Existence of sign-changing solutions to the Lane-Emden problem via parabolic approach

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The Lane-Emden problem $(\mathcal{E}_p)$

We study the superlinear elliptic boundary value problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (\mathcal{E}_p)$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ and $p > 1$.

F. De Marchis, I. Ianni & F. Pacella, JDE 2013.

F. De Marchis & I. Ianni, DCDS 2015.

F. De Marchis, I. Ianni & F. Pacella, JEMS 2015.

F. De Marchis, I. Ianni & F. Pacella, AMPA to appear.

F. De Marchis, I. Ianni & F. Pacella, preprint.

In this talk

We find solutions to $(\mathcal{E}_p)$ having some specific qualitative properties

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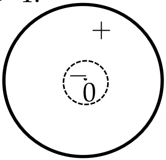
More precisely we are interested in the existence of sign-changing solutions u_p to $(\mathcal{E}_p)$ with:

- two nodal domains
- interior nodal line: $\quad NL_p \cap \partial\Omega = \emptyset, \quad NL_p := \overline{\{x \in \Omega,\ u_p(x) = 0\}}$

When Ω is a *ball*

The existence of such a solution is obvious for every $p > 1$:



least energy nodal ***radial*** solution

2 nodal domains and interior nodal line

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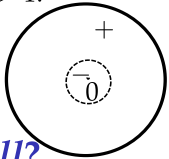

Natural question: when Ω is *not a ball*?

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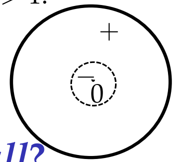


Natural question: when Ω is *not a ball*?

Least energy nodal solution

- A. Castro, J. Cossio & J.M. Neuberger, Rocky Mount. J. Math. 1997. - existence via variational methods
- T. Bartsch & T. Weth, TMNA 2003. - **2 nodal domains** and Morse index 2
- A. Aftalion & F. Pacella, C. R. Math. Acad. Sci. Paris 2004. - $\Omega =$ ball/annulus: it is not radial and its nodal set touches $\partial\Omega$
- M. Grossi, C. Grumiau & F. Pacella, Ann. IHP 2012. Under an additional assumption, show that the nodal line touches $\partial\Omega$ for p large

Other results

- P. Esposito, M. Musso & A. Pistoia, Proc. London Math. Soc. 2007 - for p large, solution with 2 nodal domain, nodal set touches the boundary

Our main result

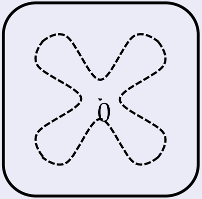
Theorem (De Marchis - I. - Pacella)

Let G be a cyclic group of rotations of the plane about the origin with $|G| \geq 4$. Assume that $\Omega \subset \mathbb{R}^2$ is simply connected and invariant under the action of G.

Then for p sufficiently large there exists an initial condition $u_0 = u_{0,p} \in H_0^1(\Omega)$ such that the solution $v(t, u_0)$ to $(\mathcal{P}_p)$ is global, sign-changing $\forall t \geq 0$, $\omega(u_0) \neq \emptyset$.

Moreover any $u_p \in \omega(u_0)$ is a sign changing G-symmetric solution to $(\mathcal{E}_p)$ which has the following properties:

a. $p \int_\Omega |\nabla u_p|^2 \leq \alpha \cdot 8\pi e$, for some $\alpha < 5$
b. 2 nodal domains
c. $NL_p \cap \partial\Omega = \emptyset$
d. $O \notin NL_p$



$|G|$ is the order of the group G, $\omega(u_0)$ is the ω-limit set

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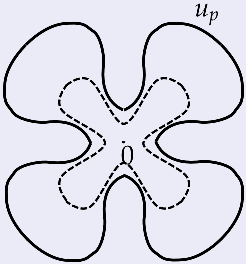

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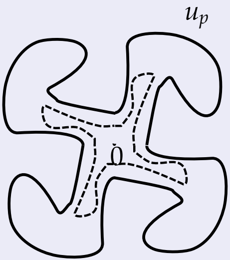

