Positive solutions of an elliptic equation with a dynamical boundary condition

Marek Fila

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F., Ishige and Kawakami – this talk

The problem

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is the limit problem of

$$\begin{cases} \varepsilon\partial_t u - \Delta u = f(u), & x \in \Omega \subset \mathbb{R}^N, \;\; t > 0, \\ \partial_t u + \partial_\nu u = g(u), & x \in \partial\Omega, \;\; t > 0, \\ u(x,0) = \phi(x), & x \in \Omega, \end{cases}$$

as $\varepsilon \to 0$, here $\varphi = \phi$ on $\partial\Omega$.

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where $u = u(x,t)$, $N \geq 2$, $\mathbb{R}^N_+ := \{x = (x', x_N) \,:\, x' \in \mathbb{R}^{N-1},\ x_N > 0\}$, Δ is the N-dimensional Laplacian (in x), $\partial_\nu := -\partial/\partial x_N$ and $p > 1$.

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write $u = v + w$ where

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\begin{cases}
-\Delta v = (v+w)^p, & -\Delta w = 0 & x \in \mathbb{R}^N_+, \ \ t > 0, \\
v = 0, & \partial_t w + \partial_\nu w = -\partial_\nu v, & x \in \partial\mathbb{R}^N_+, \ \ t > 0, \\
 & w(x,0) = \varphi(x') \geq 0, & x = (x',0) \in \partial\mathbb{R}^N_+,
\end{cases}
$$

For any $x' \in \mathbb{R}^{N-1}$ and $\lambda > 0$, let $\mathcal{P}$ be the $(N-1)$-dimensional Poisson kernel

$$\mathcal{P}(x', \lambda) := c_N \lambda^{1-N} \Lambda^{-N}(x'/\lambda) \qquad \text{with} \qquad \Lambda(x') := (1 + |x'|^2)^{1/2},$$

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$$G(x,y) := \begin{cases} k_N \left(|x-y|^{-(N-2)} - |x-y_*|^{-(N-2)} \right) & \text{if} \quad N \geq 3, \\ \dfrac{1}{4\pi} \log \left(1 + \dfrac{4x_2 y_2}{|x-y|^2} \right) & \text{if} \quad N = 2, \end{cases}$$

G is the Green function for the Laplace equation on $\mathbb{R}^N_+$ with the Dirichlet boundary condition.

Definition

Let φ be a nonnegative measurable function in $\mathbb{R}^{N-1}$. For any $\sigma > 0$, we call a nonnegative measurable function u in $\mathbb{R}^N_+ \times (0, \sigma]$ a solution in $\mathbb{R}^N_+ \times (0, \sigma]$ if u satisfies

$$u(x', x_N, t) = \int_{\mathbb{R}^{N-1}} P(x' - y', x_N, t)\varphi(y')\, dy' + \int_{\mathbb{R}^N_+} G(x, y)u(y, t)^p\, dy$$
$$+ \int_0^t \int_{\mathbb{R}^N_+} P(x' - y', x_N + y_N, t - s)u(y, s)^p\, dy\, ds < \infty$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \in [0, \infty)$ and $t \in (0, \sigma]$.

Nonuniqueness

If v is a solution of

$$\begin{cases} -\Delta v = v^p, & x \in \mathbb{R}^N_+, \\ \partial_\nu v = 0, & x \in \partial\mathbb{R}^N_+, \\ v(x) = \varphi(x') \geq 0, & x = (x', 0) \in \partial\mathbb{R}^N_+, \end{cases}$$

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then

$$u(x,t) := v(x', x_N + t)$$

satisfies

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In fact, u is the minimal solution.

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In fact, u is the minimal solution. Minimal solutions are unique.

Theorem 1

(i) If

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then there are no nontrivial local-in-time solutions;

(ii) If $p > p_*$, then, for suitable small initial data φ, there exist solutions defined for all $t > 0$ which behave like the Poisson kernel as $t \to \infty$.

Theorem 2

Let $p > p_*$ and let ψ be a nonnegative function in $\mathbb{R}^{N-1}$ such that

$$\liminf_{|x'|\to\infty} |x'|^{\frac{2}{p-1}} \psi(x') > 0.$$

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Then there exists a constant $\kappa > 0$ such that, if

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Theorem 3

Let $p > p_*$ and let φ be a nonnegative function in $\mathbb{R}^{N-1}$ such that

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Then there is no local-in-time solution.

Theorem 4

Let $p > p_*$. Then there exists $k > 0$ such that if

$$\varphi(x') \leq k(1 + |x'|)^{-\frac{2}{p-1}}, \qquad x' \in \mathbb{R}^{N-1},$$

then there is a global-in-time solution u satisfying

$$u(x,t) \leq C(1 + |x'| + x_N + t)^{-\frac{2}{p-1}}, \qquad (x,t) \in \mathbb{R}^N_+ \times (0,\infty),$$

for some constant $C > 0$.

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Furthermore, if $u = u(x,t)$ is a minimal solution of (D) and $v = v(x)$ is a minimal solution of (E) then

$$u(x', x_N, t) = v(x', x_N + t)$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \geq 0$ and $t > 0$.

Theorem 6

Let $p_* < p < \infty$ and consider the elliptic problem

$$\begin{cases} -\Delta v = v^p & \text{in} \quad \mathbb{R}^N_+, \\ v = \kappa\psi & \text{on} \quad \partial\mathbb{R}^N_+, \end{cases}$$

where $\kappa > 0$ and $\psi = \psi(x')$ is a nonnegative bounded function in $\mathbb{R}^{N-1}$ such that

$$\psi \not\equiv 0 \quad \text{in} \quad \mathbb{R}^{N-1} \qquad \text{and} \qquad \limsup_{|x'|\to\infty} |x'|^{\frac{2}{p-1}} \psi(x') < \infty.$$

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Then there exists $\kappa_* \in (0, \infty)$ such that

- If $0 < \kappa < \kappa_*$ then there is a solution.
- If $\kappa > \kappa_*$ then there is no solution.

proofs – integral representation + Phragmén-Lindelöf

Theorem 7

Let $\sigma > 0$ and let $u = u(x,t)$ satisfy

$$u(\cdot,t) \in C^2(\mathbb{R}^N_+) \cap C^1(\overline{\mathbb{R}^N_+}) \quad \text{for any} \quad t \in (0,\sigma],$$
$$u \in C(\overline{\mathbb{R}^N_+} \times (0,\sigma]), \quad \partial_t u \in C(\partial\mathbb{R}^N_+ \times (0,\sigma]),$$

and

$$-\Delta u \geq 0 \quad \text{in} \quad \mathbb{R}^N_+ \times (0,\sigma], \qquad \partial_t u + \partial_\nu u \geq 0 \quad \text{on} \quad \mathbb{R}^N_+ \times (0,\sigma].$$

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Assume that

$$\liminf_{t \to +0} \inf_{x' \in B'(0,R)} u(x',0,t) \geq 0 \quad \text{for any } R > 0,$$
$$\limsup_{R \to \infty} \inf_{|x|=R, t \in (0,\sigma]} \frac{u(x,t)}{1+x_N} \geq 0.$$

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Then $u \geq 0$ in $\mathbb{R}^N_+ \times (0,\sigma]$.

Thanks for your attention.