Transition fronts for the Fisher-KPP equation

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Time-dependent reaction-diffusion equation

$$u_t = u_{xx} + f(t,u), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}$$

with monostable Fisher-KPP type reaction

$$\begin{cases} f(t,0) = f(t,1) = 0, \quad f(t,u) \geq 0 \text{ in } \mathbb{R} \times [0,1] \\ \dfrac{f(t,u)}{u} \text{ is nonincreasing with respect to } u \in (0,1] \end{cases}$$

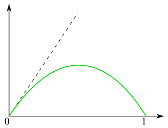


Figure : Function $f(t,\cdot)$

and there are two functions $f_\pm : [0,1] \to \mathbb{R}$ such that

$$\begin{cases} f_\pm(0) = f_\pm(1) = 0, \quad f_\pm(u) > 0 \text{ in } (0,1) \\ \dfrac{f(t,u)}{f_\pm(u)} \underset{t \to \pm\infty}{\longrightarrow} 1 \text{ uniformly for } u \in (0,1) \end{cases}$$

- Typical case:

$$f(t,u) = \widetilde{\mu}(t)\,\widetilde{f}(u)$$

where $\widetilde{f}(0) = \widetilde{f}(1) = 0$, $\widetilde{f} > 0$ on $(0,1)$,

$\widetilde{f}(u)/u$ is nonincreasing with respect to $u \in (0,1]$

and $\widetilde{\mu}(t) \to \widetilde{\mu}_\pm > 0$ as $t \to \pm\infty$.

In this case, $f_\pm(u) = \widetilde{\mu}_\pm \widetilde{f}(u)$.

- Remark: $f(t,u) > 0$ for $(t,u) \in \mathbb{R} \times (0,1)$ with large $|t|$.

  But the case $f(t,\cdot) = 0$ for some t in a compact set is not excluded.

Propagating solutions connecting the unstable steady state 0 and the stable steady state 1 ?

Standard traveling fronts when $f = f(u)$ does not depend on t

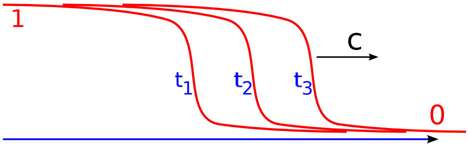
Homogeneous equation

$$u_t = u_{xx} + f(u)$$

Traveling fronts

$$u(t,x) = \varphi_c(x - ct)$$

with $\varphi_c : \mathbb{R} \to (0,1)$, $\varphi_c(-\infty) = 1$, $\varphi_c(+\infty) = 0$



Set of admissible speeds $\{c\} = [c^*, +\infty)$ with $c^* = 2\sqrt{f'(0)}$

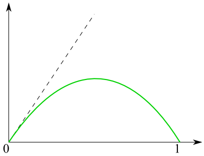
For each speed $c \geq c^*$, φ_c is decreasing and unique up to shifts

Stability of the fronts

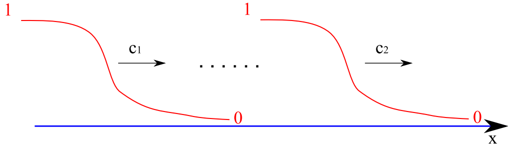
[Aronson, Weinberger] [Bramson] [Fisher] [Hamel, Nolen, Roquejoffre, Ryzhik] [Hamel, Roques] [Kamataka] [Kolmogorov, Petrovski, Piskunov] [Lau] [Sattinger] [Uchiyama]

Other propagating fronts even in the homogeneous case $f = f(u)$

$$u_t = u_{xx} + f(u) \quad \text{with concave function } f$$



(a) Function f



(b) Fronts with changing speed

For any $c_2 > c_1 \geq 2\sqrt{f'(0)}$, there are some solutions $u(t,x)$ such that

$$\left\{\begin{array}{ll} u(t,x) - \varphi_{c_1}(x - c_1 t) \to 0 & \text{as } t \to -\infty \\ u(t,x) - \varphi_{c_2}(x - c_2 t) \to 0 & \text{as } t \to +\infty \end{array}\right. \quad \text{uniformly in } x \in \mathbb{R}$$

