Multi-layer radial solutions for a supercritical Neumann problem

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Susanna Terracini (Univ. di Torino)
Let us consider the following problem

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\begin{cases}
-\Delta u + u = u^p & \text{in } B_1 \\
u > 0 & \text{in } B_1, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1
\end{cases}
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with \( N \geq 2 \).
This problem has a long history! Let us start by the Dirichlet case.
The Dirichlet case

Some classical results

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- All solutions are radial (Gidas, Ni and Nirenberg).
- There is exactly one solution for $1 < p < \frac{N+2}{N-2}$.
- There are no solutions for $p \geq \frac{N+2}{N-2}$ (Pohozaev identity).
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<tr>
<th>Dirichlet problem</th>
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<tr>
<td>(i) All solutions are radial</td>
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<td>(ii) Uniqueness for ( 1 &lt; p &lt; \frac{N+2}{N-2} )</td>
<td>(ii) Nonuniqueness results</td>
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<td>(iii) No solutions for ( p \geq \frac{N+2}{N-2} )</td>
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**Theorem, C.-S. Lin and W. M. Ni**

There exist radius $R_0$ and $R_1$ such that

- If $1 < p < \frac{N+2}{N-2}$ there exists $R_0 > 0$ such that for any $R \leq R_0$ the only solution is $u \equiv 1$.
- For any $p > 1$ there exists $R_1 = R_1(p)$ such that for any $R \geq R_1$ there exists a nonconstant solution.
In 2011 E. Serra and P. Tilli introduced the following interesting constraint,

\[ M = \{ u \in H^1_{\text{rad}}(B_R) | u \geq 0, u(r) \leq u(s) \text{ for } 0 \leq r \leq s \leq 1 \}. \]
An interesting constraint

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---

**Theorem, E. Serra and P. Tilli (2011)**

Assume that \( a(r) \in L^1(B_1) \) is a radial, increasing, positive and nonconstant function. Then for any \( p > 1 \) the problem

\[
\begin{align*}
-\Delta u + u &= a(|x|)u^p \quad \text{in } B_1 \\
u &> 0 \quad \text{in } B_1, \\
u &= 0 \quad \text{on } \partial B_1
\end{align*}
\]

admits at least one radially increasing solution.
The previous result was generalized to the case $a \equiv 1$ by

Assume that $p > \lambda_2(B_1)$. Then the problem

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where \( \Omega \) is a convex domain. Then, for any \( p > 1 \) (not too small), there exists a solution \( u_p \) satisfying,

\[
(x - x_p) \cdot \nabla u_p(x) \geq 0
\]

for some \( x_p \in \Omega \) and for any \( x \in \Omega \).
The case where $p \to +\infty$ (Dirichlet problem)

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Then, as $p \to +\infty$ we have that,

(i) $\|u_p\|_\infty \to 1$,

(ii) $u_p(r) \to \frac{G(r,r_0)}{G(r_0,r_0)}$ where $G(r,s)$ is the Green function of the operator $-u'' - \frac{N-1}{r} u'$ and $r_0$ is given by

\[
r_0 = \left( \frac{a^{2-N} + b^{2-N}}{2} \right)^{\frac{1}{2-N}}
\]
Ideas of the proof of the previous result

Why is the maximum of $u_p$ going to 1?
Let us recall that $u_p$ can be characterized as

$$I_p = \inf_{u \in H^1_0(A)} \int_{B^1} |\nabla u|^2 \left( \int_{B^1} |u|^{p+1} + 1 \right)^{p+1}$$

It is not difficult to see that $0 < C_0 \leq I_p \leq C_1$. Then,

(a) If $||u_p||_{\infty} \rightarrow M > 1$ we get that $u_p \rightarrow +\infty$ in a set of positive measure and so $I_p \rightarrow +\infty$.

(b) On the other hand if $||u_p||_{\infty} \rightarrow M < 1$ we get that $u_p \rightarrow 0$ uniformly and this is not possible since 0 is an isolated solution.

(ii) Why is $u_p$ converging to $G(r, r_0)$?
By the previous step we derive that $u_p(r) \rightarrow u < 1$ far away from the maximum point. Passing to the limit we get that $-u'' - N_0 r u' = 0$ in $(a, b)$ and the claim follows.
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Then, as $p \to +\infty$, there exists a solution $u_p$ such that

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\lim_{p \to +\infty} u_p(r) = G(r, 1),
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where $G(r, s)$ is the Green function of $-u'' - N - 1 + u$ with $u'(0) = u'(1) = 0$.

