Minimising a relaxed Willmore functional for graphs subject to Dirichlet boundary conditions

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The Willmore functional for graphs

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain and
$\varphi : \overline{\Omega} \to \mathbb{R}$ a smooth boundary datum.
We consider the minimisation of the Willmore functional

$$W(u) := \frac{1}{4} \int_\Omega H^2 \sqrt{1 + |\nabla u|^2}\, dx$$

for graphs $u : \overline{\Omega} \to \mathbb{R}$
with mean curvature

$$H := \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right)$$

(sum of principal curvature, w.r.t. upper unit normal)
subject to Dirichlet boundary conditions, i.e. in the class

$$\mathcal{M} := \{ u \in H^2(\Omega) : (u - \varphi) \in H_0^2(\Omega) \}.$$

Characteristic features: Conformal invariance & lack of comparison principle

- The Willmore functional: a cousin of the area functional?
  At least some of what follows is reminiscent of Giusti's (and other's) BV-approach to minimal graphs.
- However, there are characteristic differences:
- Conformal invariance: If $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ is conformal (i.e. a Möbius transformation), then

$$W(u) = W(\Phi(u)).$$

  In particular: Scaling invariance.
- Willmore functional involves second derivatives: No Stampacchia-tricks! No comparison principles!
- Willmore equation = Euler-Lagrange equation for the Willmore functional.
  In general, no a-priori-estimates known for solutions of the Dirichlet problem for the Willmore equation.

Previous results

A lot is known about closed Willmore surfaces but by far less about Willmore boundary value problems.

- Reiner Schätzle, The Willmore boundary problem, *Calc. Var. Partial Differential Equations*, **37**, 275–302, 2010.
  Parametric/GMT–approach: Branched Willmore minimisers in $\mathbb{S}^3 \cong \mathbb{R}^3 \cup \{\infty\}$!
  In general, little geometric information. These solutions may not be graphs or even contain ∞. No a-priori-bounds.
- Bergner, Dall'Acqua, Deckelnick, Fröhlich, Grunau, Schieweck, . . . :
  Under restrictive symmetry assumptions, suitable minimisers obey strong a-priori-bounds. Precise information of geometry of these minimisers.

Why working in the class $\mathcal{M}$ of graphs?

Recall $\mathcal{M} = \{u \in H^2(\Omega) : (u - \varphi) \in H_0^2(\Omega)\}$.

- Being a graph (= projectable) is an important geometric information.
- Working in $\mathcal{M}$ yields diameter and area bounds in terms of the Willmore functional.
- Reasoning becomes relatively "elementary", GMT can be mostly avoided.
- On the other hand: Working in $\mathcal{M}$ imposes a sort of obstacle condition to the geometric objects.

Area and diameter bounds

Recall $\mathcal{M} = \{u \in H^2(\Omega) : (u - \varphi) \in H_0^2(\Omega)\}$.
Our first key result:

Theorem

There exists a constant $C = C(\Omega, \|\varphi\|_{W^{2,1}(\partial\Omega)})$ such that for any $u \in \mathcal{M}$ we have

$$\sup_{x\in\Omega} |u(x)| + \int_\Omega \sqrt{1 + |\nabla u(x)|^2}\, dx \leq C\big(W(u)^2 + 1\big).$$

Key ingredients:

- Lemma by Leon Simon (1993), which estimates the diameter in terms of the boundary data and the L^1-norm (!) of the length of the second fundamental form.
- Integrate

$$\int_\Omega u\, H\, dx$$

by parts.

In general, no *better* estimates!

- Previous result requires to work with "smooth" graphs.
- In general, no $W^{1,p}$-estimates with $p > 1$ in terms of the Willmore energy.
- Think of a bowler hat, smooth as a surface, but with arbitrarily steep profile curve: $u \notin W^{1,p}(\Omega)$ but $W(u) < \infty$.
- This means that the area and diameter bounds determine $BV \cap L^{\infty}(\Omega)$ as solution class!
- Willmore functional is not defined there.
- Consider there the lower semicontinuous envelope of the Willmore functional.
- What do these functionals have to do with each other?
- How are the boundary conditions then encoded?