[Hamel, Nadirashvili]

More general front-like solutions (for $u_t = u_{xx} + f(t,u)$)

Definition [Berestycki, Hamel] (adapted to our equation)

A *transition front* connecting 1 and 0 is a solution $u : \mathbb{R}\times\mathbb{R} \to [0,1]$ for which there exists a family $(x_t)_{t\in\mathbb{R}}$ of real numbers such that

$$\begin{cases} u(t, x + x_t) \to 1 & \text{as } x \to -\infty \\ u(t, x + x_t) \to 0 & \text{as } x \to +\infty \end{cases} \quad \text{uniformly in } t \in \mathbb{R}$$

(the transition between 0 and 1 has a uniformly bounded width)

For a given transition front u, the real numbers x_t are defined up to a bounded function and they are at a finite distance from any given level set: for every $0 < \alpha \le \beta < 1$, there is $C = C(u, \alpha, \beta)$ such that

$$\forall\, t \in \mathbb{R}, \quad \{x \in \mathbb{R};\, \alpha \le u(t,x) \le \beta\} \subset [x_t - C, x_t + C]$$

Example: $u(t, x_t) = 1/2$

Notion of global mean speed (for $u_t = u_{xx} + f(t,u)$)

Definition [Berestycki, Hamel] (adapted to our equation)

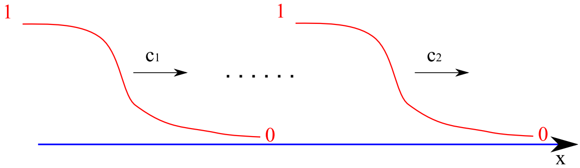
A transition front u connecting 1 and 0 has a global mean speed γ if

$$\frac{x_t - x_s}{t - s} \to \gamma \text{ as } t - s \to +\infty$$

If a given transition front u has a global mean speed γ, then γ is finite and does not depend on the precise choice of $(x_t)_{t\in\mathbb{R}}$

When $f = f(u)$, any standard traveling front $u(t,x) = \varphi_c(x - ct)$ is a transition front with global mean speed c

Further notions of speeds



Transition fronts with $x_t = c_1 t$ for $t \leq 0$ and $x_t = c_2 t$ for $t \geq 0$

They are not standard traveling fronts

No global mean speed if $c_2 > c_1$

Need of further notions of asymptotic speeds to describe the solutions, even in the homogeneous case $f = f(u)$!

Definition of asymptotic speeds (for $u_t = u_{xx} + f(t,u)$)

We say that a transition front connecting 1 and 0 has an *asymptotic past speed* c_-, respectively an *asymptotic future speed* c_+, if

$$\frac{x_t}{t} \to c_- \text{ as } t \to -\infty, \quad \text{respectively } \frac{x_t}{t} \to c_+ \text{ as } t \to +\infty$$

The asymptotic speeds, if any, are finite and do not depend on the precise choice of $(x_t)_{t\in\mathbb{R}}$

If a transition front has a global mean speed γ, then it has asymptotic past and future speeds $c_\pm = \gamma$

Questions:

- Set of transition fronts connecting 1 and 0 ?
- Conditions for a solution $0 < u(t, x) < 1$ to be a transition front ?
- Existence of asymptotic past and future speeds ?
- Set of admissible past and future speeds ?
- Asymptotic profiles as $t \to \pm\infty$?

