A critical exponent for Hénon type equation on the hyperbolic space

Shoichi Hasegawa
(Mathematical Institute, Tohoku University)

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1. Introduction

(E) $$-\Delta u = |x|^{\alpha} |u|^{p-1} u \quad \text{in} \quad \mathbb{R}^N,$$

where $N \geq 3$, $p > 1$, and $\alpha > -2$.

● Liouville theorem for the equation (E) was proved by

(1) Farina for the case $\alpha = 0$ in 2007.

(2) Dancer, Du, and Guo for the case $\alpha > -2$ in 2011.

Theorem A (Farina 2007, Dancer, Du, and Guo 2011)

Let $u \in C^2(\mathbb{R}^N)$ be a stable solution of (E). Then there exists some exponent $p(N, \alpha)$ such that if p satisfies

$$\begin{cases} 1 < p < +\infty & \text{if} \quad N \leq 10 + 4\alpha, \\ 1 < p < p(N, \alpha) & \text{if} \quad N > 10 + 4\alpha, \end{cases}$$

then $u \equiv 0$ in $\mathbb{R}^N$. Moreover, if $p \geq p(N, \alpha)$, then (E) has stable, positive, and radial solutions.

Let $\mathbb{H}^N$ be the N-dimensional hyperbolic space with the following metric:

$$ds^2 = dr^2 + (\sinh r)^2 \, d\Theta^2, r > 0, \Theta \in \mathbb{S}^{N-1}.$$

$$\text{(L)} \qquad -\Delta_g u = |u|^{p-1} u \quad \text{in} \quad \mathbb{H}^N,$$

where $N \geq 3$ and $p > 1$.

For $\beta > 0$, let $u_\beta = u_\beta(r)$ be the solution of

$$\begin{cases} u''(r) + \frac{N-1}{\tanh r} u'(r) + |u|^{p-1} u = 0, \\ u(0) = \beta, \quad u'(0) = 0. \end{cases}$$

Theorem B (Berchio, Ferrero, and Grillo, 2014)

Let $p > 1$. Then, there exists $\beta_0 = \beta_0(N, p)$ such that if $\beta \leq \beta_0$, then u_β is a stable solution of (L).

Theorem A (Farina 2007, Dancer, Du, and Guo 2011)

Let $u \in C^2(\mathbb{R}^N)$ be a stable solution of (E). Then there exists some exponent $p(N, \alpha)$ such that if p satisfies

$$\begin{cases} 1 < p < +\infty & \text{if} \quad N \leq 10 + 4\alpha, \\ 1 < p < p(N, \alpha) & \text{if} \quad N > 10 + 4\alpha, \end{cases}$$

then $u \equiv 0$ in $\mathbb{R}^N$. Moreover, if $p \geq p(N, \alpha)$, then (E) has stable, positive, and radial solutions.

Problem

Is there a critical exponent for (L) with some weight?

2. Liouville Theorem

(E) $$-\Delta u = |x|^{\alpha} |u|^{p-1} u \quad \text{in} \quad \mathbb{R}^N.$$

(L) $$-\Delta_g u = |u|^{p-1} u \quad \text{in} \quad \mathbb{H}^N.$$

(Example) $$-\Delta_g u = r^{\alpha} |u|^{p-1} u \quad \text{in} \quad \mathbb{H}^N.$$

(H) $$-\Delta_g u = w^{\alpha} |u|^{p-1} u \quad \text{in} \quad \mathbb{H}^N,$$

where $N \geq 3$, $p > 1$, $\alpha > 0$, and the weight $w = w(r, \theta)$ is non-negative and satisfies the following asymptotic behavior:

$$\limsup_{r \to +\infty} \frac{\sinh r}{w(r, \theta)} < +\infty.$$

Definition

A solution $u \in C^2(\mathbb{H}^N)$ of (H) is stable if the inequality

$$Q[u](v) := \int_{\mathbb{H}^N} \left\{ |\nabla_g v|_g^2 - p w^\alpha |u|^{p-1} v^2 \right\} dV_g \geq 0$$

holds for any $v \in C_c^1(\mathbb{H}^N)$.

Here, the above inequality means that the second variation of the following functional is nonnegative:

$$E(u) := \int_{\mathbb{H}^N} \left\{ \frac{1}{2} |\nabla_g u|_g^2 - w^\alpha \frac{|u|^{p+1}}{p+1} \right\} dV_g .$$

Theorem 1

Let $u \in C^2(\mathbb{H}^N)$ be a stable solution of (H). If p satisfies

$$\begin{cases} 1 < p < +\infty & \text{if} \quad N \leq 1 + 4\alpha, \\ 1 < p < p_c(N, \alpha) & \text{if} \quad N > 1 + 4\alpha, \end{cases}$$

then $u \equiv 0$ in $\mathbb{H}^N$. Here, $p_c(N, \alpha)$ is expressed as follows:

$$p_c(N, \alpha) := \frac{(N-1)^2 - 2\alpha(N-1) - 2\alpha^2 + 2\alpha\sqrt{2\alpha(N-1) + \alpha^2}}{(N-1)(N-4\alpha-1)}.$$

- When $p > p_c$, $\exists$non-trivial stable solution $\Rightarrow$ p_c is critical.