Finally, $u$ minimizes the following infimum,

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\inf_{u \in H^1_{rad}(B_1)} \left\{ \int_{B_1} |\nabla u|^2 + u^2 \left( \int_{B_1} |u|^p + 1 \right)^{p+1} \right\}
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subject to $u \leq c$ in $B_1 \setminus B_\rho$. 
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Some remarks on the Green function

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Properties of the Green function

The Green function $G(r, s)$ of the operator $-u'' - \frac{N-1}{r} u' + u$ with Neumann boundary condition is defined as

$$ \begin{cases} 
  -G_{rr}(r, s) - \frac{N-1}{r} G_r(r, s) + G(r, s) = \delta_s(r) & \text{in } (0, 1) \\
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If $N = 3$ the Green function can be explicitly computed,

$$G(r, s) = \begin{cases} \frac{e^r - e^{-r}}{2r} e^s s \quad \text{for} \quad r \leq s \\ \frac{e^s - e^{-s}}{2r} e^r s \quad \text{for} \quad r > s. \end{cases}$$
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(i) $u_p(r)$ converges pointwise to $\sum_{j=1}^{k} A_j G(r, \alpha_j)$, where $(A_1, \ldots, A_k)$ is a solution of the system

$$k \sum_{j=1}^{k} A_j G(\alpha_i, \alpha_j) = 1, \quad i = 1, \ldots, k.$$ 

(ii) $(\alpha_{1,p}, \ldots, \alpha_{k,p}) \to (\alpha_1, \ldots, \alpha_k)$ as $p \to \infty$ and $(\alpha_1, \ldots, \alpha_k)$ is a critical point of the function

$$\phi(\alpha_1, \ldots, \alpha_k) = \inf \{ \| u \|_{H^1_{rad}(B_1)} : u(\alpha_1) = \ldots = u(\alpha_k) = 1 \},$$

in the set $0 < \alpha_1 < \ldots < \alpha_k < 1$. We will see that $\phi(\alpha_1, \ldots, \alpha_k) = k \sum_{i=1}^{k} A_i \alpha_i^{N-1-j}$. 

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(i) $u_p(r)$ converges pointwise to $\sum_{j=1}^{k} A_j G(r, \alpha_j)$, where $(A_1, \ldots, A_k)$ is a solution of the system

$$\sum_{j=1}^{k} A_j G(\alpha_i, \alpha_j) = 1, \quad i = 1, \ldots, k.$$ 

(ii) $(\alpha_{1,p}, \ldots, \alpha_{k,p}) \to (\alpha_1, \ldots, \alpha_k)$ as $p \to \infty$ and $(\alpha_1, \ldots, \alpha_k)$ is a critical point of the function

$$\varphi(\alpha_1, .., \alpha_k) = \inf\{\|u\|_{H^1_{\text{rad}}(B_1)}^2 \colon u(\alpha_1) = \ldots = u(\alpha_k) = 1\},$$
in the set $0 < \alpha_1 < \ldots < \alpha_k < 1$. 
The main result

Let $k > 0$ be an integer. Then there exists $p(k)$ such that for any $p > p(k)$ there exists a radial solution $u_p(r)$ having exactly $k$ maximum points $\alpha_{1,p}, \ldots, \alpha_{k,p}$. Furthermore we have that

(i) $u_p(r)$ converges pointwise to $\sum_{j=1}^{k} A_j G(r, \alpha_j)$, where $(A_1, \ldots, A_k)$ is a solution of the system

$$
\sum_{j=1}^{k} A_j G(\alpha_i, \alpha_j) = 1, \quad i = 1, \ldots, k.
$$

(ii) $(\alpha_{1,p}, \ldots, \alpha_{k,p}) \to (\alpha_1, \ldots, \alpha_k)$ as $p \to \infty$ and $(\alpha_1, \ldots, \alpha_k)$ is a critical point of the function

$$
\varphi(\alpha_1, \ldots, \alpha_k) = \inf \{ \|u\|_{H^1_{rad}(B_1)}^2 : u(\alpha_1) = \ldots = u(\alpha_k) = 1 \},
$$

in the set $0 < \alpha_1 < \ldots < \alpha_k < 1$.

We will see that $\varphi(\alpha_1, \ldots, \alpha_k) = \sum_{i=1}^{k} A_i \alpha_i^{N-1}$.
The case of one peak

There exists \( \bar{\rho} \) such that for any \( \rho > \bar{\rho} \) there exists a radial solution \( u_\rho(r) \) having exactly 1 maximum point \( \alpha_1, \rho \) such that

(i) \( u_\rho(r) \to A G(r, \alpha_1) = G(r, \alpha_1) G(\alpha_1, \alpha_1) \).