Other assumptions: T-periodic reaction $f(t,u)$, pulsating fronts

$$\begin{cases} u(t,x)=\varphi(t,x-ct)\\ \varphi(t,\xi) \text{ is } T\text{-periodic in } t\\ \varphi(t,-\infty)=1,\ \varphi(t,+\infty)=0 \end{cases}$$

The profile $t\mapsto u(t,ct+\cdot)$ is T-periodic in time

Existence of a continuum of speeds $[c^*,+\infty)$ [Fréjacques] [Hamel, Roques] [Liang, Zhao] [Nadin] [Nolen, Rudd, Xin] [Weinberger]

Time almost-periodic, uniquely ergodic media [Huang, Shen] [Nadin, Rossi] [Shen]

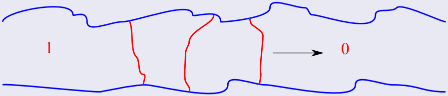
Bistable reaction $f(t,u)$

[Alikakos, Bates, Chen] [Contri] [Fang, Zhao] [Shen]

Spatial dependence $u_t=u_{xx}+f(x,u)$: pulsating and transition fronts

Berestycki, Ding, Ducrot, El Smaily, Fang, Giletti, Hamel, Heinze, Liang, Matano, Mellet, Nadin, Nadirashvili, Nolen, Roquejoffre, Roques, Ryzhik, Sire, Weinberger, Xin, Zhao, Zlatoš

Another definition of generalised fronts, by H. Matano



Example: $u_t = u_{xx} + b(x)\, f(u)$

Define $\sigma_\xi b(\cdot) = b(\cdot + \xi)$ and assume that $\mathcal{H} = \overline{\{\sigma_\xi b\}}$ is compact in $L^\infty(\mathbb{R})$

A definition by H. Matano: u is a generalised front if there exists a continuous mapping $w : \mathcal{H} \times \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} u(t, x + \xi(t)) = w(\sigma_{\xi(t)} b, x) \\ w(z,s) \to 1 \text{ as } s \to -\infty \text{ uniformly w.r.t. } z \in \mathcal{H} \\ w(z,s) \to 0 \text{ as } s \to +\infty \text{ uniformly w.r.t. } z \in \mathcal{H} \end{cases}$$

For homogeneous equations, the profile of the solution is invariant in time

In the general case, a generalized front is a transition front connecting 1 and 0

Definition of wave-like solutions in random media, by W. Shen

I. Introduction: transition fronts and asymptotic speeds

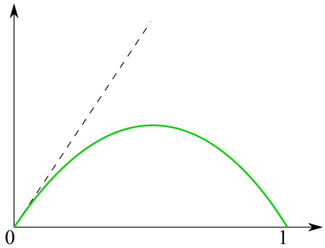
II. Homogeneous Fisher-KPP equation

III. Heterogeneous Fisher-KPP type equation

II. Homogeneous Fisher-KPP equation

$$u_t = u_{xx} + f(u)$$

with concave function f



Standard traveling fronts $\varphi_c(x - ct)$ for $c \geq c^* = 2\sqrt{f'(0)}$

Global mean speeds

Theorem

The set of admissible global mean speeds of transition fronts connecting 1 and 0 is equal to the interval $[c^*, +\infty)$.

Furthermore, if a transition front u has a global mean speed $\gamma > c^*$, then it is a standard traveling front of the type $u(t,x) = \varphi_\gamma(x - \gamma t)$.

First part follows from [Aronson, Weinberger]

Second part follows from [Hamel, Nadirashvili] and further qualitative results (see later...)

Asymptotic past and future speeds

Theorem

Transition fronts connecting 1 and 0 and having asymptotic past and future speeds $c_\pm$ exist if and only if

$$c^* \le c_- \le c_+ < +\infty$$

The sufficiency condition follows from the construction of [Hamel, Nadirashvili]

The necessity condition means that transition fronts always accelerate

Asymptotic profiles

The necessity condition of the previous theorem is a consequence of a more general result:

Theorem

For any transition front connecting 1 and 0, there holds

$$c^* \le \liminf_{t\to-\infty} \frac{x_t}{t} \le \liminf_{t\to+\infty} \frac{x_t}{t} \le \limsup_{t\to+\infty} \frac{x_t}{t} < +\infty$$