A lower semicontinuity result in $\mathcal{M}$

Recall $\mathcal{M} = \{u \in H^2(\Omega) : (u - \varphi) \in H^2_0(\Omega)\}$.

Theorem

Let $(u_k)_{k\in\mathbb{N}} \subset \mathcal{M}$ be a given sequence that satisfies

$$\liminf_{k\to\infty} W(u_k) < \infty.$$

Then there exists a $u \in BV(\Omega) \cap L^\infty(\Omega)$ such that up to selecting a subsequence

$$u_k \to u \text{ in } L^1(\Omega) \quad (k \to \infty).$$

If in addition $u \in W^{1,1}(\Omega)$ then $H = \nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \in L^2(\Omega)$ exists in the weak sense and

$$\frac{1}{4}\int_\Omega H^2 \sqrt{1 + |\nabla u|^2}\, dx \leq \liminf_{k\to\infty} W(u_k).$$

Remarks on this lower semicontinuity result in $\mathcal{M}$

- This is the second key result of our work.
- In particular, the Willmore functional is L^1-lower semicontinuous in $\mathcal{M}$.
- Proof uses the area and diameter bounds.
- Boundedness of the Willmore energy yields compactness in $H^1(\Omega)$ e.g. of $q_k = (1 + |\nabla u_k|^2)^{-5/4}$ and $v_k = q_k \nabla u_k$.
- We could have used a general lower semicontinuity result by Schätzle, but our proof relies only on classical Sobolev space theory. In any case, the mentioned bounds have to be employed.

A lower semicontinuous extension of the Willmore functional

Similar approach already used by Ambrosio, Bellettini, Dal Maso & coworkers.
Recall $\mathcal{M} = \{u \in H^2(\Omega) : (u - \varphi) \in H_0^2(\Omega)\}$.

Definition (Lower semicontinuous envelope)

$$\overline{W} : L^1(\Omega) \to [0, \infty],$$
$$\overline{W}(u) := \inf\{\liminf_{k\to\infty} W(u_k) : \mathcal{M} \ni u_k \to u \text{ in } L^1(\Omega)\}.$$

A more explicit characterisation of $\overline{W}$ is often out of reach.
The previous lower semicontinuity result, however, implies:

Theorem

For $u \in \mathcal{M}$ one has $\overline{W}(u) = W(u)$.

This means: The lower semicontinuous envelope $\overline{W}$ is indeed the largest possible *lower semicontinuous extension* of W to $L^1(\Omega)$.

Existence of a minimiser

Theorem

There exists a function $u \in BV(\Omega) \cap L^\infty(\Omega)$ such that

$$\forall v \in L^1(\Omega): \quad \overline{W}(u) \leq \overline{W}(v).$$

Attainment of the Dirichlet boundary conditions

- $u = \varphi$ in $L^1(\partial\Omega)$ in the sense of traces.
- Lebesgue decomposition $\nabla u = \nabla^s u + \nabla^a u$.
- $|\nabla^s u|(\partial\Omega) = 0$.
- $\nabla^a u$ has an approximately continuous representative which is well defined $\mathcal{H}^1$-almost everywhere on $\partial\Omega$ and coincides with $\nabla\varphi$.