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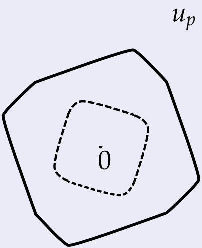

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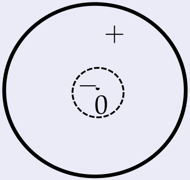

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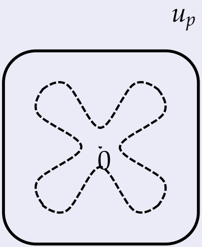

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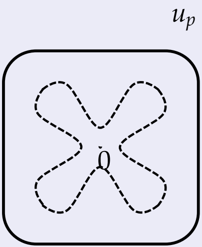

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Main ingredients of the proof

- a dynamical approach which combines the use of the *parabolic flow* $(\mathcal{P}_p)$ associated to $(\mathcal{E}_p)$ together with a *topological argument*, to obtain nodal solutions
- energy estimates for p large, to control the number of nodal domains
- geometrical arguments in presence of G-symmetry, to prove that the nodal line doesn't touch the boundary and doesn't contain the origin

Existence of a nodal solution

We consider the initial value problem for the associated semilinear heat equation

$$\begin{cases} \frac{\partial v}{\partial t} - \Delta v = |v|^{p-1}v & \text{in } \Omega \times (0, +\infty) \\ v = 0 & \text{on } \partial\Omega \times (0, +\infty) \\ v(\cdot, 0) = u_0 & \text{on } \Omega \end{cases} \qquad (\mathcal{P}_p)$$

and we look for solutions of the elliptic eq. $(\mathcal{E}_p)$ as equilibria in the *ω-limit set*.

In order to obtain nodal equilibria we need to select in a proper way u_0. We use a topological argument based on the use of the *Krasnoselskii genus*.

- A. Castro, J. Cossio & J.M. Neuberger, Rocky Mount. J. of Math. 1997. - existence via variational methods
- P. Quittner, NODEA 2004 - existence via parabolic flow
- M. Conti, L. Merizzi & S. Terracini, ARMA 2000.
- N. Ackermann & T. Bartsch, J. Dynam. Diff. Eq. 2005.
- J. Wei & T. Weth, ARMA 2008.
- I. Ianni, Top. Meth. Nonl. Anal. 2013.

$$\begin{cases} \frac{\partial v}{\partial t} - \Delta v = |v|^{p-1} v & \text{in } \Omega \times (0, +\infty) \\ v = 0 & \text{on } \partial\Omega \times (0, +\infty) \\ v(\cdot, 0) = u_0 & \text{on } \Omega \end{cases} \qquad (\mathcal{P}_p)$$

Largely studied by many authors

- F.B. Weissler, Indiana Univ. Math. J. 1980.
- T. Cazenave & P.-L. Lions, Comm. Part. Diff. Eq. 1984.
- H. Hoshino & Y. Yamada, Funkc. Ekvac. 1991.
- P. Quittner, Acta Math. Univ. Comen. 1999/ Math. Ann. 2001.
- F. Gazzola & T. Weth, Diff. and Int, Eq. 2005.

Local solvability

For every $u_0 \in H_0^1(\Omega)$ problem $(\mathcal{P}_p)$ has a unique solution $v_p(t) = v_p(t, u_0)$ $\in C^0([0,T); H_0^1(\Omega)) \cap C^1((0,T); L^2(\Omega))$ with maximal existence time $T := T(u_0) > 0$ which is a classical solution for $t \in (0,T)$.

- if $T(u_0) < \infty$ then $\|v_p(t)\| \to +\infty$ as $t \to T(u_0)$ (blow-up)
- if $u_0 \geq 0$, $u_0 \not\equiv 0$, then $v_p(t) > 0$ in $\Omega \times (0, T(u_0))$ (maximum priciple)
- $\mathcal{A} := \{(t, u_0) : 0 \leq t < T(u_0)\}$ is open in $[0,\infty) \times H_0^1(\Omega)$
- v_p is a semiflow on $\mathcal{A}$
- Continuity w.r. to initial data: For every $u_0 \in H_0^1(\Omega)$ and every $t \in (0, T(u_0))$ there is a neighborhood $U \subset H_0^1(\Omega)$ of u_0 such that
  $T(u) > t$ for $u \in U$ and $v_p(t,\cdot) : (U, \|\cdot\|) \to (C_0^1(\Omega), \|\cdot\|_{C^1})$ is continuous

