Dichotomy theorem for a degenerate one dimensional Keller-Segel model.

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The classical Keller-Segel model

We are interested in a population of cells which attract themselves through a chemical signal.

The Patlak, Keller and Segel model involves two species:

- the cell density $\rho(t,x)$,
- the chemoattractant concentration $c(t,x)$.

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \Delta\rho - \chi\nabla\cdot\left(\rho\nabla c\right) \quad t>0,\ x\in\Omega\subset\mathbb{R}^2, \\ -\Delta c = \rho \end{cases}$$

Parameters:

- The chemosensitivity coefficient χ
- The total mass of cells M, which is conserved along time.

In the whole space $\mathbb{R}^2$,

$$c(t,x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\rho(t,y)\,dy\ .$$

Dichotomy theorem in the whole space $\mathbb{R}^2$

Theorem (Blanchet, Dolbeault and Perthame)

Assume $\rho_0\big(|\log\rho_0| + (1+|x|^2)\big) \in L^1(\mathbb{R}^2)$. If $\chi M < 8\pi$ then solutions are global in time (dispersion); they blow up in finite time if $\chi M > 8\pi$ (aggregation).

In case of subcritical regime $\chi M < 8\pi$, the solution converges towards a self-similar profile (diffusion-dominating rescaling).

Main tool: Dissipated free energy.

$$\searrow \mathcal{F}[\rho] = \int_{\mathbb{R}^2} \rho(y)\log\rho(y)\, dy + \frac{\chi}{4\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \rho(x)\log|x-y|\rho(y)\, dxdy\,.$$

There are dozens of contributions (prior to this theorem), among which Nanjundiah, Jäger and Luckhaus, Nagai, Biler, Herrero and Velázquez, Gajewski and Zacharias, Horstmann, Senba and Suzuki... '92...

The variance computation

We fix the center of mass equal to zero and introduce the variance of the density:

$$\Pi_2(t) = \int |x|^2 \rho(t,x)\,dx.$$

We impose $\Pi_2(0) < +\infty$ and compute formally that

$$\begin{aligned}\frac{d}{dt}\Pi_2(t) &= \int |x|^2 \partial_t \rho(t,x)\,dx = -\int 2x\nabla\rho(t,x)\,dx \\ &+ \chi \iint 2x\left(\rho(t,x)\frac{-1}{2\pi}\frac{1}{x-y}\rho(t,y)\right)dxdy \\ &= 2d\int \rho(t,x)\,dx - \frac{\chi}{2\pi}\left(\int \rho(t,x)\right)^2 \\ &= M\left(4 - \frac{\chi M}{2\pi}\right)\end{aligned}$$

The degenerate Keller-Segel model

The Keller-Segel model with a non-linear diffusion

$$\begin{cases} \partial_t \rho(t,x) = \Delta \rho^m(t,x) - \chi \nabla \cdot (\rho(t,x) \nabla S(t,x)) \,, \quad t>0, \; x \in \mathbb{R}^d \,, \\ S(t,x) = - \int_{\mathbb{R}^d} |x-y|^{d(1-m)} \rho(t,y)\, dy \,. \end{cases}$$

with the associated free energy

$$\mathcal{F}_m[\rho] = \frac{1}{m-1} \int_{\mathbb{R}^d} \rho^m(x)\, dx \\ \qquad - \chi \frac{1}{d(m-1)} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^{d(1-m)} \rho(x) \rho(y)\, dxdy$$

Is there a dichotomy theorem ?
Blanchet Carillo Laurencot '09 in dimension $d \geq 3$.

The Variance computation

For $m > 1$, with $\Pi_2(t) = \int |x|^2 \rho(t,x)\,dx$,

$$\frac{d}{dt}\Pi_2(t) = (m-1)\mathcal{F}_m(\rho(t)).$$

$$\frac{d}{dt}\mathcal{F}_m(\rho(t)) \leq 0.$$

- If $\mathcal{F}_m \geq 0$ there is no contradiction.