An idea how to show area and diameter bounds

Recall:

Theorem

$\exists C = C(\Omega, \|\varphi\|_{W^{2,1}(\partial\Omega)}) \quad \forall u \in \mathcal{M}$:

$$\sup_{x\in\Omega} |u(x)| + \int_\Omega \sqrt{1+|\nabla u(x)|^2}\, dx \le C\big(W(u)^2+1\big).$$

Proof. Consider only $u \in C^2(\overline{\Omega})$. With $A = \frac{D^2 u}{\sqrt{1+|\nabla u|^2}}$ the second fundamental form of graph(u), we use an estimate by Leon Simon (1993):

$$\begin{aligned}\sup_{x\in\Omega} |u(x)| &\le C\Big(\int_\Omega |A|_g \sqrt{1+|\nabla u|^2}\, dx + 1\Big)\\ &\le C\Big(\int_\Omega |A|_g^2 \sqrt{1+|\nabla u|^2}\, dx\Big)^{1/2}\\ &\quad \cdot \Big(\int_\Omega \sqrt{1+|\nabla u|^2}\, dx\Big)^{1/2} + C.\end{aligned}$$

An idea how to show area and diameter bounds

Using $|A|_g^2 = H^2 - 2K$ and Gauß-Bonnet gives

$$\sup_{x\in\Omega} |u(x)| \le C\big(W(u)+1)\big)^{1/2}\Big(\int_\Omega \sqrt{1+|\nabla u|^2}\,dx\Big)^{1/2} + C. \tag{1}$$

Integration by parts (applies to graphs only!):

$$\begin{aligned}\int_\Omega uH\,dx &= \int_\Omega u\nabla\cdot\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)\,dx \\ &= -\int_\Omega \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}}\,dx + \int_{\partial\Omega} \frac{u\frac{\partial u}{\partial \nu}}{\sqrt{1+|\nabla u|^2}} ds.\end{aligned}$$

Combining this with (1) yields:

$$\begin{aligned}\int_\Omega \sqrt{1+|\nabla u|^2}\,dx &\le C + C\sup_{x\in\Omega}|u(x)|\Big(\int_\Omega H^2\,dx\Big)^{1/2} \\ &\le C + C\,W(u)^{1/2} + C\,W(u)^{1/2} \\ &\quad \cdot\Big(W(u)+1\Big)^{1/2}\Big(\int_\Omega \sqrt{1+|\nabla u|^2}\,dx\Big)^{1/2}.\end{aligned}$$

An idea how to show area and diameter bounds

It follows that

$$\int_\Omega \sqrt{1+|\nabla u|^2}\,dx \le C\big(W(u)^2+1\big).$$

Recall (1):

$$\sup_{x\in\Omega}|u(x)| \le C\big(W(u)+1\big)^{1/2}\Big(\int_\Omega \sqrt{1+|\nabla u|^2}\,dx\Big)^{1/2}+C.$$

Together, this also gives

$$\sup_{x\in\Omega}|u(x)| \le C\big(W(u)^2+1\big).$$

□

An idea how to show L^1-lower semicontinuity

Explain a weaker result:

Theorem

Assume: $u_k, u \in \mathcal{M}$, $u_k \to u$ *in* $L^1(\Omega)$, $\lim_{k\to\infty} W(u_k)$ *exists.*

Then: $W(u) \leq \lim_{k\to\infty} W(u_k)$.

Proof. Observe that we have uniform area and diameter bounds. Together with our assumption this gives

$$u_k \to u \text{ strongly in any } L^p(\Omega), \quad 1 \leq p < \infty.$$

Does boundedness of the Willmore energy yield further compactness?

Indeed,

$$\int_\Omega \frac{|D^2 u_k|^2}{\left(1+|\nabla u_k|^2\right)^{5/2}} \leq \int_\Omega |A_k|_g^2 \sqrt{1+|\nabla u_k|^2}\, dx \leq C.$$

An idea how to show L^1-lower semicontinuity

We consider the bounded mappings

$$q_k := \frac{1}{(1+|\nabla u_k|^2)^{5/4}}, \qquad v_k := q_k \nabla u_k = \frac{\nabla u_k}{(1+|\nabla u_k|^2)^{5/4}}.$$