Furthermore, if $c^* < \liminf_{t\to-\infty} x_t/t$, then the front has asymptotic past and future speeds $c_\pm$, with $c^* < c_- \le c_+$, and

$$u(t, x_t + \cdot) \to \varphi_{c_\pm} \ \text{ in } C^2(\mathbb{R}) \ \text{ as } t \to \pm\infty$$

up to a bounded shift of $(x_t)_{t\in\mathbb{R}}$

Conjecture

Same conclusion without the condition $c^* < \liminf_{t\to-\infty} x_t/t$

Sufficient condition for a solution $0 < u < 1$ to be a transition front

For any solution $0 < u(t,x) < 1$, one has $\max_{[-c|t|,c|t|]} u(t,\cdot) \to 0$ as $t \to -\infty$ for every $0 \le c < c^*$ [Aronson, Weinberger]

Theorem

Let $0 < u(t,x) < 1$ be a solution such that

$$\exists\, c > c^*, \quad \max_{[-c|t|,c|t|]} u(t,\cdot) \to 0 \;\text{ as } t \to -\infty.$$

Then the limit

$$\lambda = -\lim_{x\to+\infty} \frac{\ln u(t,x)}{x}$$

exists independently of $t \in \mathbb{R}$ and satisfies $\lambda \in \big[0, \sqrt{f'(0)}\big)$.

Furthermore, u is a transition front connecting 1 and 0 if and only if $\lambda > 0$.

Lastly, if $\lambda > 0$, then u has asymptotic speeds $c_\pm$ given by

$$c^* < c_- = \sup\Big\{\gamma \ge 0,\ \lim_{t\to-\infty} \max_{[-\gamma|t|,\gamma|t|]} u(t,\cdot) = 0\Big\} \le c_+ = \lambda + \frac{f'(0)}{\lambda}$$

and it has asymptotic profiles $\varphi_{c_\pm}$.

Transition fronts as superposition of standard traveling fronts

Standard traveling fronts $\varphi_c(x-ct)$ and $\varphi_c(-x-ct)$ for $c\geq c^*=2\sqrt{f'(0)}$

$$\begin{cases} \varphi_c(\xi) \sim e^{-\lambda_c \xi} & \text{if } c > c^* \\ \varphi_{c^*}(\xi) \sim \xi\, e^{-\lambda_{c^*} \xi} & \end{cases} \quad \text{as } \xi \to +\infty \quad \text{(up to shift in } \xi)$$

with $\lambda_c = (c - \sqrt{c^2 - 4f'(0)})/2$ $(\lambda_{c^*} = c^*/2 = \sqrt{f'(0)})$

Spatially-uniform solution $\theta'(t)=f(\theta(t))$ s.t. $\theta(t)\sim e^{f'(0)t}$ as $t\to-\infty$

Set $\mathcal{X} = (-\infty, -c^*] \cup [c^*, +\infty) \cup \{\infty\}$

Set $\mathcal{M}$ of nonnegative Borel measures μ on $\mathcal{X}$ s.t. $0<\mu(\mathcal{X})<+\infty$

One-to-one map $\mu \mapsto u_\mu$ from $\mathcal{M}$ to the set of solutions $0 < u_\mu(t,x) < 1$ [Hamel, Nadirashvili]

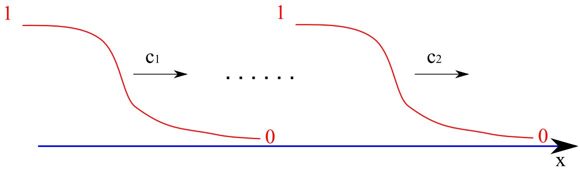
Furthermore, for a given $\mu \in \mathcal{M}$, calling $M = \mu(\mathcal{X}\backslash\{-c^*, c^*\})$,