$C_0^1(\Omega) := \{u \in C^1(\overline{\Omega}) \ : \ u = 0 \text{ on } \partial\Omega\}$ endowed with the standard norm of $C^1(\overline{\Omega})$

H. Hoshino & Y. Yamada, Funkc. Ekvac. 1991 - existence, uniqueness, regularity

F.B. Weissler, Indiana U. Math. J. 1980 - blow-up

H. Brezis & T. Cazenave, Anal. Math. 1996

P. Quittner, Houston J. Math. 2003 - continuity

An important property

The energy functional $E_p : H_0^1(\Omega) \to \mathbb{R}$

$$E_p(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p+1} \int_\Omega |u|^{p+1}$$

is a strict Lyapunov funtional

Indeed for a classical solution v_p of $(\mathcal{P}_p)$

$$\frac{d}{dt} E_p(v_p(t)) = - \left\| \frac{\partial}{\partial t} v_p(t) \right\|_2^2 \qquad t \in (0, T)$$

hence the energy is strictly decreasing along nonconstant trajectories.

Global existence, compactness, ω-limit

Let $u_{p,0} \in H_0^1(\Omega)$. If $T(u_{p,0}) = +\infty$ then:

- the map $t \mapsto \|v_p(t)\|_{C^1}$ is bounded on $[\delta, \infty)$ for all $\delta > 0$
- the set $\{v_p(t, u_{p,0}) : t \geq \delta\}$ is relatively compact in $C_0^1(\Omega)$ for all $\delta > 0$

P. Quittner, Acta Math. Univ. Comen. 1999/ Houston J. Math. 2003

$\Downarrow$ $\big(E_p$ is a strict Lyapunov functional$\big)$

Hence the ω-limit set

$$\omega(u_{p,0}) = \bigcap_{t>0} clos_{H_0^1}\left(\{v_p(s, u_{p,0}) : s \geq t\}\right)$$

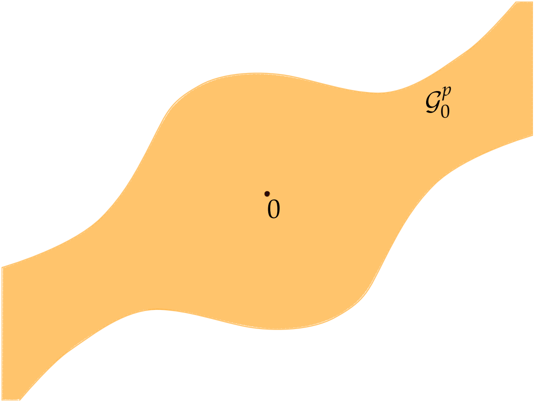
is a nonempty compact connected subset of $C_0^1(\Omega)$ consisting of solutions of $(\mathcal{E}_p)$

Existence of a nodal solution to $(\mathcal{E}_p)$

$u \equiv 0$ is an asymptotically stable solution for the parabolic problem $(\mathcal{P}_p)$

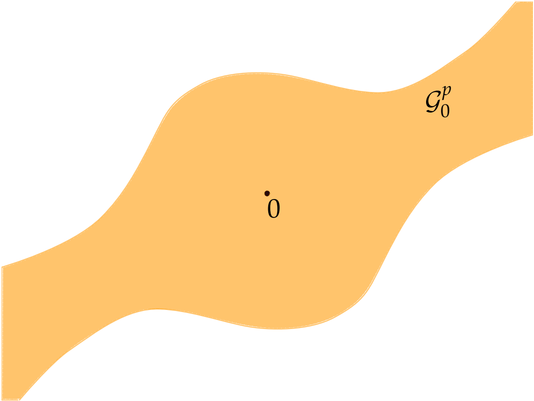
- $E_p(u)$ is a Lyapunov functional
- 0 is a strict local minimum for E_p

