\Omega,\mathbb{C})} = \left(\|\nabla u\|^2_{L^2(\Omega,\mathbb{C}^2)} + \|u\|^2_{L^2(\Omega,\mathbb{C})} + \left\|\tfrac{u}{|x-a|}\right\|^2_{L^2(\Omega,\mathbb{C})}\right)^{1/2},$$

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Hardy type inequality [A. Laptev, T. Weidl (1999)]

$$\int_{D_r(a)} |(i\nabla + A_a)u|^2\,dx \geq \Big(\min_{j\in\mathbb{Z}} |j-\gamma| \Big)^2 \int_{D_r(a)} \frac{|u(x)|^2}{|x-a|^2}\,dx,$$

for all $r > 0$, $a \in \mathbb{R}^2$ and $u \in H^{1,a}(D_r(a),\mathbb{C})$

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for all $r>0$, $a \in \mathbb{R}^2$ and $u \in H^{1,a}(D_r(a),\mathbb{C})$

Let $H^{1,a}_0(\Omega,\mathbb{C}) = \Big\{u \in H^1_0(\Omega,\mathbb{C}) : \frac{u}{|x-a|} \in L^2(\Omega,\mathbb{C})\Big\}$.

The eigenvalues

Classical spectral theory $\rightsquigarrow$ (E_a) admits a sequence of real diverging eigenvalues $\{\lambda_k^a\}_{k\geq 1}$ with finite multiplicity

$$\lambda_1^a \leq \lambda_2^a \leq \dots \leq \lambda_j^a \leq \dots \text{ (repeated according to their multiplicity)}$$

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Problem:

study the behavior of the function

$$a \mapsto \lambda_j^a$$

in a neighborhood of a fixed point $b \in \Omega$.

Up to a translation, it is not restrictive to consider $b = 0 \in \Omega$.

The case $\gamma = \frac{1}{2}$, half-integer circulation

$$A_a(x) = A_0(x-a), \quad \text{where} \quad A_0(x_1, x_2) = \frac{1}{2}\left(-\frac{x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right).$$

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- Bonnaillie-Noël, Noris, Nys, Terracini [Analysis and PDE (2014)]
  Léna [J. Math. Phys. (2015)], Noris, Nys, Terracini [CMP, to appear]
  Noris, Terracini [Indiana Univ. Math. J. (2010)]:
  a strong connection between nodal properties of eigenfunctions and the critical points of the map $a \mapsto \lambda_j^a$.

Properties of the map $a \mapsto \lambda_j^a$

Bonnaillie-Noël, Noris, Nys, Terracini (2014), Léna (2015):

- $\forall\, j \geq 1$, the map $a \mapsto \lambda_j^a$ is continuos in Ω and admits a continous extension on $\overline{\Omega}$ (as $a \to \partial\Omega$, $\lambda_j^a \to j$-th eigenvalue of $-\Delta$ in Ω)

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- **Improved regularity for simple eigenvalues:** if $\exists\, n_0 \geq 1$ s.t.

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Question:

what is the leading term in the asymptotic expansion of the map $a \mapsto \lambda_{n_0}^a$?

Let us assume that $\exists\, n_0 \geq 1$ such that $\lambda^0_{n_0}$ is simple and denote $\lambda_0 = \lambda^0_{n_0}$ and, for any $a \in \Omega$, $\lambda_a = \lambda^a_{n_0}$. Thus $\lambda_a \to \lambda_0$.
Let $\varphi_0 \in H^{1,0}_0(\Omega, \mathbb{C})$ be an eigenfunction of $(i\nabla + A_0)^2$ associated to λ_0

$$\begin{cases} (i\nabla + A_0)^2\varphi_0 = \lambda_0\varphi_0, & \text{in } \Omega, \\ \varphi_0 = 0, & \text{on } \partial\Omega, \end{cases} \quad \text{such that} \quad \int_\Omega |\varphi_0(x)|^2\, dx = 1.$$