$$\max\Big(\varphi_{c^*}(x - c^* t - c^* \ln \mu(c^*)),\ \varphi_{c^*}(-x - c^* t - c^* \ln \mu(-c^*)),$$
$$M^{-1}\int_{\mathbb{R}\backslash[-c^*,c^*]} \varphi_{|c|}\big((\mathrm{sgn}\, c)x - |c|t - |c| \ln M\big)\, d\mu(c)$$
$$+ M^{-1}\theta(t + \ln M)\,\mu(\infty)\Big)$$
$$\le\ u_\mu(t,x)\ \le\ \varphi_{c^*}(x - c^* t - c^* \ln \mu(c^*)) + \varphi_{c^*}(-x - c^* t - c^* \ln \mu(-c^*))$$
$$+ M^{-1}\int_{\mathbb{R}\backslash[-c^*,c^*]} e^{-\lambda_{|c|}((\mathrm{sgn}\, c)x - |c|t - |c| \ln M)}\, d\mu(c)$$
$$+ M^{-1} e^{f'(0)(t + \ln M)}\mu(\infty)$$

For any solution $0 < u(t,x) < 1$, one has $\max_{[-c|t|, c|t|]} u(t,\cdot) \to 0$ as $t \to -\infty$ for every $0 \le c < c^*$ [Aronson, Weinberger]

If $\max_{[-c|t|, c|t|]} u(t,\cdot) \to 0$ as $t \to -\infty$ for some $c > c^*$, then $u = u_\mu$ [Hamel, Nadirashvili]

Example



Measure $\mu = m_1\delta_{c_1} + m_2\delta_{c_2}$ with $c^* \le c_1 < c_2$

$$\left\{\begin{array}{ll} u(t,x) - \varphi_{c_1}(x - c_1 t) \to 0 & \text{as } t \to -\infty \\ u(t,x) - \varphi_{c_2}(x - c_2 t) \to 0 & \text{as } t \to +\infty \end{array}\right. \quad \text{uniformly in } x \in \mathbb{R}$$

Theorem

Let u_μ be the solution associated with a measure $\mu \in \mathcal{M}$.

Then u_μ is a transition front connecting 1 and 0 if and only if the support of μ is a compact subset of $[c^*, +\infty)$.

In such a case, u_μ is decreasing with respect to x.

- Right-moving fronts $\varphi_c(x - ct)$ are decreasing in x and connect 1 at $-\infty$ to 0 at $+\infty$
- Left-moving fronts $\varphi_c(-x - ct)$ are increasing in x and connect 0 at $-\infty$ to 1 at $+\infty$
- Faster fronts are flatter, so, for the transition zone between 1 and 0 to be uniformly bounded, it is reasonable to expect that the fronts $\varphi_c(x - ct)$ involved in u_μ have bounded speeds
- Sufficiency condition when $\text{supp}(\mu)$ is a compact subset of $(c^*, +\infty)$, with other arguments, by Zlatoš.

Theorem

Let $\mu \in \mathcal{M}$ be a measure such that

$$c^* \le c_- := \min\big(\mathrm{supp}(\mu)\big) \le \max\big(\mathrm{supp}(\mu)\big) =: c_+ < +\infty.$$

- The transition front u_μ has an asymptotic past speed equal to c_- and an asymptotic future speed equal to c_+

- The positions $(x_t)_{t\in\mathbb{R}}$ satisfy

$$\begin{cases} \limsup\limits_{t\to\pm\infty} |x_t - c_\pm t| < +\infty & \text{if } \mu(c_\pm) > 0 \\ \lim\limits_{t\to\pm\infty} (x_t - c_\pm t) = -\infty & \text{if } \mu(c_\pm) = 0 \end{cases}$$

- If $c_- > c^*$, then

$$u_\mu(t, x_t + \cdot) \to \varphi_{c_\pm} \ \text{ in } C^2(\mathbb{R}) \ \text{ as } t \to \pm\infty$$

up to a bounded shift of $(x_t)_{t\in\mathbb{R}}$

- If $c_- = c^*$ and $\mu(c^*) > 0$, then

$$u_\mu(t, c^* t + c^* \ln \mu(c^*) + \cdot) \to \varphi_{c^*} \ \text{ in } C^2(\mathbb{R}) \ \text{ as } t \to -\infty$$

Corollary

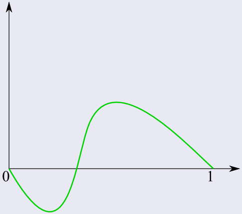
There are solutions $0 < u(t,x) < 1$ such that

$$\forall\, t \in \mathbb{R}, \quad u(t,-\infty) = 1, \quad u(t,+\infty) = 0$$

and which are not transition fronts connecting 1 and 0.

