Torsional instability in suspension bridges: 
the Tacoma Narrows Bridge case

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The most impressive failure of history is certainly the **Tacoma Narrows Bridge** collapse in November 1940.
The TNB collapse is not an isolated event; other bridges collapsed in a similar way due to hurricanes:

the **Brighton Chain Pier** (UK) collapsed in 1836;
the **Menai Straits Bridge** (UK) collapsed in 1839;
the **Wheeling Suspension Bridge** (US) collapsed in 1854;
the **Matukituki Suspension Footbridge** (NZ) collapsed in 1977.

By "similar way" we mean that unexpected and destructive torsional oscillations suddenly appeared.
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The Official Report

- O.H. Ammann, T. von Kármán, G.B. Woodruff, *The failure of the Tacoma Narrows Bridge*, Federal Works Agency (1941) considers ...the crucial event in the collapse to be the sudden change from a vertical to a torsional mode of oscillation.
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There have been many attempts to answer to the following fundamental question:

**why do torsional oscillations appear suddenly?**

- mistake in the project, structural failure
- resonance due to the frequency of the wind
- flutter theory
- vortex shedding, von Kármán vortices
- angle of attack of the wind
- parametric resonance.
The civil and aeronautical engineer Robert Scanlan writes that …the original Tacoma Narrows Bridge withstood random buffeting for some hours with relatively little harm until some **fortuitous condition** “broke” the bridge action over into its low antisymmetrical torsion flutter mode.


Joe McKenna writes that there is no consensus on what caused the sudden change to torsional motion.


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• R.H. Scanlan, *The action of flexible bridges under wind*, J.
Sound & Vibration (1978)

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torsional motion.
• P.J. McKenna, *Torsional oscillations in suspension bridges
revisited: fixing an old approximation*, Amer. Math. Monthly
(1999)

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• R. Scott, *In the wake of Tacoma. Suspension bridges and the
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To this end, we **isolate the bridge from forcing and damping**.

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The results should not depend on the particular model considered nor on the tools used to analyze it:

qualitative & quantitative – approximations & simplifications.
For a survey of the historical events and previous models:

Mathematical Models for Suspension Bridges
Nonlinear Structural Instability

This work provides a detailed and up-to-the-minute survey of the various stability problems that can affect suspension bridges. In order to deduce some experimental data and rules on the behavior of suspension bridges, a number of historical events are first described, in the course of which several questions concerning their stability naturally arise. The book then surveys conventional mathematical models for suspension bridges and suggests new nonlinear alternatives, which can potentially supply answers to some stability questions. New explanations are also provided, based on the nonlinear structural behavior of bridges. All the models and responses presented in the book employ the theory of differential equations and dynamical systems in the broader sense, demonstrating that methods from nonlinear analysis can allow us to determine the thresholds of instability.
A NEW MATHEMATICAL MODEL

The PDE's are obtained by variational methods, that is, by minimizing the energies involved.
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NOTATIONS

\[ w' = \frac{dw}{dx}, \quad \dot{w} = \frac{dw}{dt}, \quad w_x = \frac{\partial w}{\partial x}, \quad w_t = \frac{\partial w}{\partial t}. \]
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\[ L = \text{deck length} \quad 2\ell = \text{deck width} \quad (2\ell \ll L). \]

We model the deck as a degenerate plate = a beam representing the midline of the deck with cross sections which can rotate around the beam:

\[ y = \text{position of the beam}, \quad \theta = \text{angle of rotation wrt horizontal}. \]

Our results do not aim to describe the behavior of the bridge when the torsional angle becomes large.
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Then the positions of the edges of the deck are

\[ y \pm \ell \sin \theta \approx y \pm \ell \theta \]

since we aim to study what happens in a \textit{small torsional regime}.

Our results \textbf{do not} aim to describe the behavior of the bridge when the torsional angle becomes large.

Since the spacing between hangers is small, the hangers act as a continuous membrane connecting the cables and the deck.

We denote by \(-s(x)\) the position of the cables at rest, \(-s_0<0\) being the level of the left and right endpoints of the cables (the height of the towers). Each cable sustains the weight of half deck; then, \(s(x)\) satisfies:

\[
H_0 s''(x) = \left( M_2 + m\sqrt{1 + s'(x)^2} \right) g,
\]

\(s(0) = s(L) = s_0\). This problem admits a unique solution which, moreover, is symmetric with respect to \(x = L/2\).

If the cables had no mass (\(m=0\)) then the graph would be a parabola whereas if the deck had no mass (\(M=0\)) the graph would be a catenary. Since both have masses, the graph is something in between a parabola and a catenary.

