A critical exponent for Hénon type equation on the hyperbolic space

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1. Introduction

\( - \Delta u = |x|^\alpha |u|^{p-1} u \quad \text{in} \quad \mathbb{R}^N, \)

where \( N \geq 3, \ p > 1, \) and \( \alpha > -2. \)

- Liouville theorem for the equation (E) was proved by
  (1) Farina for the case \( \alpha = 0 \) in 2007.
  (2) Dancer, Du, and Guo for the case \( \alpha > -2 \) in 2011.
Theorem A (Farina 2007, Dancer, Du, and Guo 2011)

Let $u \in C^2(\mathbb{R}^N)$ be a stable solution of (E). Then there exists some exponent $p(N, \alpha)$ such that if $p$ satisfies

\[
\begin{cases}
1 < p < +\infty & \text{if } N \leq 10 + 4\alpha, \\
1 < p < p(N, \alpha) & \text{if } N > 10 + 4\alpha,
\end{cases}
\]

then $u \equiv 0$ in $\mathbb{R}^N$. Moreover, if $p \geq p(N, \alpha)$, then (E) has stable, positive, and radial solutions.
Let $\mathbb{H}^N$ be the $N$-dimensional hyperbolic space with the following metric:

$$ds^2 = dr^2 + (\sinh r)^2 d\Theta^2, \quad r > 0, \quad \Theta \in \mathbb{S}^{N-1}.$$ 

\[ \triangle u = |u|^{p-1} u \quad \text{in} \quad \mathbb{H}^N, \]

where $N \geq 3$ and $p > 1$.

For $\beta > 0$, let $u_\beta = u_\beta(r)$ be the solution of

\[
\begin{cases}
    u''(r) + \frac{N-1}{\tanh r} u'(r) + |u|^{p-1} u = 0, \\
    u(0) = \beta, \quad u'(0) = 0.
\end{cases}
\]
Theorem B (Berchio, Ferrero, and Grillo, 2014)

Let $p > 1$. Then, there exists $\beta_0 = \beta_0(N, p)$ such that if $\beta \leq \beta_0$, then $u_\beta$ is a stable solution of $(L)$.

Theorem A (Farina 2007, Dancer, Du, and Guo 2011)

Let $u \in C^2(\mathbb{R}^N)$ be a stable solution of $(E)$. Then there exists some exponent $p(N, \alpha)$ such that if $p$ satisfies

\[
\begin{cases}
1 < p < +\infty & \text{if } N \leq 10 + 4\alpha, \\
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\end{cases}
\]

then $u \equiv 0$ in $\mathbb{R}^N$. Moreover, if $p \geq p(N, \alpha)$, then $(E)$ has stable, positive, and radial solutions.

Problem

Is there a critical exponent for $(L)$ with some weight?
2. Liouville Theorem

\[(E) \quad -\Delta u = |x|^\alpha |u|^{p-1} u \quad \text{in} \quad \mathbb{R}^N.\]

\[(L) \quad -\Delta_g u = |u|^{p-1} u \quad \text{in} \quad \mathbb{H}^N.\]

\[(\text{Example}) \quad -\Delta_g u = r^\alpha |u|^{p-1} u \quad \text{in} \quad \mathbb{H}^N.\]

\[(H) \quad -\Delta_g u = w^\alpha |u|^{p-1} u \quad \text{in} \quad \mathbb{H}^N,\]

where $N \geq 3$, $p > 1$, $\alpha > 0$, and the weight $w = w(r, \theta)$ is non-negative and satisfies the following asymptotic behavior:

\[
\limsup_{r \to +\infty} \frac{\sinh r}{w(r, \theta)} < +\infty.
\]
Definition

A solution $u \in C^2(\mathbb{H}^N)$ of \((H)\) is stable if the inequality

\[
Q[u](v) := \int_{\mathbb{H}^N} \left\{ |\nabla_g v|^2_g - pw^\alpha |u|^{p-1} v^2 \right\} dV_g \geq 0
\]

holds for any $v \in C^1_c(\mathbb{H}^N)$.

Here, the above inequality means that the second variation of the following functional is nonnegative:

\[
E(u) := \int_{\mathbb{H}^N} \left\{ \frac{1}{2} |\nabla_g u|^2_g - w^\alpha \frac{|u|^{p+1}}{p+1} \right\} dV_g.
\]
Theorem 1

Let $u \in C^2(\mathbb{H}^N)$ be a stable solution of (H). If $p$ satisfies

$$
\begin{cases}
1 < p < +\infty & \text{if } N \leq 1 + 4\alpha, \\
1 < p < p_c(N, \alpha) & \text{if } N > 1 + 4\alpha,
\end{cases}
$$

then $u \equiv 0$ in $\mathbb{H}^N$. Here, $p_c(N, \alpha)$ is expressed as follows:

$$
p_c(N, \alpha) := \frac{(N - 1)^2 - 2\alpha(N - 1) - 2\alpha^2 + 2\alpha\sqrt{2\alpha(N - 1) + \alpha^2}}{(N - 1)(N - 4\alpha - 1)}.
$$

● When $p > p_c$, $\exists$ non-trivial stable solution $\Rightarrow p_c$ is critical.
3. Stable solutions

In the following, we set \( N \geq 3 \) and \( w(r, \theta) = \sinh r \).

For \( \beta > 0 \), let \( u_\beta = u_\beta(r) \) be the solution of

\[
\begin{cases}
  u''(r) + \frac{N-1}{\tanh r} u'(r) + (\sinh r)^\alpha |u|^{p-1} u = 0, \\
  u(0) = \beta, \quad u'(0) = 0.
\end{cases}
\]

Define \( p_s(N, \alpha) := (N + 2 + 2\alpha) / (N - 2) \).

**Theorem 2**

Let \( N > 1 + 4\alpha \) and

\[
p > \max\{p_s(N, \alpha), p_c(N, \alpha)\}.
\]

Then, there exists \( \beta_0 = \beta_0(N, p, \alpha) > 0 \) such that if \( \beta \in (0, \beta_0] \), then \( u_\beta \) is a positive stable solution of (H).
Theorem 3

Let $N > 1 + 4\alpha$ and $p > \max\{p_s, p_c\}$. Then, for any $0 < \beta_1 < \beta_2 \leq \beta_0$, two solutions $u_{\beta_1}$ and $u_{\beta_2}$ cannot intersect each other, i.e., $u_{\beta_1}(r) < u_{\beta_2}(r)$ for $r \geq 0$. 

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4. Criticality of $p_c$

- If $p < p_c$, there is no non-trivial stable solutions (Th.1).
- If $p > \max\{p_s, p_c\}$, $\exists$ non-trivial stable solutions (Th.2).
- Let $\alpha \in (\alpha_*, \frac{N-1}{4})$. Then $p_c$ is the critical exponent for (H).

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Theorem 4

Let $1 < p < p(N, \alpha)$. Then, there exists $\bar{\beta} = \bar{\beta}(N, p, \alpha) > 0$ such that $u_\beta$ is unstable for any $\beta > \bar{\beta}$.

- $p(N, \alpha)$ is the critical exponent on stable solutions of

$\Delta u = |x|^{\alpha} |u|^{p-1} u \quad \text{in} \quad \mathbb{R}^N.$

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Theorem 5

Let $p > 1$. Any stable radial solution to (H) has constant sign.