Proof: $u = u_\mu$ with $\mathrm{supp}(\mu) \subset [c^*, +\infty)$ and $\mathrm{supp}(\mu)$ unbounded.

Homogeneous bistable equation $u_t = u_{xx} + f(u)$



Unique standard front $\varphi(x - ct)$ [Aronson, Weinberger] [Fife, McLeod]

Any transition front connecting 1 and 0 is equal to this front, up to shifts [Hamel]

But there are solutions $0 < u(t,x) < 1$ such that

$$\forall\, t \in \mathbb{R}, \quad u(t,-\infty) = 1, \quad u(t,+\infty) = 0$$

and which *are not* transition fronts. These solutions are close to the unstable zero θ on large space intervals as $t \to -\infty$ [Morita, Ninomiya]

Some ingredients for the proof of the main theorems

Proposition

Let $\mu \in \mathcal{M}$ be supported in $[c^*, \gamma]$ for some $\gamma \in [c^*, +\infty)$.

Then, for every $(t, x) \in \mathbb{R}^2$,

$$\begin{cases} u_\mu(t, x + y) \geq \varphi_\gamma\big(\varphi_\gamma^{-1}(u_\mu(t, x)) + y\big) & \text{for all } y \leq 0, \\ u_\mu(t, x + y) \leq \varphi_\gamma\big(\varphi_\gamma^{-1}(u_\mu(t, x)) + y\big) & \text{for all } y \geq 0, \end{cases}$$

where $\varphi_\gamma^{-1} : (0, 1) \to \mathbb{R}$ denotes the reciprocal of the function φ_γ.

In other words, $u_\mu(t, \cdot)$ is steeper than φ_γ.

Consequence: $(u_\mu)_x(t, x) \leq \varphi'_\gamma\big(\varphi_\gamma^{-1}(u_\mu(t, x))\big) < 0$.

Proposition

Let $\mu \in \mathcal{M}$ be supported in $[\gamma, +\infty)$ for some $\gamma \in [c^*, +\infty)$.

Then, for every $(t, x) \in \mathbb{R}^2$,

$$\begin{cases} u_\mu(t, x + y) \le \varphi_\gamma\big(\varphi_\gamma^{-1}(u_\mu(t, x)) + y\big) & \text{for all } y \le 0 \\ u_\mu(t, x + y) \ge \varphi_\gamma\big(\varphi_\gamma^{-1}(u_\mu(t, x)) + y\big) & \text{for all } y \ge 0 \end{cases}$$

In other words, $u_\mu(t, \cdot)$ is less steep than φ_γ.

I. Introduction: transition fronts and asymptotic speeds

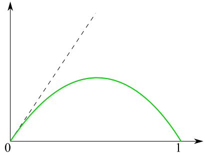
II. Homogeneous Fisher-KPP equation

III. Heterogeneous Fisher-KPP type equation

III. Heterogeneous Fisher-KPP type equation

$$u_t = u_{xx} + f(t,u), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}$$

$$\begin{cases} f(t,0) = f(t,1) = 0, \quad f(t,u) \geq 0 \text{ in } \mathbb{R} \times [0,1] \\ \dfrac{f(t,u)}{u} \text{ is nonincreasing with respect to } u \in (0,1] \end{cases}$$



and there are two functions $f_\pm : [0,1] \to \mathbb{R}$ such that

$$\begin{cases} f_\pm(0) = f_\pm(1) = 0, \quad f_\pm(u) > 0 \text{ in } (0,1) \\ \dfrac{f(t,u)}{f_\pm(u)} \underset{t \to \pm\infty}{\longrightarrow} 1 \text{ uniformly for } u \in (0,1) \end{cases}$$