So \( A_1 = 1 G(\alpha_1, \alpha_1) \),

(ii) \( \alpha_1, \rho \to \alpha_1 \) as \( \rho \to \infty \) and \( \alpha_1 \) is a critical point of the function \( \phi(\alpha_1) = \inf \{ \| u \|_{H^1_{rad}(B_1)} : u(\alpha_1) = 1 \} \), in the set \( 0 < \alpha_1 < 1 \).

Moreover we have that \( \phi(\alpha_1) = A_1 \alpha_{N-1} = \alpha_{N-1} G(\alpha_1, \alpha_1) \).
There exists $\bar{p}$ such that for any $p > \bar{p}$ there exists a radial solution $u_p(r)$ having exactly 1 maximum point $\alpha_{1,p}$ such that

\[(i)\quad u_p(r) \to A_1 G(r, \alpha_{1,p}) = G(r, \alpha_{1,p}) G(\alpha_{1,p}, \alpha_{1,p}) .\]

So $A_1 = \frac{1}{G(\alpha_{1,p}, \alpha_{1,p})}$,

\[(ii)\quad \alpha_{1,p} \to \alpha_{1} \text{ as } p \to \infty \]

and $\alpha_{1}$ is a critical point of the function $\varphi(\alpha_{1}) = \inf \{\| u \|^2_{H^1_{\text{rad}}(B_1)} : u(\alpha_{1}) = 1\}$, \hspace{1cm} in the set $0 < \alpha_{1} < 1$. Moreover we have that $\varphi(\alpha_{1}) = A_1 \frac{1}{1} = A_1 \frac{1}{1} G(\alpha_{1}, \alpha_{1})$.
There exists $\bar{p}$ such that for any $p > \bar{p}$ there exists a radial solution $u_p(r)$ having exactly 1 maximum point $\alpha_{1,p}$ such that

(i) $u_p(r) \rightarrow A_1 G(r, \alpha_1) = \frac{G(r, \alpha_1)}{G(\alpha_1, \alpha_1)}$. So

$$A_1 = \frac{1}{G(\alpha_1, \alpha_1)},$$
There exists \( \bar{p} \) such that for any \( p > \bar{p} \) there exists a radial solution \( u_p(r) \) having exactly 1 maximum point \( \alpha_{1,p} \) such that

(i) \( u_p(r) \to A_1 G(r, \alpha_1) = \frac{G(r, \alpha_1)}{G(\alpha_1, \alpha_1)} \). So

\[
A_1 = \frac{1}{G(\alpha_1, \alpha_1)},
\]

(ii) \( \alpha_{1,p} \to \alpha_1 \) as \( p \to \infty \) and \( \alpha_1 \) is a critical point of the function

\[
\varphi(\alpha_1) = \inf\{\|u\|_{H^1_{rad}(B_1)}^2 : u(\alpha_1) = 1\},
\]

in the set \( 0 < \alpha_1 < 1 \).
The case of one peak

There exists \( \bar{p} \) such that for any \( p > \bar{p} \) there exists a radial solution \( u_p(r) \) having exactly 1 maximum point \( \alpha_{1,p} \) such that

(i) \( u_p(r) \rightarrow A_1 G(r, \alpha_1) = \frac{G(r, \alpha_1)}{G(\alpha_1, \alpha_1)} \). So

\[
A_1 = \frac{1}{G(\alpha_1, \alpha_1)},
\]

(ii) \( \alpha_{1,p} \rightarrow \alpha_1 \) as \( p \rightarrow \infty \) and \( \alpha_1 \) is a critical point of the function

\[
\varphi(\alpha_1) = \inf \{ \| u \|^2_{H^1_{rad}(B_1)} : u(\alpha_1) = 1 \},
\]

in the set \( 0 < \alpha_1 < 1 \).

Moreover we have that

\[
\varphi(\alpha_1) = A_1 \alpha_1^{N-1} = \frac{\alpha_1^{N-1}}{G(\alpha_1, \alpha_1)}.
\]
The case of two peaks

There exists $\bar{p}$ such that for any $p > \bar{p}$ there exists a radial solution $u_p(r)$ having exactly 2 maximum points $\alpha_1, p$ and $\alpha_2, p$ such that

(i) $u_p(r) \to A_1 G(r, \alpha_1) + A_2 G(r, \alpha_2)$ with

$A_1 = G(\alpha_2, \alpha_2) - G(\alpha_1, \alpha_2)$
$G(\alpha_1, \alpha_1) - G(\alpha_1, \alpha_2)$

$A_2 = G(\alpha_1, \alpha_1) - G(\alpha_2, \alpha_1)$
$G(\alpha_2, \alpha_2) - G(\alpha_2, \alpha_1)$.

