# Torsional instability in suspension bridges: the Tacoma Narrows Bridge case

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The most impressive failure of history is certainly the Tacoma Narrows Bridge collapse in November 1940.

**The collapsed Tacoma Narrows Bridge**

The TNB collapse is not an isolated event; other bridges collapsed in a similar way due to hurricanes:

the Brighton Chain Pier (UK) collapsed in 1836;

the Menai Straits Bridge (UK) collapsed in 1839;

the Wheeling Suspension Bridge (US) collapsed in 1854;

the Matukituki Suspension Footbridge (NZ) collapsed in 1977.

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By "**similar way**" we mean that

**unexpected and destructive torsional oscillations suddenly appeared**.

The Official Report

• O.H. Ammann, T. von Kármán, G.B. Woodruff, *The failure of the Tacoma Narrows Bridge*, Federal Works Agency (1941)

considers ...the crucial event in the collapse to be the sudden change from a vertical to a torsional mode of oscillation.

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There have been many attempts to answer to the following fundamental question:

**why do torsional oscillations appear suddenly?**

- mistake in the project, structural failure
- resonance due to the frequency of the wind
- flutter theory
- vortex shedding, von Kármán vortices
- angle of attack of the wind
- parametric resonance.

The civil and aeronautical engineer Robert Scanlan writes that *...the original Tacoma Narrows Bridge withstood random buffeting for some hours with relatively little harm until some* **fortuitous condition** *"broke" the bridge action over into its low antisymmetrical torsion flutter mode*.

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*there is* **no consensus** *on what caused the sudden change to torsional motion*.
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The civil engineer Richard Scott writes that
*opinion on the exact cause of the Tacoma Narrows Bridge collapse is even today* **not unanimously shared**.
• R. Scott, *In the wake of Tacoma. Suspension bridges and the quest for aerodynamic stability*, ASCE Press (2001)

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To this end, we **isolate the bridge from forcing and damping**.

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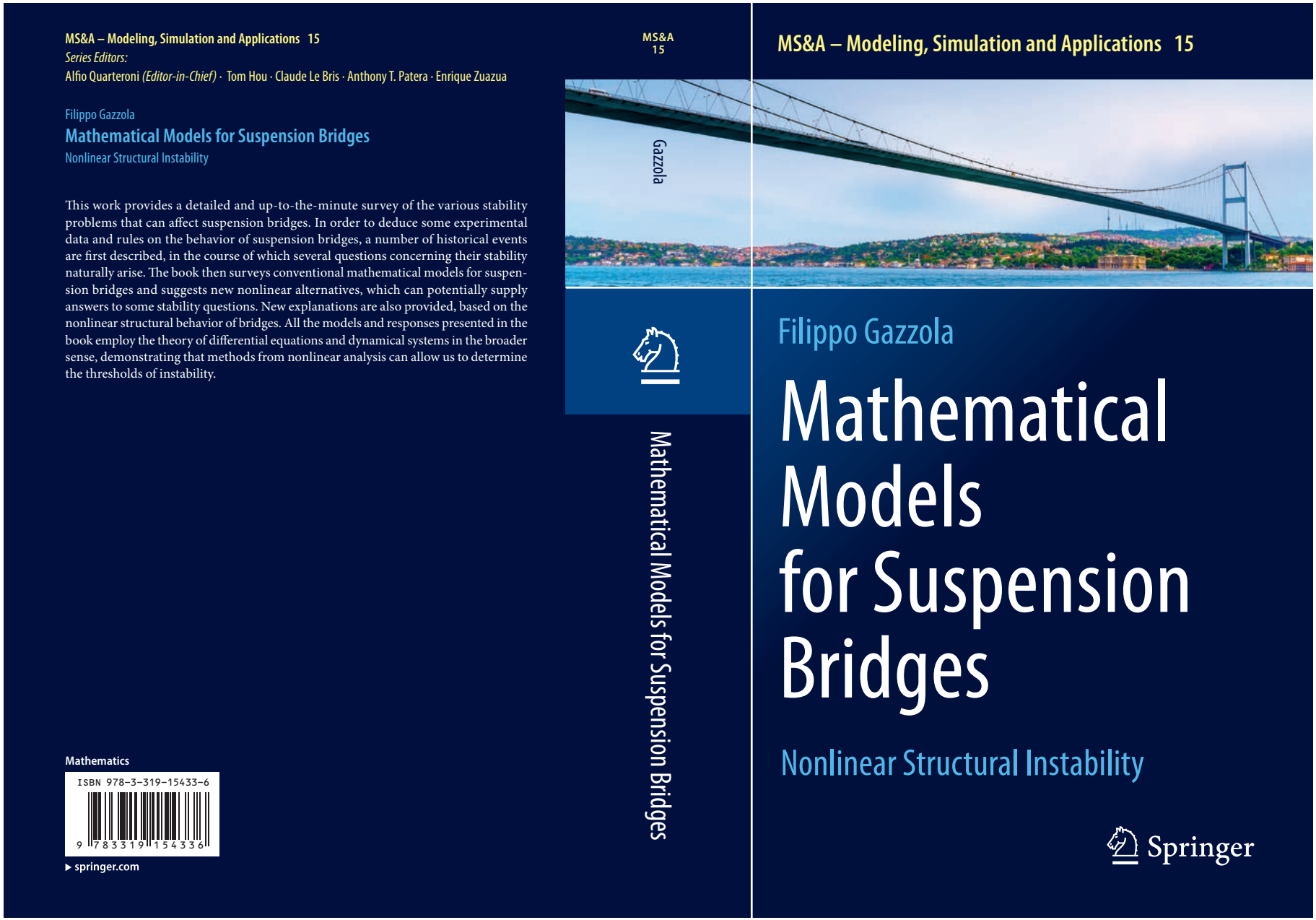
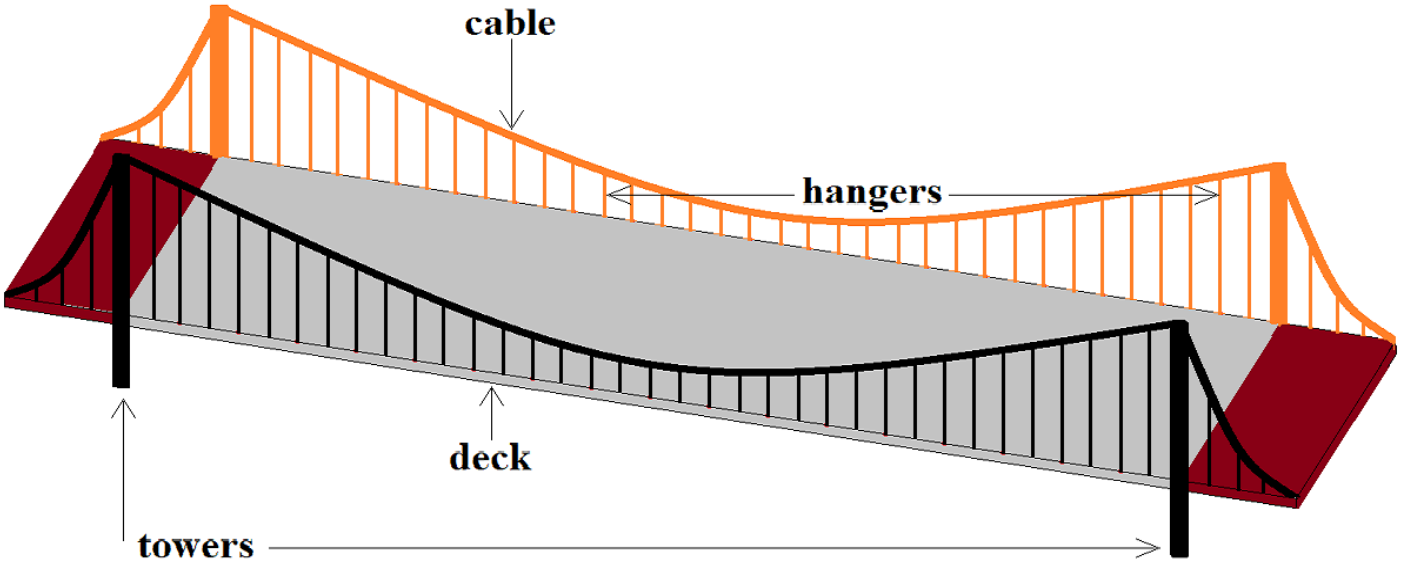
The results **should not depend** on the particular model considered nor on the tools used to analyze it:

qualitative & quantitative – approximations & simplifications.

# For a survey of the historical events and previous models:



# A NEW MATHEMATICAL MODEL



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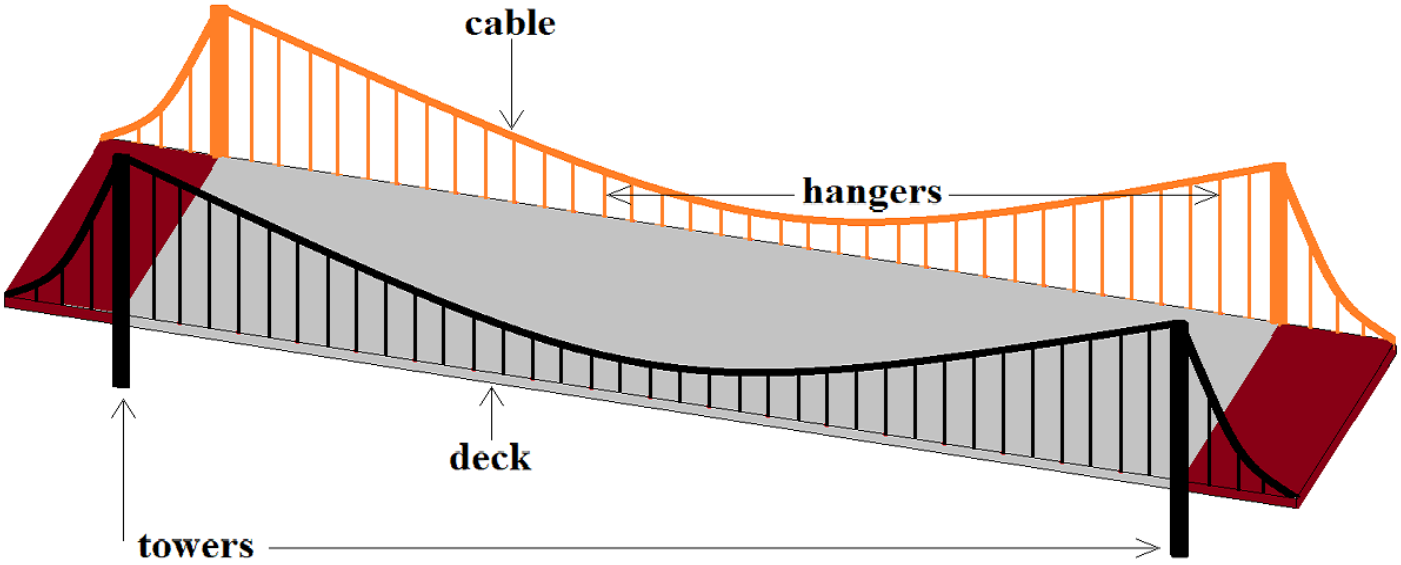


# NOTATIONS

$$w' = \frac{dw}{dx}\,, \quad \dot{w} = \frac{dw}{dt}\,, \quad w_x = \frac{\partial w}{\partial x}\,, \quad w_t = \frac{\partial w}{\partial t}\,.$$