Notation: $\mu_\pm := f_\pm'(0) > 0, \quad \underline{\mu} := \min(\mu_-, \mu_+) > 0$

Theorem (existence)

Let $c_\pm$ be any two real numbers such that

$$c_- \geq 2\sqrt{\mu_-} \quad \text{and} \quad c_+ \geq \kappa + \frac{\mu_+}{\kappa}$$

with

$$\kappa = \min\left(\sqrt{\mu_+}, \frac{c_- - \sqrt{c_-^2 - 4\mu_-}}{2}\right) > 0$$

Then there are some transition fronts u connecting 1 and 0 with asymptotic past and future speeds $c_\pm$.

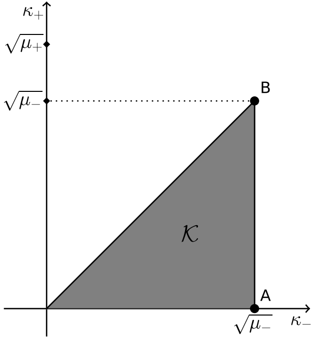
Furthermore, u satisfies $u_x(t,x) < 0$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}$.

Lastly, in all cases, except possibly when $\mu_+ > \mu_-$ and $c_\pm$ satisfy $c_- = 2\sqrt{\mu_-}$ and $c_+ = \sqrt{\mu_-} + \mu_+/\sqrt{\mu_-}$, then

$$u(t, x_t + \cdot) \to \varphi^\pm_{c_\pm} \ \text{ in } C^2(\mathbb{R}) \ \text{ as } t \to \pm\infty$$

up to a bounded shift of $(x_t)_{t\in\mathbb{R}}$, where $\varphi^\pm_{c_\pm}(x - c_\pm t)$ are standard traveling fronts connecting 1 and 0 for the limiting equations with nonlinearities $f_\pm$.

- $c_\pm \geq 2\sqrt{\mu_\pm}$
- Range of admissible past speeds $= [2\sqrt{\mu_-}, +\infty)$
- Range of admissible future speeds $= [2\sqrt{\mu_+}, +\infty)$ if $\mu_+ \leq \mu_-$
- Range of admissible future speeds $= \left[\sqrt{\mu_-} + \frac{\mu_+}{\sqrt{\mu_-}}, +\infty\right)$ if $\mu_+ > \mu_-$
- Equivalent formulation: $c_\pm = \kappa_\pm + \frac{\mu_\pm}{\kappa_\pm}$ with $\kappa_- \in (0, \sqrt{\mu_-}]$ and $\kappa_+ \in (0, \min(\kappa_-, \sqrt{\mu_+})]$



(c) Case $\mu_+ > \mu_-$

(d) Case $\mu_+ \leq \mu_-$

- If $\mu_+ > \mu_-$, then $c_+ > c_-$ (acceleration)
- If $\mu_+ \geq \mu_-$, then $c_+ \geq c_-$
- If $\mu_+ < \mu_-$, then c_+ may be less than c_- (slow down)
- [Berestycki, Hamel] and [Nadin, Rossi]

  Other assumptions on $f(t,u)$

  Proof of the existence of fronts which would correspond to the case $\kappa_+ = \kappa_- \in (0, \sqrt{\underline{\mu}})$

Theorem (optimality of the asymptotic speeds)

Assume that f_- is concave and there is $\zeta \in L^1(-\infty,0)$ such that

$$\sup_{s\in(0,1)}\Big|\frac{f(t,s)}{f_-(s)}-1\Big|\le\zeta(t)\quad\text{for all } t<0.$$

Let u be any transition front connecting 1 and 0.