(ii) $(\alpha_1, p, \alpha_2, p) \to (\alpha_1, \alpha_2)$ as $p \to \infty$ and $(\alpha_1, \alpha_2)$ is a critical point of the function $\phi(\alpha_1, \alpha_2) = \inf \{\|u\|_{H^1_{rad}(B_1)} : u(\alpha_1) = u(\alpha_2) = 1\}$, in the set $0 < \alpha_1 < \alpha_2 < 1$.

Moreover we have that $\phi(\alpha_1, \alpha_2) = A_1 \alpha_1^{N-1} + A_2 \alpha_2^{N-1} = (G(\alpha_2, \alpha_2) - G(\alpha_1, \alpha_2)) \alpha_1^{N-1} G(\alpha_1, \alpha_1) - G(\alpha_1, \alpha_2) G(\alpha_2, \alpha_1) + (G(\alpha_1, \alpha_1) - G(\alpha_2, \alpha_1)) \alpha_2^{N-1} G(\alpha_1, \alpha_1) - G(\alpha_1, \alpha_2) G(\alpha_2, \alpha_1)$. 


The case of two peaks

There exists $\bar{p}$ such that for any $p > \bar{p}$ there exists a radial solution $u_p(r)$ having exactly 2 maximum points $\alpha_{1,p}$ and $\alpha_{2,p}$ such that

(i) $u_p(r) \rightarrow A_1 G(\alpha_{1,p}) + A_2 G(\alpha_{2,p})$ with

$$A_1 = G(\alpha_{2,p}) - G(\alpha_{1,p})$$
$$A_2 = G(\alpha_{1,p}) - G(\alpha_{2,p})$$

(ii) $(\alpha_{1,p}, \alpha_{2,p}) \rightarrow (\alpha_{1,\infty}, \alpha_{2,\infty})$ as $p \rightarrow \infty$ and $(\alpha_{1,\infty}, \alpha_{2,\infty})$ is a critical point of the function $\varphi(\alpha_{1,\infty}, \alpha_{2,\infty}) = \inf \{ \| u \|_{H^1_{rad}(B_1)} : u(\alpha_{1,\infty}) = u(\alpha_{2,\infty}) = 1 \}$,
in the set $0 < \alpha_{1,\infty} < \alpha_{2,\infty} < 1$.

Moreover we have that $\varphi(\alpha_{1,\infty}, \alpha_{2,\infty}) = A_1 \alpha_{N-1}^1 + A_2 \alpha_{N-1}^2 = (G(\alpha_{2,\infty}) - G(\alpha_{1,\infty})) \alpha_{N-1}^1 G(\alpha_{1,\infty}) G(\alpha_{2,\infty}) - G(\alpha_{1,\infty}) G(\alpha_{2,\infty})$.
The case of two peaks

There exists \( \overline{p} \) such that for any \( p > \overline{p} \) there exists a radial solution \( u_p(r) \) having exactly 2 maximum points \( \alpha_{1,p} \) and \( \alpha_{2,p} \) such that

(i) \( u_p(r) \to A_1 G(r, \alpha_1) + A_2 G(r, \alpha_2) \) with

\[
A_1 = \frac{G(\alpha_2, \alpha_2) - G(\alpha_1, \alpha_2)}{G(\alpha_1, \alpha_1) G(\alpha_2, \alpha_2) - G(\alpha_1, \alpha_2) G(\alpha_2, \alpha_1)},
\]

\[
A_2 = \frac{G(\alpha_1, \alpha_1) - G(\alpha_2, \alpha_1)}{G(\alpha_1, \alpha_1) G(\alpha_2, \alpha_2) - G(\alpha_1, \alpha_2) G(\alpha_2, \alpha_1)}.
\]
There exists $\bar{p}$ such that for any $p > \bar{p}$ there exists a radial solution $u_p(r)$ having exactly 2 maximum points $\alpha_{1,p}$ and $\alpha_{2,p}$ such that

(i) $u_p(r) \to A_1 G(r, \alpha_1) + A_2 G(r, \alpha_2)$ with

\[
A_1 = \frac{G(\alpha_2, \alpha_2) - G(\alpha_1, \alpha_2)}{G(\alpha_1, \alpha_1)G(\alpha_2, \alpha_2) - G(\alpha_1, \alpha_2)G(\alpha_2, \alpha_1)},
\]

\[
A_2 = \frac{G(\alpha_1, \alpha_1) - G(\alpha_2, \alpha_1)}{G(\alpha_1, \alpha_1)G(\alpha_2, \alpha_2) - G(\alpha_1, \alpha_2)G(\alpha_2, \alpha_1)}.
\]