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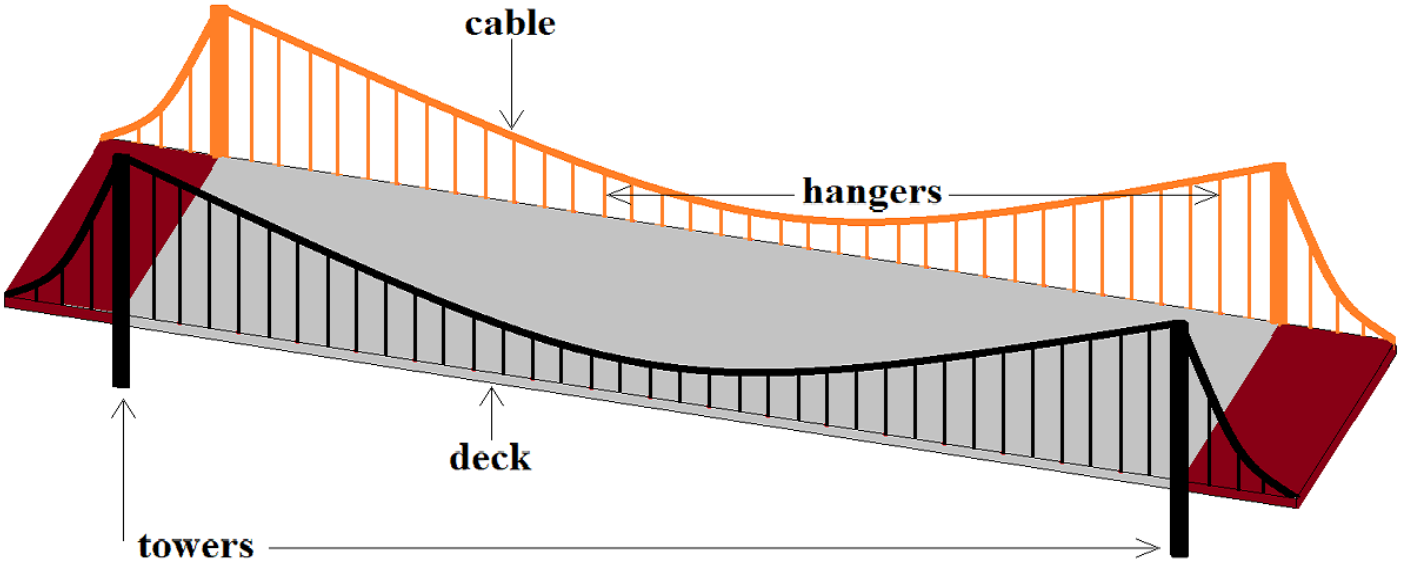


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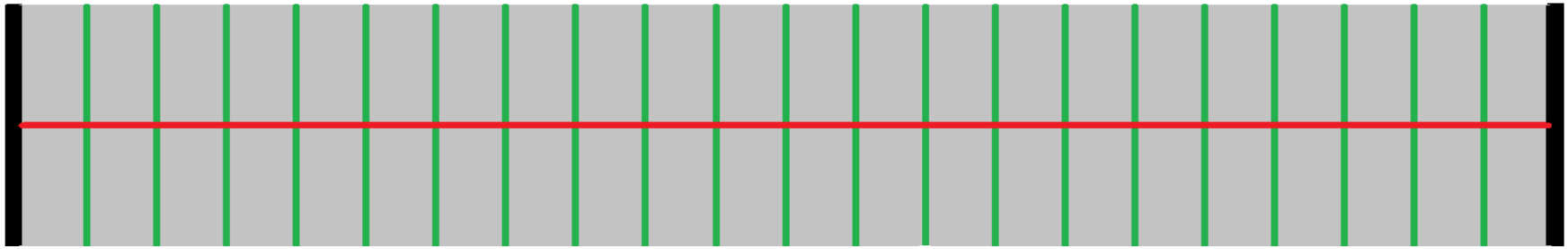
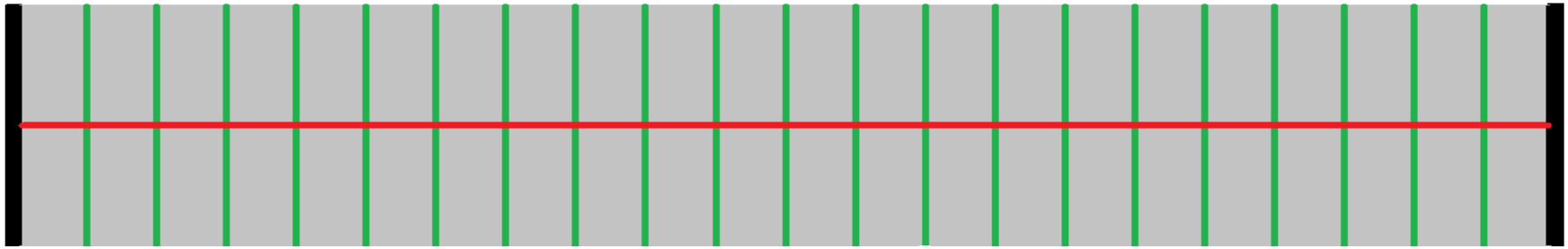
The PDE's are obtained by variational methods, that is, by minimizing the energies involved.



$L$ = deck length $\qquad 2\ell$ = deck width ($2\ell \ll L$).

We model the deck as a degenerate plate = a beam representing the midline of the deck with cross sections which can rotate around the beam:

$y$ = position of the beam, $\theta$ = angle of rotation wrt horizontal.