Theorem [F., Ferrero, Terracini (2011)]

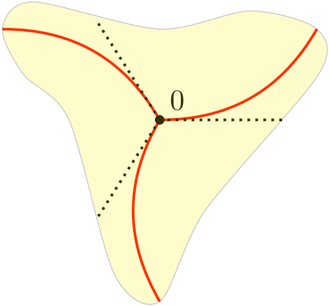
There exists $k \in \mathbb{N}$ odd s.t.

$$r^{-k/2}\varphi_0(r(\cos t, \sin t)) \to \beta_1 \frac{e^{i\frac{t}{2}}}{\sqrt{\pi}} \cos\left(\tfrac{k}{2}t\right) + \beta_2 \frac{e^{i\frac{t}{2}}}{\sqrt{\pi}} \sin\left(\tfrac{k}{2}t\right) \quad \text{as } r \to 0^+$$

in $C^{1,\tau}([0,2\pi], \mathbb{C})$, with $\beta_1, \beta_2 \in \mathbb{C}$, $(\beta_1, \beta_2) \neq (0,0)$.

[Helffer-Hoffmann-Ostenhof-Hoffmann-Ostenhof-Owen (1999)]: $e^{-i\frac{t}{2}}\varphi_0(r(\cos t, \sin t))$ is a multiple of a real-valued function and therefore either $\beta_1 = 0$ or $\frac{\beta_2}{\beta_1}$ is real.

- φ_0 has at 0 a zero or order $\frac{k}{2}$ for some odd $k \in \mathbb{N}$
- φ_0 has got exactly k nodal lines meeting at 0 and dividing the whole angle into k equal parts.



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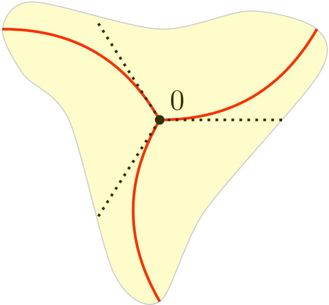

Noris, Terracini (2010), Bonnaillie-Noël, Noris, Nys, Terracini (2014):
The behavior of the eigenvalue λ_a is strongly related to the structure of the nodal lines of the associated eigenfunction.

rate of convergence of λ_a to λ_0 $\longleftrightarrow$ order of vanishing of φ_0 at 0

Theorem [Bonnaillie-Noël, Noris, Nys, Terracini (2014)]

If $k \geq 3$, then

- $|\lambda_a - \lambda_0| \leq C|a|^{\frac{k+1}{2}}$ as $a \to 0$ for a constant $C > 0$ independent of a

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The control $|a|^{\frac{k+1}{2}}$ is not the optimal one.

Our goal:

to establish the exact order of the asymptotic expansion of $\lambda_a - \lambda_0$

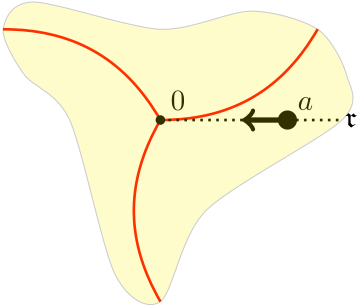
- proving that along a suitable direction the exact order is $|a|^k$, where k is the number of nodal lines of φ_0 at 0
- to find the leading term in the Taylor expansion of the eigenvalue function, detecting its exact coefficients.