3. Stable solutions

In the following, we set $N \geq 3$ and $w(r,\theta) = \sinh r$.
For $\beta > 0$, let $u_\beta = u_\beta(r)$ be the solution of

$$\begin{cases} u''(r) + \frac{N-1}{\tanh r} u'(r) + (\sinh r)^\alpha |u|^{p-1} u = 0, \\ u(0) = \beta, \quad u'(0) = 0. \end{cases}$$

Define $p_s(N,\alpha) := (N + 2 + 2\alpha) / (N - 2)$.

Theorem 2

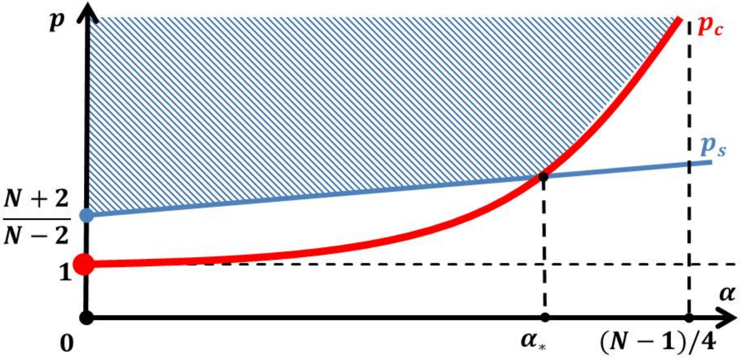
Let $N > 1 + 4\alpha$ and

$$p > \max\{p_s(N,\alpha), p_c(N,\alpha)\}.$$

Then, there exists $\beta_0 = \beta_0(N,p,\alpha) > 0$ such that if $\beta \in (0, \beta_0]$, then u_β is a positive stable solution of (H).

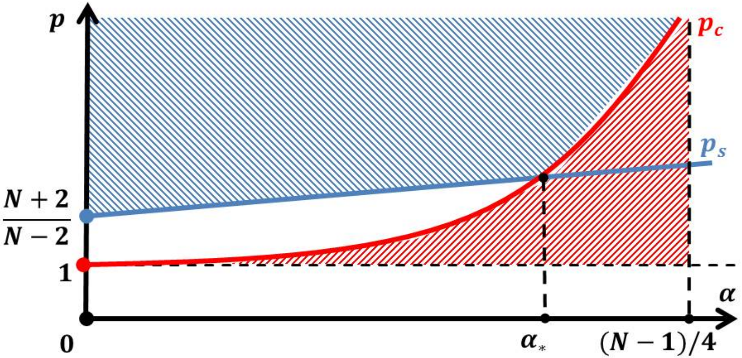
Theorem 3

Let $N > 1 + 4\alpha$ and $p > \max\{p_s, p_c\}$. Then, for any $0 < \beta_1 < \beta_2 \leq \beta_0$, two solutions u_{β_1} and u_{β_2} cannot intersect each other, i.e., $u_{\beta_1}(r) < u_{\beta_2}(r)$ for $r \geq 0$.



4. Criticality of p_c

- If $p < p_c$, there is no non-trivial stable solutions (Th.1).
- If $p > \max\{p_s, p_c\}$, $\exists$non-trivial stable solutions (Th.2).
- Let $\alpha \in \left(\alpha_*, \frac{N-1}{4}\right)$. Then p_c is the critical exponent for (H).

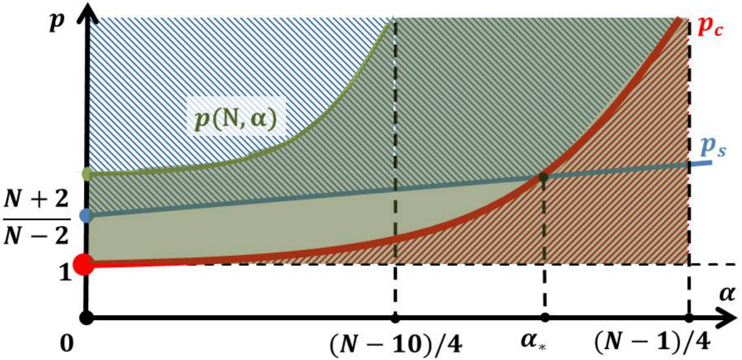


Theorem 4

Let $1 < p < p(N, \alpha)$. Then, there exists $\bar{\beta} = \bar{\beta}(N, p, \alpha) > 0$ such that u_β is unstable for any $\beta > \bar{\beta}$.

● $p(N, \alpha)$ is the critical exponent on stable solutions of

(E) $$-\Delta u = |x|^\alpha |u|^{p-1} u \quad \text{in} \quad \mathbb{R}^N.$$



Theorem 5

Let $p > 1$. Any stable radial solution to (H) has constant sign.