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Then the positions of the edges of the deck are

$$y \pm \ell \sin\theta \approx y \pm \ell\theta$$

since we aim to study what happens in a **small torsional regime**.

Our results **do not** aim to describe the behavior of the bridge when the torsional angle becomes large.

• E. Berchio, F. Gazzola, *A qualitative explanation of the origin of torsional instability in suspension bridges*, Nonlin. Anal. TMA 2015

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Each cable sustains the weight of half deck; then, $s(x)$ satisfies:

$$H_0 s''(x) = \left( \frac{M}{2} + m\sqrt{1 + s'(x)^2} \right) g \,, \qquad s(0) = s(L) = s_0 \,.$$

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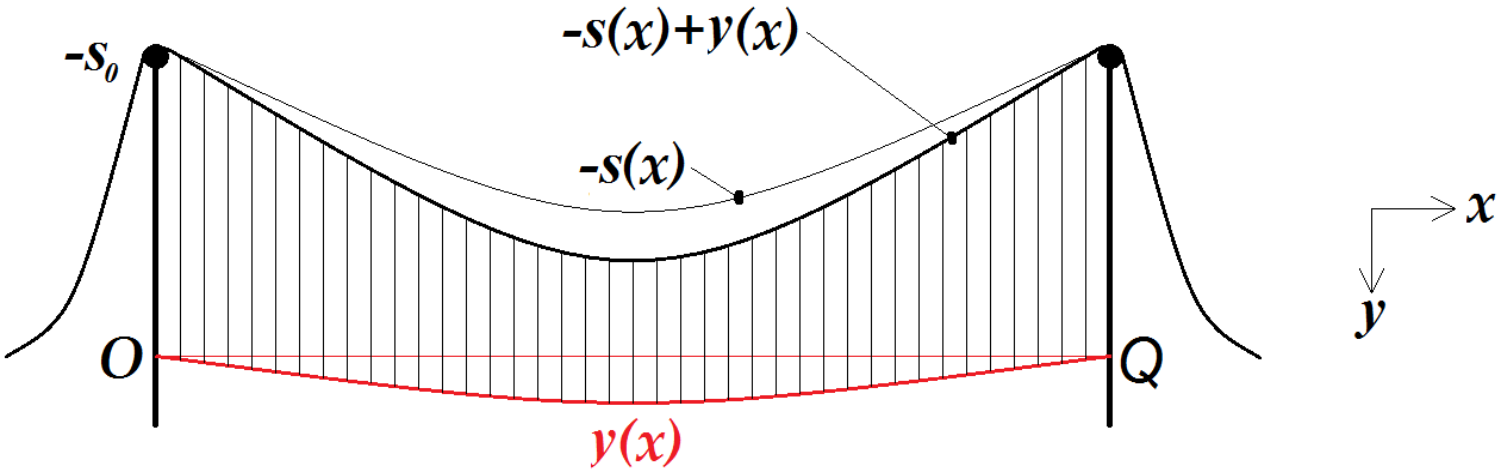
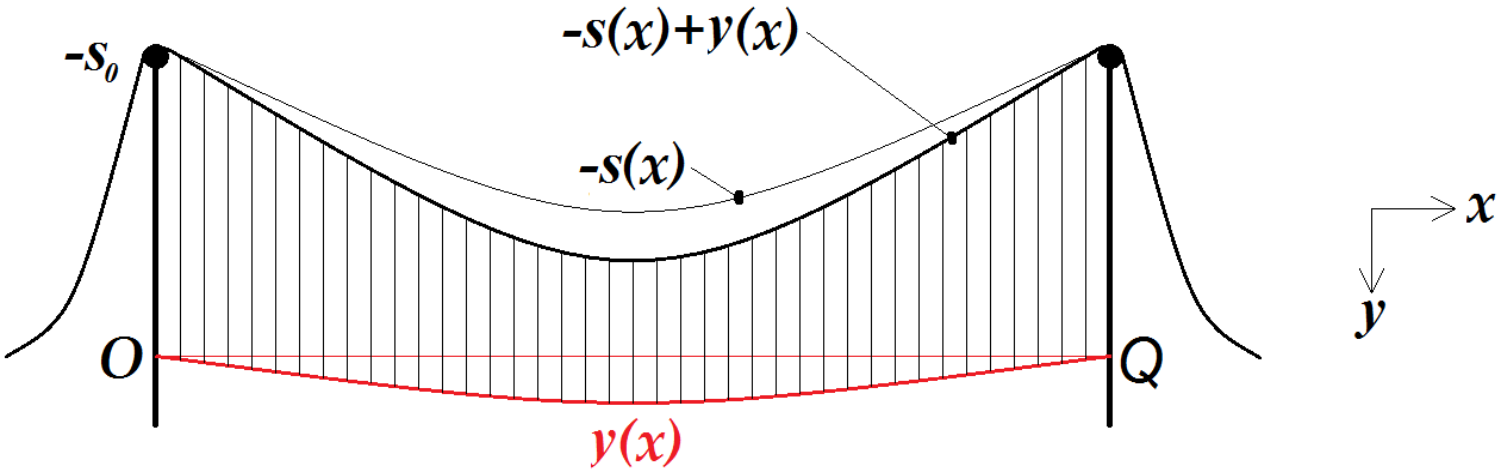
If the cables had no mass ($m = 0$) then the graph would be a parabola whereas if the deck had no mass ($M = 0$) the graph would be a catenary. Since both have masses, the graph is something in between a parabola and a catenary.
In order to simplify notations we set

$$\xi(x) := \sqrt{1 + s'(x)^2} = \text{ the local length of the cable at rest.}$$




When the deck is installed, the hangers are in tension and reach the length $s(x)$; if no additional load acts on the system, the equilibrium position of the deck is horizontal and the position of the cables at equilibrium is $-s(x)$.