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When the deck is installed, the hangers are in tension and reach the length $s(x)$; if no additional load acts on the system, the equilibrium position of the deck is horizontal and the position of the cables at equilibrium is $-s(x)$. 

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This picture should be reproduced twice and with $\pm \theta$. 
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The nonlinear contribution of the hangers is mainly due to their slackening which occurred at the TNB only after that the large torsional oscillations appeared. Our purpose is to understand how negligible torsional oscillations of the deck suddenly become dangerous ones, that is, to describe what happens before the slackening starts.
THE NONLINEAR NONLOCAL SYSTEM OF PDE's

\[(M + 2m\xi)y_{tt} = -Ely_{xxxx} + H_0 \left( \frac{2y_x}{\xi^2} + \frac{3s'(y_x^2 + \ell^2\theta_x^2)}{\xi^4} \right) - \frac{AE}{L_c} \left[ \int_0^L y_x^2 + \ell^2\theta_x^2 \right] \frac{s''}{\xi^3} \]

\[-2AE \left[ \int_0^L \frac{s'' y}{\xi^3} \right] \left( \frac{s'}{\xi} - \frac{y_x}{\xi^3} \right) x + 2AE\ell^2 \left[ \int_0^L \frac{s'' \theta}{\xi^3} \right] \left( \frac{\theta_x}{\xi^3} \right) x , \]

\[(M + 2m\xi)\theta_{tt} = \frac{GK}{\ell^2} \theta_{xx} + 2H_0 \left( \frac{\theta_x}{\xi^2} + \frac{3s' \theta_x \theta_x}{\xi^4} \right) x - \frac{AE}{L_c} \left[ \int_0^L y_x \theta_x \right] \frac{s''}{\xi^3} \]

\[-2AE \left[ \int_0^L \frac{s'' \theta}{\xi^3} \right] \left( \frac{s'}{\xi} - \frac{y_x}{\xi^3} \right) x + 2AE \left[ \int_0^L \frac{s'' y}{\xi^3} \right] \left( \frac{\theta_x}{\xi^3} \right) x . \]
\begin{align*}
(M+2m\xi)y_{tt} &= -El y_{xxxx} + H_0 \left( \frac{2y_x}{\xi^2} + \frac{3s'(y_x^2 + \ell^2 \theta_x^2)}{\xi^4} \right)_x - \frac{AE}{L_c} \left[ \int_0^L \frac{y_x^2 + \ell^2 \theta_x^2}{\xi^3} \right] \frac{s''}{\xi^3} \\
- 2AE \left[ \int_0^L \frac{s''}{\xi^3} \right] \left( \frac{s'}{\xi} - \frac{y_x}{\xi^3} \right)_x + 2AE\ell^2 \left[ \int_0^L \frac{s''}{\xi^3} \right] \left( \frac{\theta_x}{\xi^3} \right)_x,
\end{align*}

\begin{align*}
(M_3+2m\xi)\theta_{tt} &= \frac{GK}{\ell^2} \theta_{xx} + 2H_0 \left( \frac{\theta_x}{\xi^3} + \frac{3s' y_x \theta_x}{\xi^4} \right)_x - 2AE \left[ \int_0^L \frac{y_x \theta_x}{\xi^3} \right] \frac{s''}{\xi^3} \\
- 2AE \left[ \int_0^L \frac{s''}{\xi^3} \right] \left( \frac{s'}{\xi} - \frac{y_x}{\xi^3} \right)_x + 2AE \left[ \int_0^L \frac{s''}{\xi^3} \right] \left( \frac{\theta_x}{\xi^3} \right)_x.
\end{align*}

Boundary conditions

\begin{align*}
y(0, t) = y(L, t) = y_{xx}(0, t) = y_{xx}(L, t) = \theta(0, t) = \theta(L, t) = 0 \quad \forall t \geq 0.
\end{align*}
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(M/3 + 2m\xi) \theta_{tt} = \frac{GK}{\ell^2} \theta_{xx} + 2H_0 \left( \frac{\theta_x}{\xi^2} + \frac{3s'y_x \theta_x}{\xi^4} \right)_x - \frac{2AE}{L_c} \left[ \int_0^L \frac{y_x \theta_x}{\xi^3} \right] \frac{s''}{\xi^3} - 2AE \frac{L_c}{\ell^2} \left[ \int_0^L \frac{s'' \theta}{\xi^3} \right] \left( \frac{s'}{\xi^3} - \frac{y_x}{\xi^3} \right)_x + 2AE \left[ \int_0^L \frac{s'' y}{\xi^3} \right] \left( \frac{\theta_x}{\xi^3} \right)_x.
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**THEOREM** This problem admits a unique global weak solution for any initial data.
If \( \theta(x, 0) = \theta_t(x, 0) = 0 \) then \( \theta(x, t) \equiv 0 \) and \( y \) solves