Moving along directions of nodal lines

Theorem (F.-Abatangelo, 2015)

Let $\mathfrak{r}$ *be the half-line tangent to a nodal line of eigenfunction* φ_0 *associated to* λ_0 *ending at* 0*. Then, as* $a \to 0$ *with* $a \in \mathfrak{r}$,

$$\frac{\lambda_0 - \lambda_a}{|a|^k} \to -4\frac{|\beta_1|^2 + |\beta_2|^2}{\pi}\,\mathfrak{m}_k.$$



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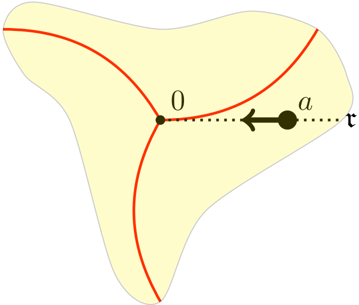

Here

- $(\beta_1, \beta_2) \neq (0,0)$ is s.t. $r^{-\frac{k}{2}}\varphi_0(r(\cos t, \sin t)) \to \beta_1 \frac{e^{i\frac{t}{2}}}{\sqrt{\pi}} \cos\left(\frac{k}{2}t\right) + \beta_2 \frac{e^{i\frac{t}{2}}}{\sqrt{\pi}} \sin\left(\frac{k}{2}t\right)$ as $r \to 0^+$
- $\mathfrak{m}_k < 0$ is a negative constant depending only on k.

The constant $\mathfrak{m}_k$

For every odd $k \in \mathbb{N}$, $\psi_k(r\cos t, r\sin t) = r^{k/2}\sin\left(\frac{k}{2}\,t\right)$, $r \geq 0$, $t \in [0, 2\pi]$, is the unique (up to a multiplicative constant) function which is

- vanishing on $s_0 = [0, +\infty) \times \{0\}$
- harmonic on $\mathbb{R}^2 \setminus s_0$ and homogeneous of degree $k/2$

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Let $s := [1, +\infty) \times \{0\}$ and $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0)\}$.
Let $\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$ be the completion of $C^\infty_c(\overline{\mathbb{R}^2_+} \setminus s)$ w.r.t. $\left(\int_{\mathbb{R}^2_+} |\nabla u|^2\,dx\right)^{1/2}$.

$$\mathfrak{m}_k = \min_{u \in \mathcal{D}^{1,2}_s(\mathbb{R}^2_+)} J_k(u)$$

where $J_k(u) = \frac{1}{2}\int_{\mathbb{R}^2_+} |\nabla u(x)|^2\,dx - \int_0^1 u(x_1, 0)\frac{\partial \psi_k}{\partial x_2}(x_1, 0)\,dx_1$.

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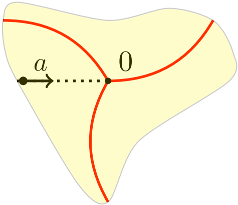
$$\begin{aligned}\mathfrak{m}_k = J_k(w_k) &= -\frac{1}{2}\int_{\mathbb{R}^2_+} |\nabla w_k(x)|^2\,dx \\ &= -\frac{1}{2}\int_0^1 \frac{\partial_+ \psi_k}{\partial x_2}(x_1, 0)\,w_k(x_1, 0)\,dx_1 < 0\end{aligned}$$

Consequences

Analyticity of $a \mapsto \lambda_a \rightsquigarrow$

$$\frac{\lambda_0 - \lambda_a}{|a|^k} \to 4\frac{|\beta_1|^2 + |\beta_2|^2}{\pi}\,\mathfrak{m}_k$$

as $a \to 0$ along the half-line opposite to the tangent to a nodal line of φ_0.



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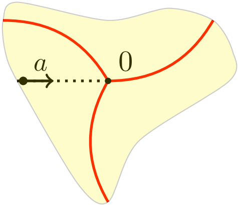

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The restriction of the function $\lambda_0 - \lambda_a$ on the straight line tangent to a nodal line changes sign at 0: is positive on the side of the nodal line and negative on the opposite side

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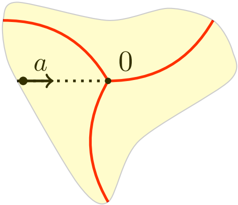

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if λ_0 is simple, then 0 cannot be an extremal point of the map $a \mapsto \lambda_a$.