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This picture should be reproduced **twice** and with $\pm\theta$.

The action of the cables is the main cause of the nonlinearity of the restoring force.

• G. Bartoli, P. Spinelli, *The stochastic differential calculus for the determination of structural response under wind*, J. Wind Eng. Industrial Aerodyn. (1993)

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The flexibility of the hangers has a significant effect when the deck is very stiff. The TNB had a very flexible deck: we assume that **the hangers are rigid**.

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The nonlinear contribution of the hangers is mainly due to their slackening which occurred at the TNB only **after** that the large torsional oscillations appeared. Our purpose is to understand how negligible torsional oscillations of the deck suddenly become dangerous ones, that is, to describe what happens **before** the slackening starts.

# THE NONLINEAR NONLOCAL SYSTEM OF PDE's

$$
\begin{aligned}
(M+2m\xi)\,y_{tt} \;=\;& -EIy_{xxxx}+H_0\left(\frac{2y_x}{\xi^2}+\frac{3s'(y_x^2+\ell^2\theta_x^2)}{\xi^4}\right)_x-\frac{AE}{L_c}\left[\int_0^L\frac{y_x^2+\ell^2\theta_x^2}{\xi^3}\right]\frac{s''}{\xi^3}\\
&-\frac{2AE}{L_c}\left[\int_0^L\frac{s''y}{\xi^3}\right]\left(\frac{s'}{\xi}-\frac{y_x}{\xi^3}\right)_x+\frac{2AE\ell^2}{L_c}\left[\int_0^L\frac{s''\theta}{\xi^3}\right]\left(\frac{\theta_x}{\xi^3}\right)_x\,,\\
\left(\frac{M}{3}+2m\xi\right)\theta_{tt} \;=\;& \frac{GK}{\ell^2}\theta_{xx}+2H_0\left(\frac{\theta_x}{\xi^2}+\frac{3s'y_x\theta_x}{\xi^4}\right)_x-\frac{2AE}{L_c}\left[\int_0^L\frac{y_x\theta_x}{\xi^3}\right]\frac{s''}{\xi^3}\\
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Boundary conditions

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y(0,t)=y(L,t)=y_{xx}(0,t)=y_{xx}(L,t)=\theta(0,t)=\theta(L,t)=0 \quad \forall t\geq 0\,.
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**THEOREM** This problem admits a unique global weak solution for any initial data.

If $\theta(x,0) = \theta_t(x,0) = 0$ then $\theta(x,t) \equiv 0$ and $y$ solves

$$(M + 2m\xi)y_{tt} + EIy_{xxxx} =$$

$$H_0 \left( \frac{2y_x}{\xi^2} + \frac{3s' y_x^2}{\xi^4} \right)_x - \frac{AE}{L_c} \left[ \int_0^L \frac{y_x^2}{\xi^3} \right] \frac{s''}{\xi^3} - \frac{2AE}{L_c} \left[ \int_0^L \frac{s'' y}{\xi^3} \right] \left( \frac{s'}{\xi} - \frac{y_x}{\xi^3} \right)_x .$$

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We call **nonlinear longitudinal modes** the **periodic solutions** $y$ of this equation but ...

**do periodic solutions exist**?

Periodic solutions exist for related nonlinear equations such as

$$\begin{aligned} &u_{tt} - u_{xx} + f(x,u) = 0 \quad (x,t) \in (0,\pi) \times \mathbb{R}_+ \,, \\ &u(0,t) = u(\pi,t) = 0 \quad t \in \mathbb{R}_+ \,, \end{aligned}$$

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$$\begin{aligned} &u_{tt} - \left(a + b \int_0^\pi u_x^2\right) u_{xx} = 0 \quad (x,t) \in (0,\pi) \times \mathbb{R}_+ \,, \\ &u(0,t) = u(\pi,t) = 0 \quad t \in \mathbb{R}_+ \,. \end{aligned}$$

- P.H. Rabinowitz, CPAM (1978)
- C. Lee, Nonlin. Anal. (2000)
- J.Q. Liu, Nonlin. Anal. (2002-2004)
- T. Cazenave, F.W. Weissler, Quart. Appl. Math. (1996)
- G. Gentile, V. Mastropietro, M. Procesi, Comm. Math. Phys. (2005)

We proceed numerically. We first seek small periodic solutions close to those of the linearized problem

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We seek approximate periodic solutions in the form

$$y(x,t) = \sum_{k=1}^{n} y_k(t) \sin\left(\frac{k\pi x}{L}\right) .$$

We put $Y(t) = \{y_k(t)\}_{k=1,\dots,n}$ with initial conditions $Y(0) = Y_0$ and $\dot{Y}(0) = Y_1$, all in $\mathbb{R}^n$. The PDE is so reduced to a system of nonlinear ODEs

$$\ddot{Y}(t) = G\big(Y(t)\big)\,, \quad Y(0) = Y_0\,, \quad \dot{Y}(0) = Y_1\,.$$

We use Newton's method to find initial vectors $Y_0$ and $Y_1$ which lead to periodic solutions.

Thus, we find numerically

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**$k$-th nonlinear longitudinal modes**.

♣ When little energy is inserted into the bridge, it oscillates close to a linear mode with small amplitude.

♣ On each branch, the period is increasing w.r.t. the energy.

♣ For any $k$-th nonlinear longitudinal mode, the $k$-th Fourier component is much larger than the other ones.