\[
(M + 2m\xi)y_{tt} + Ely_{xxxx} =
\]

\[
H_0 \left( \frac{2y_x}{\xi^2} + \frac{3s'y_x^2}{\xi^4} \right)_x - \frac{AE}{L_c} \left[ \int_0^L \frac{y_x^2}{\xi^3} \frac{s''}{\xi^3} \right] - \frac{2AE}{L_c} \left[ \int_0^L \frac{s''y}{\xi^3} \right] \left( \frac{s'}{\xi} - \frac{y_x}{\xi^3} \right)_x.
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\]

We call **nonlinear longitudinal modes** the periodic solutions \( y \) of this equation but ... 

**do periodic solutions exist?**
Periodic solutions exist for related nonlinear equations such as

\[
\begin{align*}
  u_{tt} - u_{xx} + f(x, u) &= 0 \quad (x, t) \in (0, \pi) \times \mathbb{R}_+, \\
  u(0, t) &= u(\pi, t) = 0 \quad t \in \mathbb{R}_+,
\end{align*}
\]

\[
\begin{align*}
  u_{tt} + u_{xxxx} + f(x, u) &= 0 \quad (x, t) \in (0, \pi) \times \mathbb{R}_+, \\
  u(0, t) &= u(\pi, t) = u_{xx}(0, t) = u_{xx}(\pi, t) = 0 \quad t \in \mathbb{R}_+,
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\[
\begin{align*}
  u_{tt} - \left( a + b \int_0^\pi u_x^2 \right) u_{xx} &= 0 \quad (x, t) \in (0, \pi) \times \mathbb{R}_+, \\
  u(0, t) &= u(\pi, t) = 0 \quad t \in \mathbb{R}_+.
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\]

- P.H. Rabinowitz, CPAM (1978)
We proceed numerically. We first seek small periodic solutions close to those of the linearized problem

$$(M + 2m\xi)y_{tt} + Ely_{xxxx} = 2H_0 \left(\frac{y_x}{\xi^2}\right)_x - \frac{2AE}{L_c} \left[\int_0^L \frac{s''y}{\xi^3}\right]\frac{s''}{\xi^3}$$

that may be obtained by separating variables.

We seek approximate periodic solutions in the form

$$y(x, t) = \sum_{k=1}^{n} y_k(t) \sin\left(\frac{k\pi x}{L}\right).$$

We put $Y(t) = \{y_k(t)\}_{k=1}^{n}$, $\dot{Y}(0) = Y_0$, $\ddot{Y}(0) = Y_1$, all in $\mathbb{R}^n$. The PDE is so reduced to a system of nonlinear ODEs

$$\ddot{Y}(t) = G(Y(t)), \hspace{1cm} Y(0) = Y_0, \hspace{1cm} \dot{Y}(0) = Y_1.$$
We proceed numerically. We first seek small periodic solutions close to those of the linearized problem

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y(x, t) = \sum_{k=1}^n y_k(t) \sin \left(\frac{k\pi x}{L}\right).
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We put $Y(t) = \{y_k(t)\}_{k=1,...,n}$ with initial conditions $Y(0) = Y_0$ and $\dot{Y}(0) = Y_1$, all in $\mathbb{R}^n$. The PDE is so reduced to a system of nonlinear ODEs

$$
\ddot{Y}(t) = G(Y(t)), \quad Y(0) = Y_0, \quad \dot{Y}(0) = Y_1.
$$

We use Newton’s method to find initial vectors $Y_0$ and $Y_1$ which lead to periodic solutions.
Thus, we find numerically

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For each $k$, we follow the branch of periodic solutions which starts on the $k$-th linear mode: these are the

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♣ When little energy is inserted into the bridge, it oscillates close to a linear mode with small amplitude.
♣ On each branch, the period is increasing w.r.t. the energy.
♣ For any $k$-th nonlinear longitudinal mode, the $k$-th Fourier component is much larger than the other ones.
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STRATEGY:

• we take $\theta \equiv 0$ in the original system → nonlinear PDE for $y$

• we approximate the nonlinear PDE with a finite system of nonlinear ODEs and we obtain approximate periodic solutions

• we put these solutions in the equation for $\theta$ → linear PDE for $\theta$

• we approximate the PDE with a finite system of linear ODEs

• we study the stability of this system. This is the so-called linear stability.