Consequences

Proposition [Bonnaillie-Noël, Noris, Nys, Terracini (2014)]

Fix any $j \in \mathbb{N}$. If 0 is an extremal point of $a \mapsto \lambda_j^a$, then

(i) either λ_j^0 is not simple

(ii) or the eigenfunction of $(i\nabla + A_0)^2$ associated to λ_j^0 has at 0 a zero of order $k/2$ with $k \geq 3$ odd.

Our theorem $\rightsquigarrow$

- we can exclude alternative (ii): If 0 is an extremal point of $a \mapsto \lambda_j^a \implies \lambda_j^0$ is not simple.

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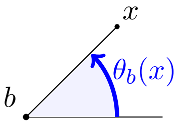
- if λ_0 is simple and $k \geq 3$, 0 is a saddle point for the map $a \mapsto \lambda_a$.
- if λ_0 is simple and $k = 1$, the gradient of the map $a \mapsto \lambda_a$ in 0 is different from zero, then 0 is not a stationary point, a fortiori not even an extremal point (Noris, Terracini (2010)).

Rotate the axes in such a way that the positive x_1-axis is tangent to one of the k nodal lines of φ_0 ending at 0 $\rightsquigarrow$ it is not restrictive to assume that $\beta_1 = 0$.

For every $b \in \mathbb{R}^2$, we define $\theta_b : \mathbb{R}^2 \setminus \{b\} \to [0, 2\pi)$ so that $\theta_b(b + r(\cos t, \sin t)) = t \ \forall\, r > 0, t \in [0, 2\pi)$.

$$\nabla\Big(\frac{\theta_b}{2}\Big) = A_b, \quad e^{-i\frac{\theta_b}{2}}(i\nabla + A_b)^2 u = -\Delta\Big(e^{-i\frac{\theta_b}{2}} u\Big)$$



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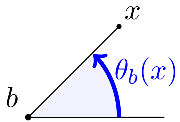

For all $a \in \Omega$, let $\varphi_a \in H^{1,a}_0(\Omega, \mathbb{C}) \setminus \{0\}$ be an eigenfunction of (E_a) associated to the eigenvalue λ_a such that

$$\int_\Omega |\varphi_a(x)|^2\, dx = 1 \quad \text{and} \quad \int_\Omega e^{\frac{i}{2}(\theta_0 - \theta_a)(x)} \varphi_a(x) \overline{\varphi_0(x)}\, dx \text{ is positive real.}$$

We have that $\varphi_a \to \varphi_0$ in $H^1(\Omega, \mathbb{C})$ and in $C^2_{\rm loc}(\Omega \setminus \{0\}, \mathbb{C})$ and

$$(i\nabla + A_a)\varphi_a \to (i\nabla + A_0)\varphi_0 \quad \text{in } L^2(\Omega, \mathbb{C}).$$

Blow-up

Theorem

$$\frac{\varphi_a(|a|x)}{|a|^{k/2}} \to \frac{\beta_2}{\sqrt{\pi}}\Psi_k \quad \text{as } a = (|a|, 0) \to 0,$$

in $H^{1,\mathbf{e}}(D_R, \mathbb{C})$ $\forall R > 1$, *a.e. and in* $C^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{\mathbf{e}\}, \mathbb{C})$, *where* $\mathbf{e} = (1, 0)$.

Ψ_k is the unique function in $H^{1,\mathbf{e}}_{\mathrm{loc}}(\mathbb{R}^2)$ satisfying

$$(i\nabla + A_{\mathbf{e}})^2 \Psi_k = 0, \quad \text{in } \mathbb{R}^2,$$

and

$$\int_{\mathbb{R}^2} |(i\nabla + A_{\mathbf{e}})(\Psi_k - e^{\frac{i}{2}\theta_{\mathbf{e}}}\psi_k)|^2 < +\infty.$$

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If $w_k \in \mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$ minimizes the functional J_k, $\Psi_k = e^{i\frac{\theta_{\mathbf{e}}}{2}}(\psi_k + w_k)$.

