The Nehari manifold method in a nonhomogeneous setting

Giovany M. Figueiredo, UFPA-Amazônia, Brazil (joint work

with Humberto Ramos Quoirin, Universidad de Santiago de Chile)

Introduction

In this work we investigate the existence of ground states for **non-homogeneous operators** that are associated a superlinear type problems such as:

1. Quasilinear type problems:

$$-div\left(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\right) = f(u), \quad u \in W_0^{1,p}(\Omega).$$

2. Kirchoff type problems:

$$-M\left(\int_\Omega |\nabla u|^2\right)\Delta u = f(u), \quad u \in H_0^1(\Omega),$$

3. Anisotropic type problems:

$$-\sum_{i=1}^{N} \partial_i\left(|\partial_i u|^{p_i-2}\partial_i u\right) = f(u), \quad u \in \mathcal{D}_0^{1,\overrightarrow{p}}(\Omega).$$

In the above problems $\Omega \subset \mathbb{R}^N$ is a bounded domain.

Ground states

Let X be a Banach space and $\Phi : X \to \mathbb{R}$ a C^1 functional.

$u_0 \in X$ is a ground state of Φ if

$$\Phi'(u_0) = 0, \quad \Phi(u_0) = \min_{\Phi'(u)=0} \Phi(u).$$

Natural method: The Nehari manifold

$$\mathcal{N} := \{u \in X \setminus \{0\} : \Phi'(u)u = 0\}.$$

If $\Phi(u_0) = \min_{\mathcal{N}} \Phi$ then u_0 is a ground state of Φ (under some conditions on Φ).

The Nehari manifold method

Prototype [Bartsch-Wang-Willem, Liu-Wang]:

$$\begin{aligned}\Phi(u) &:= \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(u), \quad u \in X = H_0^1(\Omega)\\ &= \frac{1}{2}\|u\|^2 - \int_\Omega F(u)\end{aligned}$$

where $F(s) = \int_0^s f(t)dt$ and $f : \mathbb{R} \to \mathbb{R}$ is a C^1 class function and satisfies:

1. $\lim_{s\to\infty} \frac{f(s)}{s^{q-1}} = 0$ for some $q < 2^* = \frac{2N}{N-2}$.
2. $\lim_{s\to 0} \frac{f(s)}{s} = 0$ and $\lim_{s\to\infty} \frac{f(s)}{s} = \infty$.
3. $\frac{f(s)}{s}$ is increasing in $(0, \infty)$.

Then Φ has a non-negative and nontrivial *ground state* u_0, which is a weak solution of

$$-\Delta u = f(u), \quad u \in H_0^1(\Omega). \tag{1}$$

Abstract setting [Szulkin-Weth]

Let X be a uniformly convex Banach space such that $\|\cdot\|$ is a $\mathcal{C}^1$ functional on $X \setminus \{0\}$. Let Φ be such that $\Phi(0) = 0$ and $\Phi = I_0 - I$ where I_0, I are $\mathcal{C}^1$ functionals on X satisfying, for some $p > 1$:

1. $I'(u) = o(\|u\|^{p-1})$ as $u \to 0$.
2. $s \mapsto \frac{I'(su)}{s^{p-1}}$ is strictly increasing in $(0, \infty)$ for every $u \neq 0$.
3. $\frac{I(su)}{s^p} \to \infty$ uniformly for u on weakly compact subsets of $X \setminus \{0\}$ as $s \to \infty$.
4. I' is completely continuous.
5. I_0 is weakly lower semicontinuous, positively **homogeneous** of degree p, i.e. $\mathbf{I_0(su) = s^p I_0(u)}$, and satisfies

$$C_0\|u\|^p \leq I_0(u) \leq C_0^{-1}\|u\|^p$$

$$\left(I_0'(v) - I_0'(w)\right)(v - w) \geq C_1 \left(\|v\|^{p-1} - \|w\|^{p-1}\right)\left(\|v\| - \|w\|\right)$$

for some $C_0, C_1 > 0$ and every $u, v \in X$.