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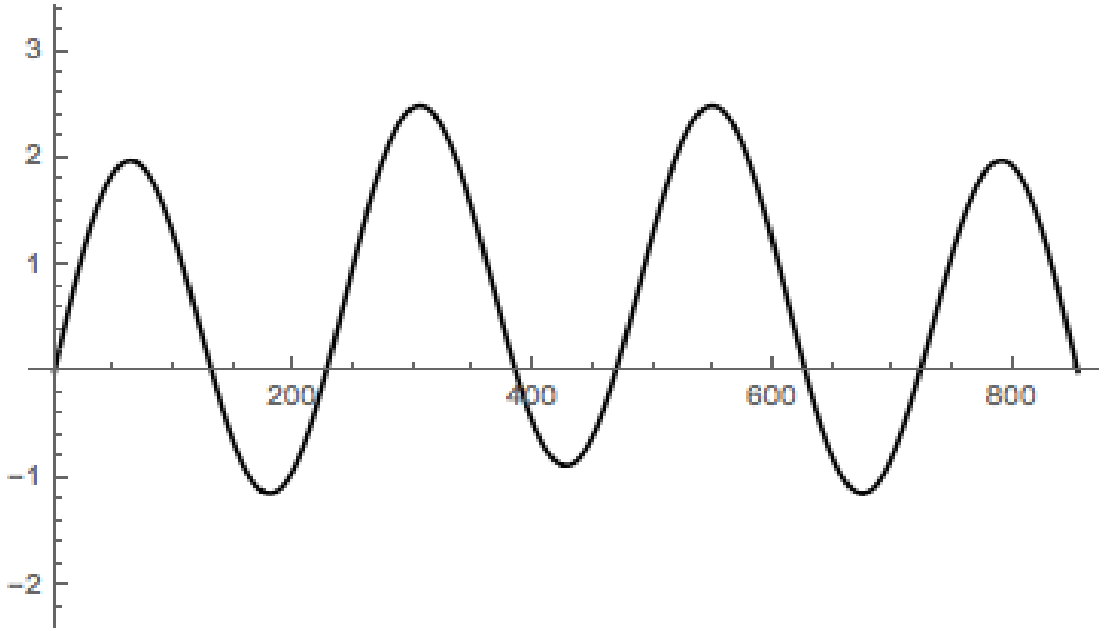


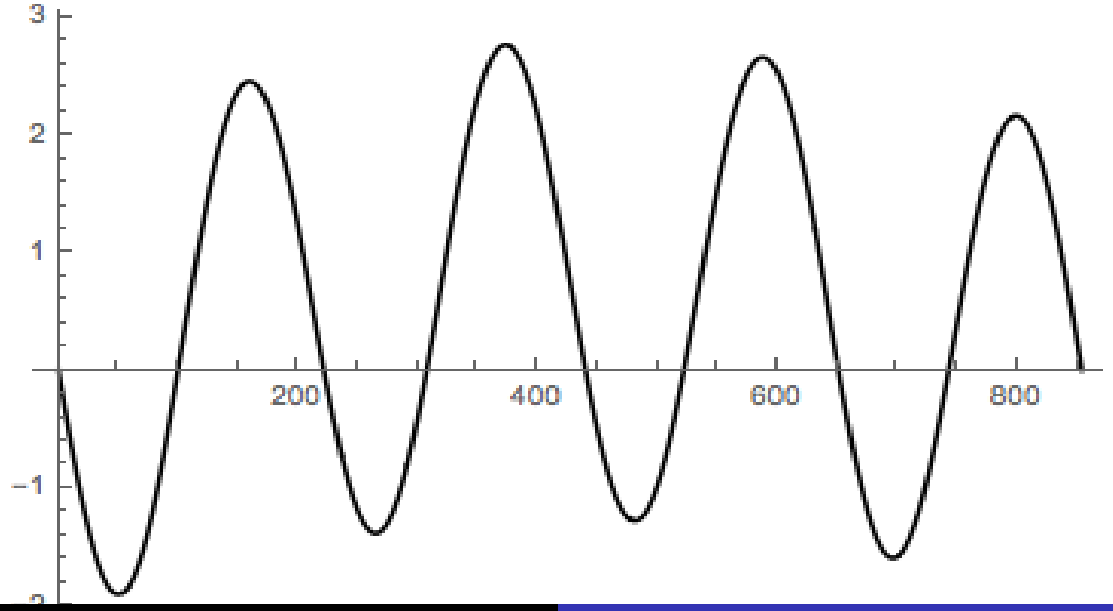

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- R.W. Dickey, *Stability of periodic solutions of the nonlinear string*, Quart. Appl. Math. (1980)

For "most equations" it is equivalent to the Lyapunov stability.

- M. Ghisi, M. Gobbino, *Stability of simple modes of the Kirchhoff equation*, Nonlinearity (2001)

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**Remark:** small longitudinal oscillations = small energy.

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$$u_{tt} - \left(a + b \int_0^\pi u_x^2\right) u_{xx} = 0 \quad (x,t) \in (0,\pi) \times \mathbb{R}_+ \,,$$

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This is true for some Hamiltonian systems

• E. Berchio, F. Gazzola, C. Zanini, *Which residual mode captures the energy of the dominating mode in second order Hamiltonian systems?*, preprint

We insert the parameters of the TNB from the Official Report

• O.H. Ammann, T. von Kármán, G.B. Woodruff, *The failure of the Tacoma Narrows Bridge*, Federal Works Agency (1941)