Upper bound for $\lambda_0 - \lambda_a$: the Rayleigh quotient for λ_0

For all $1 \leq j \leq n_0$ and $a \in \Omega$, let $\varphi_j^a \in H_0^{1,a}(\Omega, \mathbb{C}) \setminus \{0\}$ be an eigenfunction of problem (E_a) associated to the eigenvalue λ_j^a such that

$$\int_\Omega |\varphi_j^a(x)|^2 \, dx = 1 \quad \text{and} \quad \int_\Omega \varphi_j^a(x) \overline{\varphi_\ell^a(x)} \, dx = 0 \text{ if } j \neq \ell.$$

For $j = n_0$, we choose $\varphi_{n_0}^a = \varphi_a$.

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Courant-Fisher *minimax characterization* of the eigenvalue λ_0, we have that

$$\lambda_0 = \min \left\{ \max_{u \in F \setminus \{0\}} \frac{\int_\Omega |(i\nabla + A_0)u|^2 dx}{\int_\Omega |u|^2 \, dx} : \begin{array}{l} F \text{ is a subspace of } H_0^{1,0}(\Omega, \mathbb{C}), \\ \dim F = n_0 \end{array} \right\}.$$

For $R > 2$ and $a = (|a|, 0)$ with $|a|$ small, define $v_{j,R,a}$ as follows:

$$v_{j,R,a} = \begin{cases} v^{ext}_{j,R,a}, & \text{in } \Omega \setminus D_{R|a|}, \\ v^{int}_{j,R,a}, & \text{in } D_{R|a|}, \end{cases} \qquad j = 1, \dots, n_0,$$

$$v^{ext}_{j,R,a} := e^{\frac{i}{2}(\theta_0 - \theta_a)} \varphi^a_j \text{ in } \Omega \setminus D_{R|a|}, \quad (i\nabla + A_0)^2 v^{ext}_{j,R,a} = \lambda^a_j v^{ext}_{j,R,a} \text{ in } \Omega \setminus D_{R|a|},$$

$$\begin{cases} (i\nabla + A_0)^2 v^{int}_{j,R,a} = 0, & \text{in } D_{R|a|}, \\ v^{int}_{j,R,a} = e^{\frac{i}{2}(\theta_0 - \theta_a)} \varphi^a_j, & \text{on } \partial D_{R|a|}. \end{cases}$$

For $R > 2$ and $a = (|a|, 0)$ with $|a|$ small, define $v_{j,R,a}$ as follows:

$$v_{j,R,a} = \begin{cases} v_{j,R,a}^{ext}, & \text{in } \Omega \setminus D_{R|a|}, \\ v_{j,R,a}^{int}, & \text{in } D_{R|a|}, \end{cases} \qquad j = 1, \dots, n_0,$$

$$v_{j,R,a}^{ext} := e^{\frac{i}{2}(\theta_0 - \theta_a)} \varphi_j^a \text{ in } \Omega \setminus D_{R|a|}, \quad (i\nabla + A_0)^2 v_{j,R,a}^{ext} = \lambda_j^a v_{j,R,a}^{ext} \text{ in } \Omega \setminus D_{R|a|},$$

$$\begin{cases} (i\nabla + A_0)^2 v_{j,R,a}^{int} = 0, & \text{in } D_{R|a|}, \\ v_{j,R,a}^{int} = e^{\frac{i}{2}(\theta_0 - \theta_a)} \varphi_j^a, & \text{on } \partial D_{R|a|}. \end{cases}$$

$$\dim\left(\operatorname{span}\{v_{1,R,a}, \dots, v_{n_0,R,a}\}\right) = n_0$$

For $R > 2$ and $a = (|a|, 0)$ with $|a|$ small, define $v_{j,R,a}$ as follows:

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$$\begin{cases} (i\nabla + A_0)^2 v_{j,R,a}^{int} = 0, & \text{in } D_{R|a|}, \\ v_{j,R,a}^{int} = e^{\frac{i}{2}(\theta_0 - \theta_a)} \varphi_j^a, & \text{on } \partial D_{R|a|}. \end{cases}$$

$\dim\left(\mathrm{span}\{v_{1,R,a}, \dots, v_{n_0,R,a}\}\right) = n_0$ $\rightsquigarrow$ After a Gram-Schmidt normalization, use $v_{j,R,a}$ as test functions in the Rayleigh quotient

$$\Downarrow$$

$$\lambda_0 - \lambda_a \le \int_{D_{R|a|}} \left|(i\nabla + A_0) v_{n_0,R,a}^{int}\right|^2 dx - \int_{D_{R|a|}} \left|(i\nabla + A_a)\varphi_a\right|^2 dx + o(|a|^k)$$

as $|a| \to 0$.

Blow-up and let $R \to +\infty \rightsquigarrow$

$$\limsup_{|a|\to 0} \frac{\lambda_0 - \lambda_a}{|a|^k} \leq \frac{|\beta_2|^2}{\pi} k\sqrt{\pi}(\xi(1) - \sqrt{\pi})$$

where

$$\xi(r) := \frac{1}{\sqrt{\pi}} \int_0^{2\pi} e^{-\frac{i}{2}\theta_{\mathbf{e}}(r\cos t, r\sin t)} \Psi_k(r\cos t, r\sin t) \sin\left(\tfrac{k}{2}t\right) dt.$$

In a similar way we obtain a lower bound for $\lambda_0 - \lambda_a$ estimating the Rayleigh quotient for $\lambda_a \rightsquigarrow$

$$\liminf_{|a|\to 0} \frac{\lambda_0 - \lambda_a}{|a|^k} \geq \frac{|\beta_2|^2}{\pi} k\sqrt{\pi}(\xi(1) - \sqrt{\pi})$$

Hence

$$\lim_{|a|\to 0} \frac{\lambda_0 - \lambda_a}{|a|^k} = \frac{|\beta_2|^2}{\pi} k\sqrt{\pi}(\xi(1) - \sqrt{\pi})$$

$$\xi(1) - \sqrt{\pi} = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} w_k(\cos t, \sin t) \sin\left(\tfrac{k}{2} t\right) dt = \omega(1)$$

where

$$\omega(r) := \frac{1}{\sqrt{\pi}} \int_0^{2\pi} w_k(r\cos t, r\sin t) \sin\left(\tfrac{k}{2} t\right) dt.$$

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$w_k \in \mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$ minimizes $J_k \rightsquigarrow \ldots \rightsquigarrow$

$$\xi(1) - \sqrt{\pi} = \omega(1) = -\frac{4}{k\sqrt{\pi}} \mathfrak{m}_k > 0$$

$$\frac{\lambda_0 - \lambda_a}{|a|^k} \to \frac{|\beta_2|^2}{\pi} k\sqrt{\pi}(\xi(1) - \sqrt{\pi}) = -4\frac{|\beta_2|^2}{\pi}\mathfrak{m}_k$$

as $a \to 0$ tangentially to any of the k nodal lines of φ_0.

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as $a \to 0$ tangentially to any of the k nodal lines of φ_0.

The Taylor polynomials of the function $a \mapsto \lambda_0 - \lambda_a$ with center 0 and degree strictly smaller than k vanish, since they vanish on the k independent directions corresponding to the nodal lines of φ_0

$$\Downarrow$$

$$\lambda_0 - \lambda_a = P(a) + o(|a|^k), \quad \text{as } |a| \to 0^+,$$

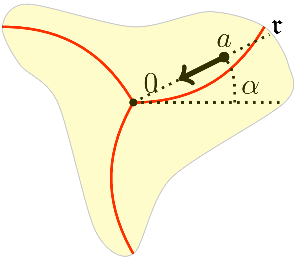
for some

$$P \not\equiv 0, \quad P(a) = P(a_1, a_2) = \sum_{j=0}^{k} c_j a_1^{k-j} a_2^j$$

homogeneous polynomial of degree k.