Then Φ has a ground state. Moreover, if Φ is even then it has infinitely many critical points.

Main application:

$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega F(u), \quad u \in X = W_0^{1,p}(\Omega), \; p < N \tag{2}$$

$$= \frac{1}{p}\|u\|^p - \int_\Omega F(u) \tag{3}$$

where f is a C^1 class function that now satisfies:

1. $\displaystyle\lim_{s\to\infty} \frac{f(s)}{s^{q-1}} = 0$ for some $q < p^* = \frac{Np}{N-p}$.
2. $\displaystyle\lim_{s\to 0} \frac{f(s)}{s^{p-1}} = 0$ and $\displaystyle\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = \infty$.
3. $\frac{f(s)}{s^{p-1}}$ is increasing in $(0, \infty)$.

Then Φ has a non-negative and nontrivial *ground state* u_0, which is a weak solution of

$$-\Delta_p u \equiv -div(|\nabla u|^{p-2}\nabla u) = f(u), \quad u \in W_0^{1,p}(\Omega).$$

Main ingredients

Geometrical features of Φ:

(A2) For any $w \in X \setminus \{0\}$ the map $t \mapsto \Phi(tw)$, defined for $t > 0$, has a unique critical point $t_w > 0$ which satisfies

$$\Phi(t_w w) = \max_{t>0} \Phi(tw).$$

(A3) t_w is uniformly bounded away from zero for $w \in \mathcal{S}$ (the unit sphere in X), i.e. there exists $\delta > 0$ such that $t_w \geq \delta$ for every $w \in \mathcal{S}$. Moreover, t_w is bounded from above for w in a compact subset of $\mathcal{S}$, i.e. given a compact set $\mathcal{W} \subset \mathcal{S}$ there exists $C_{\mathcal{W}} > 0$ such that $t_w \leq C_{\mathcal{W}}$ for every $w \in \mathcal{W}$.

Consequence: $\mathcal{N}$ is homeomorphic to S through the projection $w \mapsto t_w w$.

We intend to show that $(A2)$ and $(A3)$ can be extended to $\Phi = I_0 - I$ with I_0 **not necessarily homogeneous**, e.g.

1. $I_0(u) = \int_\Omega \left(\frac{1}{p} |\nabla u|^p + \frac{1}{q} |\nabla u|^q \right), \quad u \in W_0^{1,p}(\Omega), \; p > q > 1.$
2. $I_0(u) = \tilde{M}(\|u\|^2), \quad u \in H_0^1(\Omega).$
3. $I_0(u) = \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i u|^{p_i}, \quad u \in \mathcal{D}_0^{1,\vec{p}}(\Omega)$, the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm $\|u\| = \sum_{i=1}^{N} \|\partial_i u\|_{p_i}$.

Main abstract result

Theorem

(Indiana-2015) Let X be a reflexive Banach space such that $\|\cdot\|$ is a $\mathcal{C}^1$ functional on $X \setminus \{0\}$ and $\Phi : X \to \mathbb{R}$ be a $\mathcal{C}^1$ weakly lower semicontinuous functional such that $\Phi(0) = 0$. In addition, we assume that there exist $p, r > 1$ such that:

1. $\liminf\limits_{u \to 0} \dfrac{\Phi'(u)u}{\|u\|^r} > 0$
2. *For every $u \in X$ we have $\Phi(u) \geq C_0\|u\|^r - I(u)$ where $C_0 > 0$ and I is a weakly continuous functional on X.*
3. $\lim\limits_{t \to \infty} \dfrac{\Phi(tu)}{t^p} = -\infty$ *uniformly for u on weakly compact subsets of $X \setminus \{0\}$.*
4. *For every $u \in X \setminus \{0\}$ the map $t \mapsto \dfrac{\Phi'(tu)u}{t^{p-1}}$ is decreasing*

Then Φ has a nontrivial ground state. If, in addition, Φ is even then we may choose $u_0 \geq 0$.