Then

$$\begin{cases} 2\sqrt{\mu_-}\le c_- := \liminf\limits_{t\to-\infty}\dfrac{x_t}{t}\le\limsup\limits_{t\to-\infty}\dfrac{x_t}{t}<+\infty\\ \kappa+\dfrac{\mu_+}{\kappa}\le c_+ := \liminf\limits_{t\to+\infty}\dfrac{x_t}{t}\le\limsup\limits_{t\to+\infty}\dfrac{x_t}{t}<+\infty.\end{cases}$$

Furthermore, if $c_->2\sqrt{\mu_-}$, then u has asymptotic past and future speeds $c_\pm$.

Lastly, if $c_->2\sqrt{\mu_-}$ and there is $\widetilde{\zeta}\in L^1(0,+\infty)$ such that

$$\sup_{s\in(0,1)}\Big|\frac{f(t,s)}{f_+(s)}-1\Big|\le\widetilde{\zeta}(t)\quad\text{for all } t>0,$$

then convergence to the limiting profiles $\varphi^\pm_{c_\pm}$.

Example: $f(t,u) = \widetilde{\mu}(t)\,\widetilde{f}(u)$ with $\widetilde{\mu} - \widetilde{\mu}(\pm\infty) \in L^1(\mathbb{R}_\pm)$

Corollary

Same assumptions as in the previous theorem.

Transition fronts connecting 1 and 0 and having asymptotic past and future speeds $c_\pm$ exist if and only if c_- and c_+ satisfy

$$c_- \geq 2\sqrt{\mu_-} \quad \text{and} \quad c_+ \geq \kappa + \frac{\mu_+}{\kappa}$$

with

$$\kappa = \min\left(\sqrt{\mu_+}, \frac{c_- - \sqrt{c_-^2 - 4\mu_-}}{2}\right) > 0.$$

Transition fronts connecting 1 and 0 and having a global mean speed γ exist if and only if $\mu_+ \leq \mu_-$ and $\gamma \geq 2\sqrt{\mu_-}$.

A sufficient condition for an entire solution to be a transition front

Theorem

Same assumptions as in the previous theorem.

Let $0 < u(t,x) < 1$ be an entire solution such that

$$\exists\, c > 2\sqrt{\mu_-}, \quad \max_{[-c|t|,c|t|]} u(t,\cdot) \to 0 \text{ as } t \to -\infty.$$

Then the following limit exists independently of $t \in \mathbb{R}$:

$$\lambda = -\lim_{x\to+\infty} \frac{\ln u(t,x)}{x} \in [0, \sqrt{\mu_-}).$$

Furthermore, u is a transition front connecting 1 and 0 if and only if $\lambda > 0$.

Lastly, if $\lambda > 0$, then u has asymptotic speeds $c_\pm$ given by

$$\begin{cases} 2\sqrt{\mu_-} < c_- = \sup\Big\{\gamma \geq 0, \lim_{t\to-\infty} \max_{[-\gamma|t|,\gamma|t|]} u(t,\cdot) = 0\Big\} \\ c_+ = \min(\lambda, \sqrt{\mu_+}) + \dfrac{\mu_+}{\min(\lambda, \sqrt{\mu_+})} \end{cases}$$

and it has asymptotic profiles $\varphi^{\pm}_{c_\pm}$.

Time-dependent diffusivity

$$u_t = \sigma(t) u_{xx} + f(t,u)$$

with $0 < a \leq \sigma(t) \leq b < +\infty$, $\sigma(t) \to \sigma_\pm$ as $t \to \pm\infty$ and $\sigma - \sigma_- \in L^1(\mathbb{R}_-)$

Corollary

Transition fronts connecting 1 and 0 having asymptotic past and future speeds $c_\pm$ exist if and only

$$c_- \geq 2\sqrt{\sigma_-\mu_-} \quad \text{and} \quad c_+ \geq \kappa + \frac{\sigma_+\mu_+}{\kappa}$$

where

$$\kappa = \min\left(\sqrt{\sigma_+\mu_+}, \frac{\sigma_+}{\sigma_-} \times \frac{c_- - \sqrt{c_-^2 - 4\sigma_-\mu_-}}{2}\right)$$