Existence of a nodal solution to $(\mathcal{E}_p)$



$$\mathcal{G}_0^p := \{u \in H_0^1(\Omega) : T(u) = +\infty \text{ and } v_p(t,u) \to 0 \text{ in } H_0^1(\Omega) \text{ as } t \to +\infty\}$$

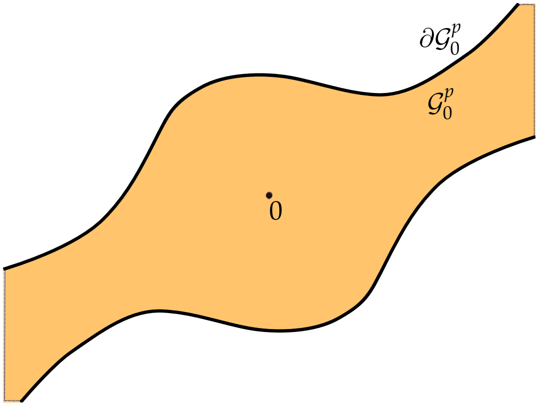
Existence of a nodal solution to $(\mathcal{E}_p)$



$\mathcal{G}_0^p$ is an *open* neighborhood of 0 in $H_0^1(\Omega)$, because:

- 0 is asymptotically stable
- continuity of trajectories for fixed time with respect to the initial data
- semiflow properties

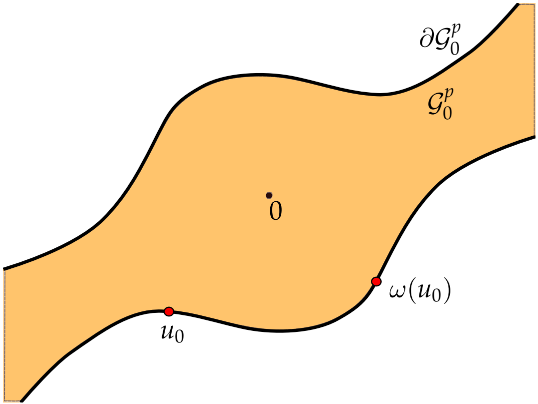
Existence of a nodal solution to $(\mathcal{E}_p)$



$\partial\mathcal{G}_0^p$ is positively invariant and symmetric

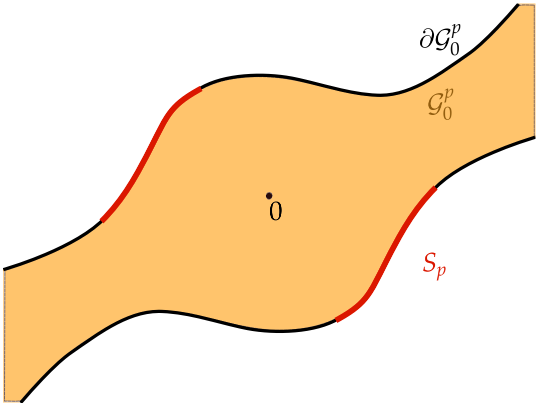
$T(u_0) = +\infty$, for any initial condition $u_0 \in \partial\mathcal{G}_0^p$

Existence of a nodal solution to $(\mathcal{E}_p)$



For any initial condition $u_0 \in \partial\mathcal{G}_0^p$
the ω-limit set for the parabolic problem $(\mathcal{P}_p)$
$\emptyset \neq \omega(u_0) \subset \partial\mathcal{G}_0^p$ and
any $u \in \omega(u_0)$ is a solution for the elliptic problem $(\mathcal{E}_p)$