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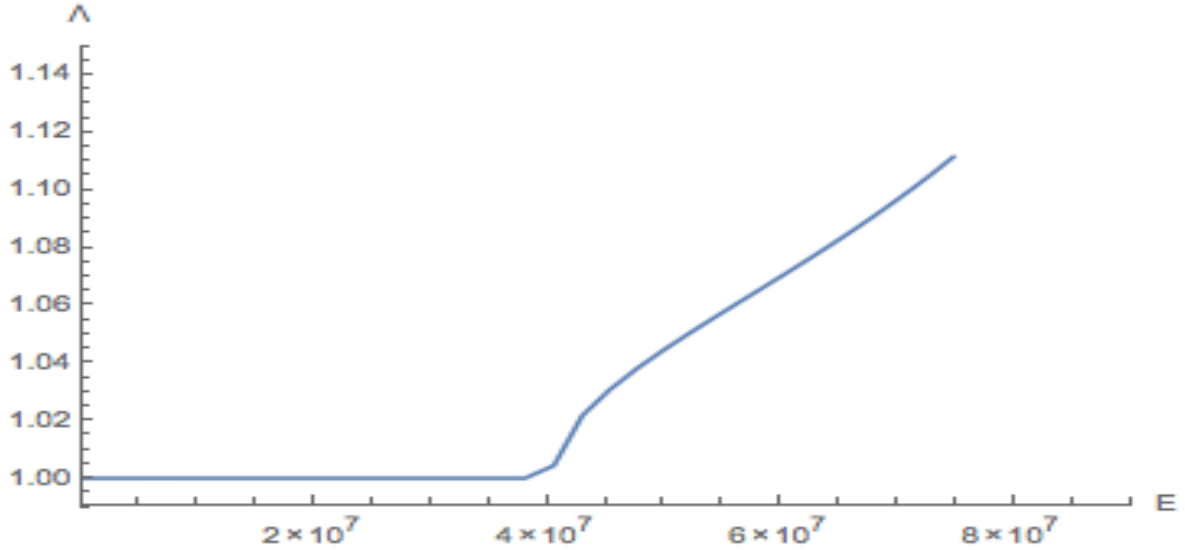
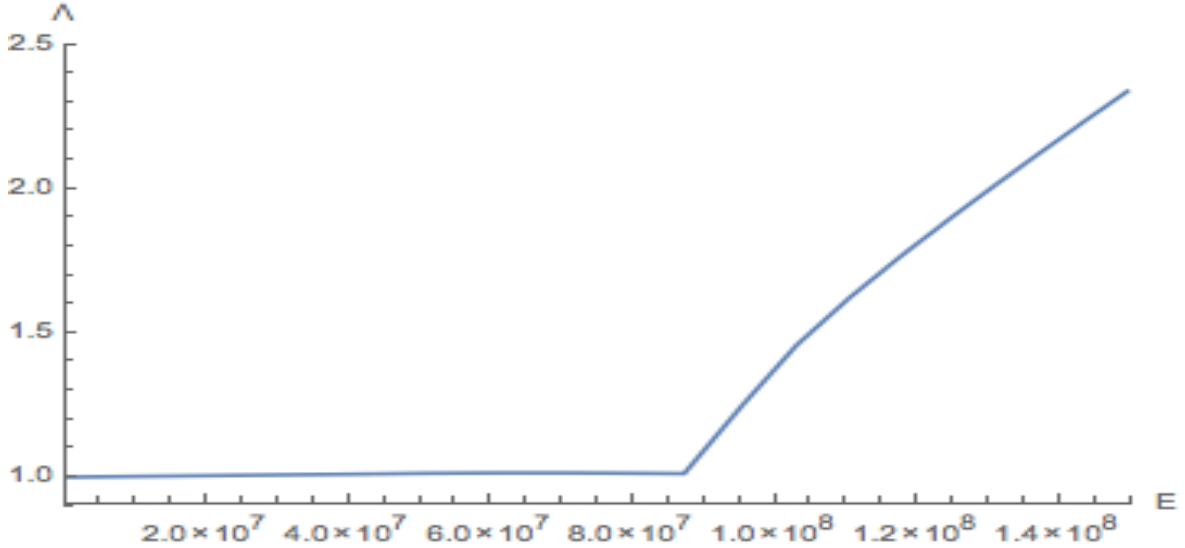
• when such value is larger than 1 the longitudinal mode is torsionally unstable.

# THE PLOTS FOR k=1 AND k=9



## Thresholds of instability

energy threshold, period at the threshold, amplitude of oscillation

$$\Delta := \max_{x,t} y(x,t) - \min_{x,t} y(x,t)\,.$$

| Branch | Energy ($MJ$) | Period ($s$) | $\Delta$ ($m$) |
|---|---|---|---|
| 1 | 38. | 11.22 | 5.8 |
| 2 | 51.8 | 8.46 | 10 |
| 3 | 15.5 | 5.48 | 6.5 |
| 4 | 53.7 | 3.97 | 7.8 |
| 5 | 74.1 | 3.14 | 6.9 |
| 6 | 56.6 | 2.53 | 4.5 |
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♣ A careful look at the video and the data in the Official Report confirm that the oscillations prior to the TNB collapse were of the order of a few meters, as in our numerical results.

# AN EFFECTIVE MEASURE OF TORSIONAL INSTABILITY

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The **Floquet Theorem** states that if $\Psi(t)$ is a fundamental matrix solution of a linear system $\dot{Z} = A(t)Z$ ($A$ is $T$-periodic) then there exists a matrix $B$ and a $T$-periodic matrix $P$ such that $\Psi(t) = P(t)e^{tB}$ (both $B$ and $P$ may be complex-valued).

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**DEFINITION** We call **expansion rate** the largest rate of growth:

$$\mathcal{ER} := \max_j \ |\lambda_j|^{1/T} .$$

# Expansion rates versus energy (in MJ)

| Mode | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|---|
| 1 | 1. | 1. | 1.0662 | — | — | — | — |
| 2 | 1. | 1.00365 | 1.03904 | — | — | — | — |
| 3 | 1.00614 | 1.02071 | 1.02961 | 1.08141 | 1.20949 | — | — |
| 4 | 1. | 1. | 1.01287 | — | — | — | — |
| 5 | 1. | 1. | 1.00001 | 1.01521 | 1.50051 | — | — |
| 6 | 1. | 1. | 1. | 1.09919 | 1.16332 | — | — |
| 7 | 1. | 1. | 1. | 1. | 1.09852 | 1.58567 | 1.97158 |
| 8 | 1. | 1. | 1. | 1. | 1.00112 | 1.66552 | — |
| 9 | 1. | 1. | 1.01322 | 1.01353 | 1.24852 | 1.76429 | 2.12488 |
| 10 | 1. | 1. | 1. | 1. | 1.25447 | 1.73715 | 2.05263 |

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The **energy** of a longitudinal mode is related to the **amplitude** of oscillation $\Delta$ and, for a given energy, higher modes have smaller amplitudes. Therefore, for the same amplitude of oscillation, the expansion rate is larger for the 9th and 10th longitudinal mode than for the 5th.