Otherwise:

- If $\mathcal{F}_m(\rho_0) < 0$, the solution blows-up in finite time.
- If $\mathcal{F}_m(\rho_0) \geq 0$??

The critical parameter is given by

$$\chi_c = \inf\left[\chi \geq 0 \,|\, \exists \rho \text{ with } \mathbb{F}_m(\rho) < 0\right]$$

Do we always have blow-up in the supercritical case ?

A one dimensional particles scheme

A gradient flow interpretation

Assume without loss of generality $M = 1$.

The Keller-Segel system with a logarithmic weight:

$$\partial_t \rho = \Delta\rho + \frac{\chi}{d\pi}\nabla\cdot\left(\rho\nabla\log|z|*\rho\right)\,,$$
$$\partial_t u = \Delta u + \nabla\cdot\left(-uy + \frac{\chi}{d\pi}\nabla\log|z|*u\right)\,,$$

is formally the (generalized) gradient flow of the free energy functional in the space of probability measures endowed with the Wasserstein metric.

Reformulation in 1D

We reformulate the degenerate Keller-Segel equations in dimension 1.

The energy becomes for any increasing function $X \in L^2(0,1)$:

$$F(X) = \frac{1}{m-1} \int_{(0,1)} \left(\frac{dX}{dp} \right)^{1-m} dp \\ - \frac{\chi}{m-1} \iint_{(0,1)^2} |X(p) - X(q)|^{1-m}, dpdq.$$

Particles approximation

We take a discretization step h_N ($h_N = \frac{1}{N}$), and define for $i = 0..N+1$, $X_i = X(ih_N)$. Each particle carries a mass h_N. It leads to the discrete energy $F_N : \mathbb{R}^N \to \mathbb{R}$:

$$F_N(X) = \sum_{i=1}^{N-1} (X_{i+1} - X_i)^{1-m} - \chi \sum_{1 \le i \ne j \le N} |X_i - X_j|^{1-m},$$

and the associate gradient flow equation:

Definition (Gradient flow equation.)

$$\begin{cases} \dot{X}(t) = -\nabla F_N(X(t)) & t \in \mathbb{R} \\ X(0) = X^0 & X^0 \in \mathbb{R}^N. \end{cases}$$

The scheme was studied by DEVYS '10 FOR THE CLASSICAL KELLER-SEGEL CASE.

Critical parameters

Definition

Let $p \in \mathbb{N}$ and

$$C_p = \min \left\{ \sum_{i=1}^{p-1} (X_{i+1} - X_i)^{1-m} \,\middle|\, \sum_{1 \le i \ne j \le p} |X_i - X_j|^{1-m} = 1 \right\}.$$

- $C_p > \frac{1}{p}$ (variant of the Hardy-Littlewood-Sobolev inequality).
- F_p is bounded by below iff $\chi \le C_p$.

Dichotomy theorem

Theorem

Let $\chi > C_N$, and $2 > m > 1$. Let X be a solution of the the discrete gradient flow

- *If for all $p \in [1, N]$, $\chi \neq C_p$ then X blows up in finite time.*
- *If there exists $p \in [1, N]$ such that $\chi = C_p$ then either X blows up in finite or $Y = \frac{X}{|X|}$ blows up (at infinity) + rigidity.*

Idea of the proof

We define $f_{m+1}(X(t)) = |X(t)|^{m+1}$.

- f_{m+1} is concave.
- $f'_{m+1}(X(t)) = (m+1)(m-1)F_N(X(t))|X(t)|^{m-1}$.
- $f''_{m+1}(X(t)) = -\frac{(m+1)(m-1)}{f_{m+1}(X(t))} H^N_{m+1}\left(\frac{X(t)}{|X(t)|}\right)$.

where for any Y with $|Y| = 1$,

$$H_N(Y) = \left[|\nabla F_N(Y)|^2 - ((m-1)F_N(Y))^2\right] \geq 0.$$

- Convergence to a critical point of H_N (and F_N).
- Critical points of F_N are minimizers.

Compactness issue.

Perspectives

- The 1D/2D continuous case.
- Blow-up in finite time.
- Rigidity of the blow-up