We have

$$\begin{aligned} \partial_i q_k &= -\frac{5}{2}\sum_{\ell=1}^{2} \frac{\partial_\ell u_k \, \partial_i \partial_\ell u_k}{(1+|\nabla u_k|^2)^{9/4}} = O\Big(\frac{|D^2 u_k|}{(1+|\nabla u_k|^2)^{7/4}}\Big) \\ \partial_i v_k &= (\partial_i q_k)\nabla u_k + q_k \, \partial_i \nabla u_k = O\Big(\frac{|D^2 u_k|}{\left(1+|\nabla u_k|^2\right)^{5/4}}\Big). \end{aligned}$$

So, $(q_k)_{k\in\mathbb{N}}$ and $(v_k)_{k\in\mathbb{N}}$ are bounded in $H^1(\Omega)$.
$\exists q, v \in H^1(\Omega)$ such that, up to a subsequence,

$$q_k \rightharpoonup q, \quad v_k \rightharpoonup v \text{ in } H^1(\Omega),$$

$$q_k \to q, \quad v_k \to v \text{ in any } L^p(\Omega) \text{ and a.e.}$$

An idea how to show L^1-lower semicontinuity

Claim:

$$q > 0 \text{ almost everywhere in } \Omega.$$

Indeed, consider $E := \{x \in \Omega : q(x) = 0\}$. By means of Lebesgue's theorem we find

$$\begin{aligned} |E| &\leq \left(\int_E \sqrt{1+|\nabla u_k|^2}\,dx\right)^{1/2}\left(\int_E \left(1+|\nabla u_k|^2\right)^{-1/2}dx\right)^{1/2} \\ &\leq C\left(\int_E \underbrace{q_k^{2/5}}_{\leq 1, \to q^{2/5} \text{ a.e.}}\,dx\right)^{1/2} \to C\left(\int_E q^{2/5}\,dx\right)^{1/2} = 0. \end{aligned}$$

Claim:

$$v = q\nabla u.$$

An idea how to show L^1-lower semicontinuity

Indeed, we have for all $\eta \in C_0^\infty(\Omega, \mathbb{R}^2)$

$$
\begin{aligned}
\int_\Omega \eta \cdot v \, dx &= \int_\Omega \eta \cdot v_k \, dx + o(1) = \int_\Omega q_k \eta \cdot \nabla u_k \, dx + o(1) \\
&= -\int_\Omega (\operatorname{div} \eta) \underbrace{q_k}_{\to q \text{ in } L^2} \underbrace{u_k}_{\to u \text{ in } L^2} \, dx \\
&\quad - \int_\Omega \underbrace{(\eta \cdot \nabla q_k)}_{O(1) \text{ in } L^2} \cdot \underbrace{(u_k - u)}_{\to 0 \text{ in } L^2} \, dx \\
&\quad - \int_\Omega (\eta u) \cdot \underbrace{\nabla q_k}_{\rightharpoonup \nabla q \text{ in } L^2} \, dx + o(1) \\
&= -\int_\Omega (\operatorname{div} \eta) q u \, dx - \int_\Omega (\eta \cdot \nabla q) \, u \, dx + o(1) \\
&= \int_\Omega (q \nabla u) \cdot \eta \, dx + o(1).
\end{aligned}
$$

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An idea how to show L^1-lower semicontinuity

From $v = q\nabla u$ and $q > 0$ a.e. we conclude that

$$\nabla u_k = \frac{1}{q_k} v_k \quad \to \quad \frac{1}{q} v = \nabla u \qquad \text{a.e.}$$

$$\sqrt{1 + |\nabla u_k|^2} \quad \to \quad \sqrt{1 + |\nabla u|^2} \qquad \text{a.e.}$$

Discuss now $H_k = \operatorname{div}\left(\frac{\nabla u_k}{\sqrt{1+|\nabla u_k|^2}}\right)$, $H = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$.