For “most equations” it is equivalent to the Lyapunov stability.

M. Ghisi, M. Gobbino, Stability of simple modes of the Kirchhoff equation, Nonlinearity (2001)
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G. Arioli & F. Gazzola - DipMat - PoliMi

Torsional Instability at the TNB
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For “most equations” it is equivalent to the Lyapunov stability.

The study of the torsional stability of a suspension bridge is so reduced to that of the linear system of ODE’s:

\[ \ddot{\theta}_k(t) + \sum_{j=1}^{\nu} \chi_{jk}(t) \theta_j(t) = 0, \quad (k = 1, \ldots, \nu) \]

where the coefficients \( \chi_{jk} \) are periodic and depend on \( y(x, t) \).

**Theorem**

Let \( \nu = 2 \). If the longitudinal mode \( y \) is sufficiently small (that is, \( \|y\|_\infty \) is small), then the system is stable. This means that if longitudinal oscillations are sufficiently small then they are stable, that is, no large torsional oscillations appear.

Remark: small longitudinal oscillations = small energy.
The study of the torsional stability of a suspension bridge is so reduced to that of the linear system of ODE’s:

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This means that if longitudinal oscillations are sufficiently small then they are stable, that is, no large torsional oscillations appear.

**Remark:** small longitudinal oscillations = small energy.
ARE LARGE LONGITUDINAL OSCILLATIONS UNSTABLE?

This is true in the case of the Kirchhoff equation
\[ u_{tt} - \left( a + b \int_0^\pi u^2(x) \, dx \right) u_{xx} = 0 \quad (x, t) \in (0, \pi) \times \mathbb{R}^+ \]


This is true for some Hamiltonian systems
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We proceed **numerically**:

- take a $k$-th nonlinear longitudinal mode $y$ having period $T_y$;
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THE PLOTS FOR $k=1$ AND $k=9$
Thresholds of instability

energy threshold, period at the threshold, amplitude of oscillation

\[ \Delta := \max_{x,t} y(x, t) - \min_{x,t} y(x, t). \]

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♣ A careful look at the video and the data in the Official Report confirm that the oscillations prior to the TNB collapse were of the order of a few meters, as in our numerical results.
The Floquet Theorem states that if $\Psi(t)$ is a fundamental matrix solution of a linear system $\dot{Z} = A(t)Z$ (both $A$ and $P$ may be complex-valued), there exists a matrix $B$ and a $T$-periodic matrix $P$ such that $\Psi(t) = Pe^{tB}$, where the eigenvalues $\lambda_1, ..., \lambda_{2\nu}$ of $e^{TB}$ are called the characteristic multipliers: if $V_1, \ldots, V_m$ denote the corresponding normalized eigenvectors, then the solution $Z_j$ satisfying the initial condition $Z_j(0) = V_j$ (for $j = 1, \ldots, m$) also satisfies $Z_j(T) = \lambda_j Z_j(0)$ and, in turn, $Z_j(nT) = \lambda_n^j Z_j(0)$ for any integer $n \geq 1$.

Whence $|\lambda_j^1|/T$ is the rate of growth of the amplitude of oscillation of $Z_j$.

**DEFINITION**

We call expansion rate the largest rate of growth:

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AN EFFECTIVE MEASURE OF TORSIONAL INSTABILITY

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- On the day of the collapse, the bridge appeared to be behaving in the customary manner and the motions were considerably less than had occurred many times before.

- On the day of the collapse, the torsional oscillations started suddenly and the motions, which a moment before had involved a number of waves (nine or ten) had shifted almost instantly to two.

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This Table explains why torsional oscillations did not appear earlier even in presence of larger longitudinal oscillations:

there are longitudinal modes which have a very low expansion rate;

the 9th and 10th mode appear more prone to generate torsional oscillations because they have large expansion rate.
OUR EXPLANATION OF THE TNB COLLAPSE

It is clear that in absence of wind or external loads the deck of a bridge remains still. When the wind hits a bluff body (such as the deck of the TNB) the flow is modified and goes around the body. Behind the deck, or a "hidden part" of it, the flow creates vortices which are, in general, asymmetric. This asymmetry generates a forcing lift which starts the vertical oscillations of the deck.
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**NEXT STEPS:** to obtain a more accurate description of the dynamics of suspension bridges, take into account also the behavior of the aerodynamic and dissipation effects after the appearance of the torsional oscillations.
THANK YOU FOR YOUR ATTENTION
Torsional Instability at the TNB