Existence of a nodal solution to $(\mathcal{E}_p)$

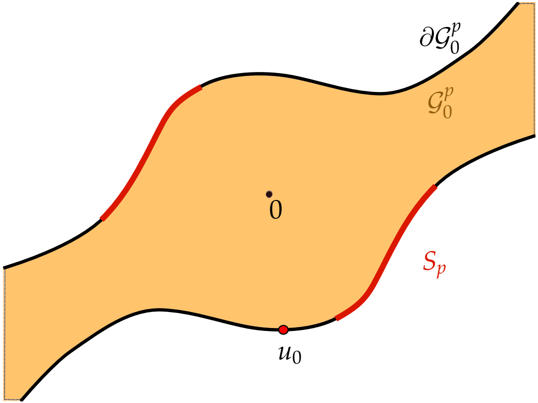


Next we select $u_0 \in \partial\mathcal{G}_0^p$ so that any $u \in \omega(u_0)$ is a nodal solution to $(\mathcal{E}_p)$

$$S_p := \{u \in \partial\mathcal{G}_0^p \ : \ u \geq 0 \text{ or } u \leq 0\}$$

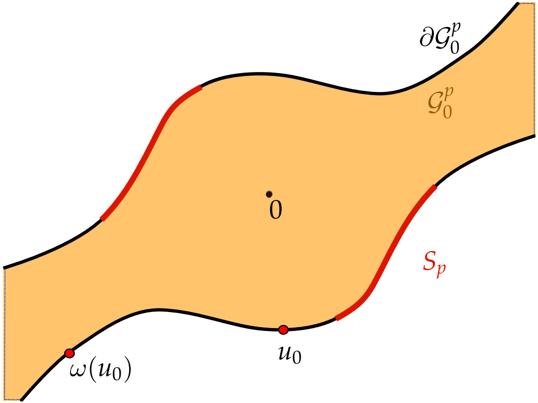
S_p is (symmetric, closed in $H_0^1(\Omega)$ and) positively invariant (maximum principle)

Existence of a nodal solution to $(\mathcal{E}_p)$



we prove that $\exists\, u_0\ (= u_{p,0}) \in \partial\mathcal{G}_0^p \setminus S_p$ s.t. $\omega(u_0) \subseteq \partial\mathcal{G}_0^p \setminus S_p$

Existence of a nodal solution to $(\mathcal{E}_p)$



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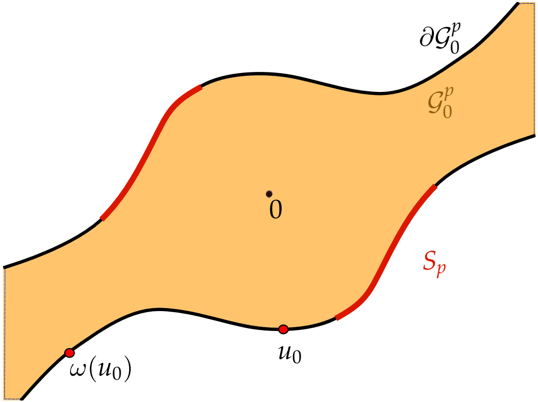

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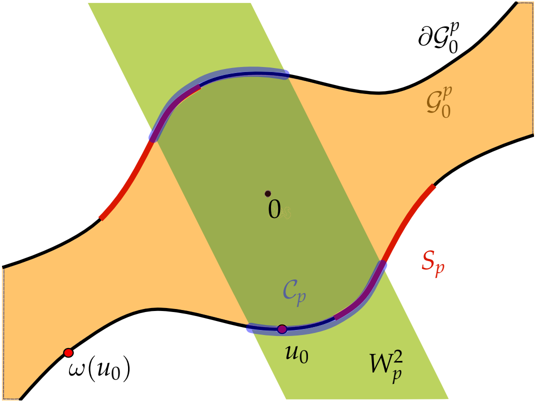
by contradiction, since $genus(S_p) \leq 1$

Existence of a nodal solution to $(\mathcal{E}_p)$



$$\mathcal{C}_p := \partial(W_p^2 \cap \mathcal{G}_0^p), \qquad W_p^2 := \{s_1 Z_{p,1} + s_2 Z_{p,2} : s_1, s_2 \in \mathbb{R}\}$$

for any $Z_{p,1}, Z_{p,2} \in H_0^1(\Omega)$ s.t. $supp\ Z_{p,1} \cap supp\ Z_{p,2} = \emptyset$,