(ii) $(\alpha_{1,p}, \alpha_{2,p}) \to (\alpha_1, \alpha_2)$ as $p \to \infty$ and $(\alpha_1, \alpha_2)$ is a critical point of the function

\[
\varphi(\alpha_1, \alpha_2) = \inf\{\|u\|_{H^1_{\text{rad}}(B_1)}^2 : u(\alpha_1) = u(\alpha_2) = 1\},
\]

in the set $0 < \alpha_1 < \alpha_2 < 1$. 
The case of two peaks

There exists \( \bar{p} \) such that for any \( p > \bar{p} \) there exists a radial solution \( u_p(r) \) having exactly 2 maximum points \( \alpha_{1,p} \) and \( \alpha_{2,p} \) such that

(i) \( u_p(r) \rightarrow A_1 G(r, \alpha_1) + A_2 G(r, \alpha_2) \) with

\[
A_1 = \frac{G(\alpha_2, \alpha_2) - G(\alpha_1, \alpha_2)}{G(\alpha_1, \alpha_1) G(\alpha_2, \alpha_2) - G(\alpha_1, \alpha_2) G(\alpha_2, \alpha_1)},
\]

\[
A_2 = \frac{G(\alpha_1, \alpha_1) - G(\alpha_2, \alpha_1)}{G(\alpha_1, \alpha_1) G(\alpha_2, \alpha_2) - G(\alpha_1, \alpha_2) G(\alpha_2, \alpha_1)}.
\]

(ii) \((\alpha_{1,p}, \alpha_{2,p}) \rightarrow (\alpha_1, \alpha_2)\) as \( p \rightarrow \infty \) and \((\alpha_1, \alpha_2)\) is a critical point of the function

\[
\varphi(\alpha_1, \alpha_2) = \inf\{\|u\|_{H^{1}_{rad}(B_1)}^2 : u(\alpha_1) = u(\alpha_2) = 1\},
\]

in the set \( 0 < \alpha_1 < \alpha_2 < 1 \).

Moreover we have that \( \varphi(\alpha_1, \alpha_2) = A_1 \alpha_1^{N-1} + A_2 \alpha_2^{N-1} = \)

\[
\frac{(G(\alpha_2, \alpha_2) - G(\alpha_1, \alpha_2)) \alpha_1^{N-1}}{G(\alpha_1, \alpha_1) G(\alpha_2, \alpha_2) - G(\alpha_1, \alpha_2) G(\alpha_2, \alpha_1)} + \frac{(G(\alpha_1, \alpha_1) - G(\alpha_2, \alpha_1)) \alpha_2^{N-1}}{G(\alpha_1, \alpha_1) G(\alpha_2, \alpha_2) - G(\alpha_1, \alpha_2) G(\alpha_2, \alpha_1)}.
\]
Here there is the graph of the solution,
Sketch of the proof

Construction of the 1-peak solution “gluing” an increasing solution in $(0, \alpha)$ and a decreasing solution in $(\alpha, 1)$.

Construction of the 2-peak solution “gluing” the 1-peak solutions in $(0, \alpha)$ and $(\alpha, 1)$ respectively.

The general case follows by a degree argument.
Construction of the 1-peak solution “gluing” an increasing solution in $(0, \alpha)$ and a decreasing solution in $(\alpha, 1)$.
Construction of the 1-peak solution “gluing” an increasing solution in \((0, \alpha)\) and a decreasing solution in \((\alpha, 1)\).

Construction of the 2-peak solution “gluing” the 1-peak solutions in \((0, \alpha)\) and \((\alpha, 1)\) respectively.
Construction of the 1-peak solution “gluing” an increasing solution in \((0, \alpha)\) and a decreasing solution in \((\alpha, 1)\).

Construction of the 2-peak solution “gluing” the 1-peak solutions in \((0, \alpha)\) and \((\alpha, 1)\) respectively.

The general case follows by a degree argument.
Comparison with perturbative methods

Question

Is it possible to get the same result using the Lyapounov-Schmidt reduction? Namely,

\[ u_p(r) = \sum_{i=1}^{k} PU_{\lambda_p,\alpha_p}(r) + \phi_p(r) \]

where \( PU_{\lambda,\alpha} \) is the projection of a function \( U_{\lambda,\alpha} \) solving some limit problem and \( \phi_p \to 0 \) as \( p \to +\infty \)?

The answer (very likely) is YES, but if needs many computations! In our opinion this approach is simpler.