Moving along any direction

For $\alpha \in [0, 2\pi)$, $\mathbf{p} = (\cos\alpha, \sin\alpha)$
and $a = |a|\mathbf{p} = |a|(\cos\alpha, \sin\alpha)$



$$\frac{\varphi_a(|a|x)}{|a|^{k/2}} \to \frac{\beta_2}{\sqrt{\pi}}\Psi_{\mathbf{p}} \quad \text{as } a = |a|\mathbf{p} \to 0,$$

in $H^{1,\mathbf{p}}(D_R, \mathbb{C})$ for every $R > 1$ and
in $C^2_{\rm loc}(\mathbb{R}^2 \setminus \{\mathbf{p}\}, \mathbb{C})$.

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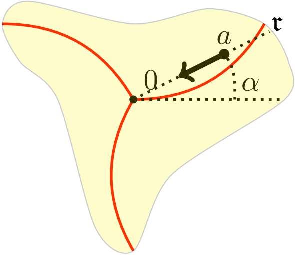

$\Psi_{\mathbf{p}}$ is the unique function in $H^{1,\mathbf{p}}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{C})$ satisfying

$$\begin{cases} (i\nabla + A_{\mathbf{p}})^2\Psi_{\mathbf{p}} = 0, & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2\setminus D_r} \left|(i\nabla + A_{\mathbf{p}})(\Psi_{\mathbf{p}} - e^{\frac{i}{2}(\theta_{\mathbf{p}} - \theta_0^{\mathbf{p}})} e^{\frac{i}{2}\theta_0}\psi_k)\right|^2 dx < +\infty, \ \forall\, r > 1. \end{cases}$$

$$\theta_{\mathbf{p}}(\mathbf{p} + r(\cos t, \sin t)) = t, \quad \theta_0^{\mathbf{p}}(r(\cos t, \sin t)) = t, \quad \forall\, r > 0, \ t \in [\alpha, \alpha + 2\pi)$$

For $\alpha \in [0, 2\pi)$, $\mathbf{p} = (\cos\alpha, \sin\alpha)$ and $a = |a|\mathbf{p} = |a|(\cos\alpha, \sin\alpha)$

$$\lim_{|a|\to 0} \frac{\lambda_0 - \lambda_a}{|a|^k} = \frac{|\beta_2|^2}{\pi}\, k\sqrt{\pi} f(\alpha),$$

where

$$f : [0, 2\pi) \to \mathbb{R}, \quad f(\alpha) = (\xi_{\mathbf{p}}(1) - \sqrt{\pi}),$$

$$\xi_{\mathbf{p}}(r) := \frac{1}{\sqrt{\pi}} \int_0^{2\pi} e^{-\frac{i}{2}(\theta_{\mathbf{p}} - \theta_0^{\mathbf{p}})(r\cos t, r\sin t)} \Psi_{\mathbf{p}}(r\cos t, r\sin t) e^{-i\frac{t}{2}} \sin\left(\tfrac{k}{2}t\right) dt.$$

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We already know that $f(0) = \xi_{\mathbf{e}}(1) - \sqrt{\pi} = \xi(1) - \sqrt{\pi} = -\frac{4}{k\sqrt{\pi}}\mathfrak{m}_k > 0$

Symmetry properties of $f(\alpha)$

Rotation of $\frac{2\pi}{k}$:

$$\mathcal{R}_1(x_1, x_2) = \begin{pmatrix} \cos\frac{2\pi}{k} & -\sin\frac{2\pi}{k} \\ \sin\frac{2\pi}{k} & \cos\frac{2\pi}{k} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Reflexion through the x_1-axis:

$$\mathcal{R}_2(x) = \mathcal{R}_2(x_1, x_2) = (x_1, -x_2),$$

Problem:

- how does the limit profile $\Psi_{\mathbf{p}}$ change when the above trasformations act on $\mathbf{p}$?