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These observations show that **the transfer of energy (from longitudinal to torsional) depends on the particular excited mode**.

| Mode | $\Delta$ ($m$) | 4 MJ | 6 MJ | 8 MJ | 10 MJ | 12 MJ | 14 MJ |
|---|---|---|---|---|---|---|---|
| 1 | 5.8 | 1. | 1.0662 | — | — | — | — |
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| 8 | 4.6 | 1. | 1. | 1. | 1.00112 | 1.66552 | — |
| 9 | 3.8 | 1. | 1.01322 | 1.01353 | 1.24852 | 1.76429 | 2.12488 |
| 10 | 3.3 | 1. | 1. | 1. | 1.25447 | 1.73715 | 2.05263 |

This Table explains **why torsional oscillations did not appear earlier even in presence of larger longitudinal oscillations**:

there are longitudinal modes which have a very low expansion rate;

the 9th and 10th mode appear more prone to generate torsional oscillations because they have large expansion rate.

# OUR EXPLANATION OF THE TNB COLLAPSE

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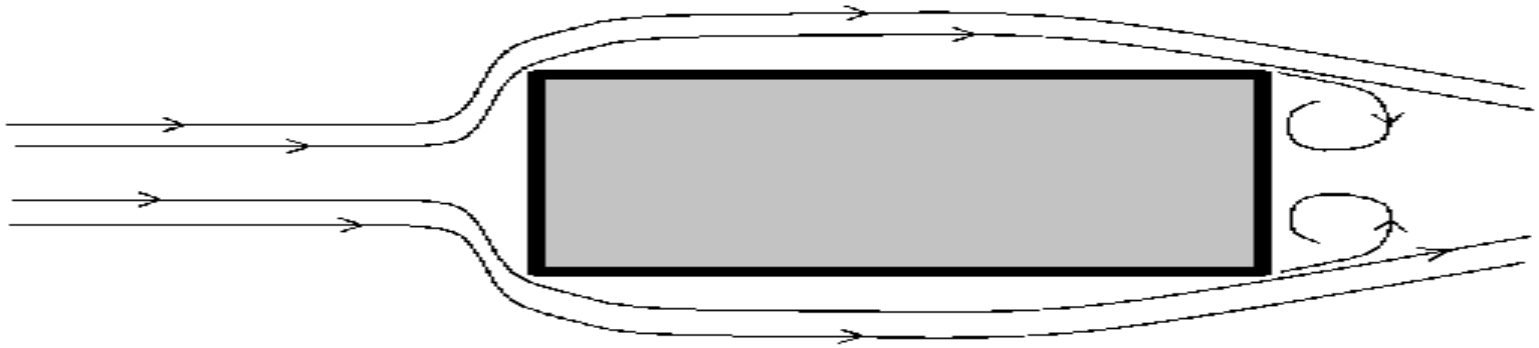
When the wind hits a bluff body (such as the deck of the TNB) the flow is modified and goes around the body. Behind the deck, or a "hidden part" of it, the flow creates vortices which are, in general, asymmetric.



This asymmetry generates a forcing lift which starts the vertical oscillations of the deck.

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This is the point where our analysis starts, that is, when the longitudinal oscillations of the structure reach a periodic motion which is maintained in amplitude by a somehow perfect equilibrium between the input of energy from the wind and internal dissipation.

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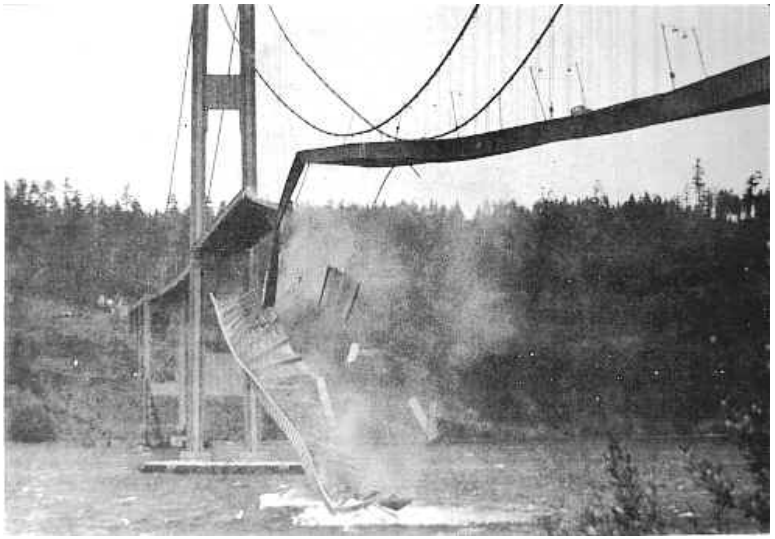
We have seen that, in such situation, if the longitudinal oscillation is sufficiently large then a structural instability appears: this is the onset of torsional oscillations.

**NEXT STEPS:** to obtain a more accurate description of the dynamics of suspension bridges, take into account also the behavior of the aerodynamic and dissipation effects after the appearance of the torsional oscillations.



**THANK YOU FOR YOUR ATTENTION**