Proof steps:

- Existence and uniqueness of $t(u)$ such that $t(u)u \in \mathcal{N}$:

$$\liminf_{u\to 0} \frac{\Phi'(u)u}{\|u\|^r} > 0, \lim_{t\to\infty} \frac{\Phi(tu)}{t^p} = -\infty \Rightarrow \begin{cases} \Phi(tu) > 0, & t \sim 0 \\ \Phi(tu) < 0, & t \sim \infty. \end{cases}$$

$$\Rightarrow \exists t_0 > 0 : \Phi(t_0 u) = \max_{t>0} \Phi(tu)$$

$$\frac{\Phi'(tu)u}{t^{p-1}} \text{ decreasing } \Rightarrow t_0 \text{ is the only critical point of } \Phi(tu)$$

  Moreover, $\Phi(u) > 0$ for every $u \in \mathcal{N}$.

- $\mathcal{N}$ is bounded away from zero: $\liminf_{u\to 0} \frac{\Phi'(u)u}{\|u\|^r} > 0$.

- Minimizing sequences in $\mathcal{N}$ are bounded and converge, up to a subsequence, to some $u_0 \not\equiv 0$:
  Assume $\Phi(u_n) \to c := \inf_{\mathcal{N}} \Phi$, $\|u_n\| \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|}$ and assume $v_n \rightharpoonup v_0$.
  - $v_0 \equiv 0$:

    $$\Phi(u_n) \geq \Phi(tv_n) \geq C_0 t^r - I(tv_n) \to C_0 t^r - I(0), \quad \forall t > 0.$$

    Contradiction with $\Phi(u_n) \leq C$.
  - $v_0 \not\equiv 0$:

    $$\frac{\Phi(u_n)}{\|u_n\|^p} = \frac{\Phi(\|u_n\| v_n)}{\|u_n\|^p} \to -\infty$$

    Contradiction with $\Phi(u_n) > 0$ for every n.

  Conclusion: (u_n) is bounded and, up to a subsequence, $u_n \rightharpoonup u_0 \not\equiv 0$ (same argument as in the case $v_0 \equiv 0$).

- c is achieved:

$$c \leq \Phi(t_0 u_0) \leq \liminf \Phi(t_0 u_n) \leq \liminf \Phi(u_n) = c,$$

since Φ is w.l.s.c.

- c is a critical value of Φ:
Φ satisfies $(A2)$, $(A3)$. By [Szulkin-Weth], $c = \inf_{\mathcal{S}} \Psi$, where Ψ is defined by

$$\Psi(w) = \Phi(t_w w) \quad \text{for } w \in \mathcal{S}.$$

Moreover Ψ is a $\mathcal{C}^1$ functional on $\mathcal{S}$, which is a $\mathcal{C}^1$ submanifold of X, and w is a critical point of Ψ if and only if $t_w w$ is a critical point of Φ. □

Infinitely many solutions

Corollary

Under the assumptions of Theorem 1, assume in addition that Φ *is even and satisfies the Palais-Smale condition on* $\mathcal{N}$. *Then* Φ *has infinitely many pairs of critical points.*

Applications

Quasilinear problems

Let $a : [0, \infty) \to [0, \infty)$ be $\mathcal{C}^1$ in $(0, \infty)$. Let $A(t) = \int_0^t a(s)\, ds$ for $t \in \mathbb{R}$. We consider

$$-\, div\left(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\right) = f(u), \quad u \in W_0^{1,p}(\Omega), \qquad (4)$$

Corollary

Under the above assumptions on a, assume in addition that there exist $p \geq q > 1$ *such that:*