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Problem:

- how does the limit profile $\Psi_{\mathbf{p}}$ change when the above trasformations act on $\mathbf{p}$?
- how does the coefficient $\xi_{\mathbf{p}}(1)$ and hence $f(\alpha)$ change when the above trasformations act on $\mathbf{p}$?

The function ψ_k (to which the limit profile is asymptotic at ∞ up to suitable phases) is invariant (up to phases) under the actions $\mathcal{R}_1, \mathcal{R}_2$:

$$e^{\frac{i}{2}(\theta_0 \circ \mathcal{R}_1)}(\psi_k \circ \mathcal{R}_1) = -e^{i\frac{\pi}{k}} e^{\frac{i}{2}\theta_0} \psi_k, \quad \psi_k \circ \mathcal{R}_2 = \psi_k.$$

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$$\Downarrow$$

$$\Psi_{\mathcal{R}_1^{-1}(\mathbf{p})} = -e^{-i\frac{\pi}{k}} \big(\Psi_{\mathbf{p}} \circ \mathcal{R}_1\big), \qquad \Psi_{\mathcal{R}_2(\mathbf{p})} = -e^{i\theta_{\mathcal{R}_2(\mathbf{p})}} \big(\Psi_{\mathbf{p}} \circ \mathcal{R}_2\big).$$

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$$\Downarrow$$

$$\xi_{\mathcal{R}_1^{-1}(\mathbf{p})}(1) = \xi_{\mathbf{p}}(1), \qquad \xi_{\mathcal{R}_2(\mathbf{p})}(1) = \xi_{\mathbf{p}}(1),$$

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$$\Downarrow$$

$$f(\alpha) = f\big(\alpha - \tfrac{2\pi}{k}\big) \qquad f(\alpha) = f(2\pi - \alpha)$$

$$g:\mathbb{R}\to\mathbb{R},\quad g(\alpha):=P(\cos\alpha,\sin\alpha)$$

- $g(\alpha)=\sum_{j=0}^{k}c_j(\cos\alpha)^{k-j}(\sin\alpha)^j$
- $g(\alpha)=\frac{|\beta_2|^2}{\sqrt{\pi}}kf(\alpha)\Rightarrow g(\alpha)=g\left(\alpha+\frac{2\pi}{k}\right),\ g(\alpha)=g(2\pi-\alpha)\ \forall\alpha\in\mathbb{R}$
- $c_0=g(0)=-4\frac{|\beta_2|^2}{\pi}\mathfrak{m}_k>0$

$$\Downarrow$$

$$g(\alpha)=c_0\cos(k\alpha)\quad\text{for all } \alpha\in\mathbb{R}.$$

Theorem [F.-Abatangelo (2015)]

If $\alpha \in [0, 2\pi)$, then

$$\frac{\lambda_0 - \lambda_a}{|a|^k} \to C_0 \cos(k\alpha) \text{ as } a \to 0 \text{ with } a = |a|(\cos\alpha, \sin\alpha),$$

with $C_0 = -4\frac{|\beta_2|^2}{\pi}\mathfrak{m}_k$.

Theorem [F.-Abatangelo (2015)]

If $\alpha \in [0, 2\pi)$, then

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with $C_0 = -4\frac{|\beta_2|^2}{\pi}\mathfrak{m}_k$.

The leading term in the Taylor expansion of $a \mapsto \lambda_0 - \lambda_a$ is then given by

$$P(|a|(\cos\alpha, \sin\alpha)) = C_0|a|^k \cos(k\alpha).$$

Hence $P(a_1, a_2) = C_0\Re\big((a_1 + i\, a_2)^k\big)$ and then

the polynomial P is *harmonic*.