In view of the bounded Willmore energy

$$\int_\Omega H_k^2 \, dx \le \int_\Omega H_k^2 \sqrt{1 + |\nabla u_k|^2} \, dx \le C.$$

Hence $\exists f, \tilde{H} \in L^2(\Omega)$ such that up to a subsequence

$$H_k \left(1 + |\nabla u_k|^2\right)^{1/4} \rightharpoonup f,$$

$$H_k \rightharpoonup \tilde{H} \text{ in } L^2(\Omega).$$

An idea how to show L^1-lower semicontinuity

Claim: $\tilde{H} = H$ and $f = H\big(1 + |\nabla u|^2\big)^{1/4}$.
Indeed, for any $\zeta \in C_0^\infty(\Omega, \mathbb{R})$

$$\begin{aligned}
\int_\Omega \tilde{H}\zeta\, dx &= \int_\Omega H_k \zeta\, dx + o(1) \\
&= -\int_\Omega \underbrace{\frac{\nabla u_k}{\sqrt{1 + |\nabla u_k|^2}}}_{|\,.\,|\le 1, \quad \to \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \text{ a.e.}} \cdot \nabla\zeta + o(1) \\
&= -\int_\Omega \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \cdot \nabla\zeta\, dx + o(1) \\
&= \int_\Omega H\zeta\, dx + o(1)
\end{aligned}$$

by Lebesgue's theorem. Conclusion: $H = \tilde{H}$ in $L^2(\Omega)$. In particular, $H_k \rightharpoonup H$ in $L^2(\Omega)$.

An idea how to show L^1-lower semicontinuity

Moreover, we have for any $\zeta \in C_0^\infty(\Omega, \mathbb{R})$ that $\left(1 - \frac{(1+|\nabla u|^2)^{1/4}}{(1+|\nabla u_k|^2)^{1/4}}\right)\zeta \to 0$ a.e. and so, by Lebesgue's theorem and $\frac{(1+|\nabla u|^2)^{1/4}}{(1+|\nabla u_k|^2)^{1/4}} \le (1+|\nabla u|^2)^{1/4}$, in $L^2(\Omega)$.

Hence

$$\int_\Omega \zeta \left(H_k(1+|\nabla u_k|^2)^{1/4} - H(1+|\nabla u|^2)^{1/4}\right) dx$$

$$= \int_\Omega \underbrace{H_k(1+|\nabla u_k|^2)^{1/4}}_{O(1) \text{ in } L^2} \underbrace{\left(1 - \frac{(1+|\nabla u|^2)^{1/4}}{(1+|\nabla u_k|^2)^{1/4}}\right)\zeta}_{\to 0 \text{ in } L^2} dx$$

$$+ \int_\Omega \underbrace{\zeta(1+|\nabla u|^2)^{1/4}}_{\in L^2} \underbrace{(H_k - H)}_{\rightharpoonup 0 \text{ in } L^2} dx$$

$$\to \quad 0 \text{ for } k \to \infty.$$

An idea how to show L^1-lower semicontinuity

We conclude that for all $\zeta \in C_0^\infty(\Omega, \mathbb{R})$

$$\int_\Omega \zeta \left(f - H\left(1 + |\nabla u|^2\right)^{1/4} \right) dx$$

$$= \lim_{k\to\infty} \int_\Omega \zeta \left(H_k\left(1 + |\nabla u_k|^2\right)^{1/4} - H\left(1 + |\nabla u|^2\right)^{1/4} \right) dx = 0.$$

This proves that $f = H\left(1 + |\nabla u|^2\right)^{1/4}$ and so finally that

$$H_k\left(1 + |\nabla u_k|^2\right)^{1/4} \rightharpoonup H\left(1 + |\nabla u|^2\right)^{1/4} \text{ in } L^2(\Omega).$$

By weak lower semicontinuity of the L^2-norm

$$\begin{aligned} W(u) &= \frac{1}{4} \int_\Omega H^2 \sqrt{1 + |\nabla u|^2}\, dx \\ &\leq \frac{1}{4} \liminf_{k\to\infty} \int_\Omega H_k^2 \sqrt{1 + |\nabla u_k|^2}\, dx = \liminf_{k\to\infty} W(u_k). \end{aligned}$$

□