1. $k_0 \left(1 + t^{\frac{q-p}{p}}\right) \leq a(t) \leq k_1 \left(1 + t^{\frac{q-p}{p}}\right), \quad \forall t > 0$, *where* k_0, k_1 *are positive constants.*
2. a *is non-increasing.*
3. $t \mapsto a(t^p)t^p$ *and* $t \mapsto A(t^p) - a(t^p)t^p$ *are convex in* $(0, \infty)$.
4. $\lim_{t \to 0} \frac{f(t)}{t^{q-1}} = 0$, $\lim_{t \to \infty} \frac{f(t)}{t^{p-1}} = \infty$, *and* $\lim_{t \to \infty} \frac{f(t)}{t^{\alpha-1}} = 0$ *for some* $\alpha \in (p, p^*)$.
5. $t \mapsto \frac{f(t)}{t^{p-1}}$ *is increasing on* $(0, \infty)$.

Then (4) has a nontrivial and non-negative ground state.

Some examples

1. $a \equiv 1$:[Szulkin-Weth]

$$-\Delta_p u = f(u), \quad u \in W_0^{1,p}(\Omega).$$

2. $a(t) = 1 + t^{\frac{q-p}{p}}$, $1 < q < p$:

$$-\Delta_p u - \Delta_q u = f(u), \quad u \in W_0^{1,p}(\Omega).$$

In this case:

$$\Phi(u) = \frac{1}{p}\|u\|^p + \frac{1}{q}\int_\Omega |\nabla u|^q - \int_\Omega F(u), \quad u \in W_0^{1,p}(\Omega).$$

Kirchhoff type problems

Let $N = 3$, so that $2^* = 6$. We assume that $M : [0, \infty) \to [0, \infty)$ is a $\mathcal{C}^1$ function and we set $\hat{M}(t) = \int_0^t M(s)\, ds$ for $t \in \mathbb{R}$. We consider

$$-M\left(\int_\Omega |\nabla u|^2\right)\Delta u = f(u), \quad u \in H_0^1(\Omega), \tag{5}$$

Corollary

Under the above assumptions on M, assume in addition:

1. *M is increasing and $M(0) := m_0 > 0$.*
2. *$t \mapsto \dfrac{M(t)}{t}$ is decreasing.*
3. *$\lim_{t \to 0} \dfrac{f(t)}{t} = 0$, $\lim_{t \to \infty} \dfrac{f(t)}{t^3} = \infty$, and $\lim_{t \to \infty} \dfrac{f(t)}{t^{\alpha-1}} = 0$ for some $\alpha \in (4, 6)$.*
4. *$t \mapsto \frac{f(t)}{t^3}$ is increasing.*

Then (5) has a nontrivial and non-negative ground state.

Examples of M:

1. $M(t) = at + b$, with $a, b > 0$.
2. $M(t) = a + \sum_{i=1}^{k} b_i t^{\gamma_i}$, where $a > 0$, $b_i \geq 0$ and $\gamma_i \in (0, 1]$ for $i = 1, ..., k$, with $b_i > 0$ for at least one i.
3. $M(t) = a + \ln(1 + t)$, with $a > 0$.

A Anisotropic problem

Let $1 < p_1 \leq p_2 \leq ... \leq p_N$ be such that $\sum_{i=1}^{N} \frac{1}{p_i} > 1$ and $p_N < p^*$, where $p^* = \frac{N}{\left(\sum_{i=1}^{N} \frac{1}{p_i}\right) - 1}$. We consider

$$-\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i - 2} \partial_i u \right) = f(u), \quad u \in \mathcal{D}_0^{1,\overrightarrow{p}}(\Omega). \tag{6}$$

Corollary

Under the above assumptions, assume in addition:

1. $\lim_{t\to 0^+} \frac{f(t)}{t^{p_1-1}} = 0$, $\lim_{t\to\infty} \frac{f(t)}{t^{p_N-1}} = \infty$, *and* $\lim_{t\to\infty} \frac{f(t)}{t^{\alpha-1}} = \infty$ *for some* $\alpha \in (p_N, p^*)$.
2. $t \mapsto \frac{f(t)}{t^{p_N-1}}$ *is increasing.*

Then (6) *has a nontrivial and non-negative ground state.*

This is the end, thank you!