$\forall p > 1$ there exists $u_0 = s_1 Z_{p,1} + s_2 Z_{p,2} \in \mathcal{C}_p \setminus S_p$ $(\subset \partial\mathcal{G}_0^p \setminus S_p)$ such that the solution $v_p(t, u_0)$ to $(\mathcal{P}_p)$, is global, nodal for every t, $\omega(u_0) \neq \emptyset$ and any $u_p \in \omega(u_0)$ is a nodal solution to $(\mathcal{E}_p)$

Energy estimate & control on the number of nodal domains

Since E_p is a Lyapunov functional:

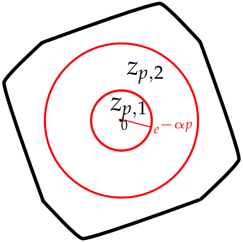
$$E_p(u_p) \leq E_p(u_0)$$

$$u_0 \in W_p^2 = span\{Z_{p,1}, Z_{p,2}\}$$

We chose the space W_p^2:

$Z_{p,1}$= the unique positive radial solution to $(\mathcal{E}_p)$ in the ball

$Z_{p,2}$= the unique positive radial solution to $(\mathcal{E}_p)$ in the annulus



$$pE_p(u_0) \underset{\text{disj. } supp}{\overset{Z_{p,1},\, Z_{p,2} \,\in\, \mathcal{N}_p}{\leq}} pE_p(Z_{p,1}) + pE_p(Z_{p,2}) \overset{\downarrow}{\leq} 4.97 \cdot 4\pi e, \quad \text{for } p \text{ large}$$

Energy estimate & control on the number of nodal domains

The nodal solution u_p to $(\mathcal{E}_p)$ obtained satisfies

$$pE_p(u_p) \leq 4.97 \cdot 4\pi e, \quad \text{for } p \text{ large}$$

$$\Downarrow$$

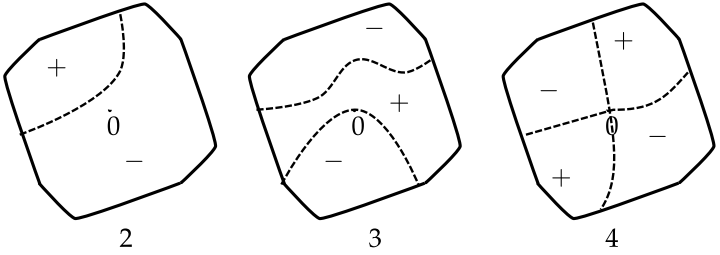
u_p has at most 4 nodal domains, for p large

Proof. Let D_p be a nodal domain of u_p, then $u_p \chi_{D_p} \in \mathcal{N}_p$ and so

$$pE_p(u_p \chi_{D_p}) \geq \inf_{\mathcal{N}_p} (pE_p) \stackrel{p \to +\infty}{\longrightarrow} 4\pi e$$

X. Ren & J. Wei, Trans. Amer. Math. Soc. 1994.

The role of the G-symmetry



All the results up to here hold for any smooth bounded domain Ω.

Assume now that $\Omega \subseteq \mathbb{R}^2$ is simply connected and G-invariant where G is a cyclic group of rotations about the origin with $|G| \geq 4$.

Since the initial condition u_0 is G-symmetric (it's radial), it follows that the solution $\varphi_p(t, u_0)$ is G-symmetric for all $t \geq 0$, and so $u_p \in \omega(u_0)$ is G-symmetric

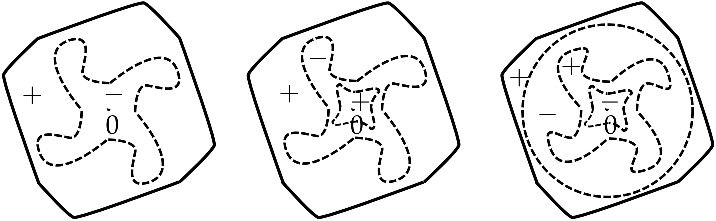
Lemma A.B.C.

Let $\Omega \subseteq \mathbb{R}^2$ be simply connected and G-invariant with $|G| \geq 4$.
Let u_p be a G-symmetric sign-changing solution to $(\mathcal{E}_p)$ with at most 4 nodal regions then:

A. $O \notin NL_p$

B. each nodal region is G-symmetric

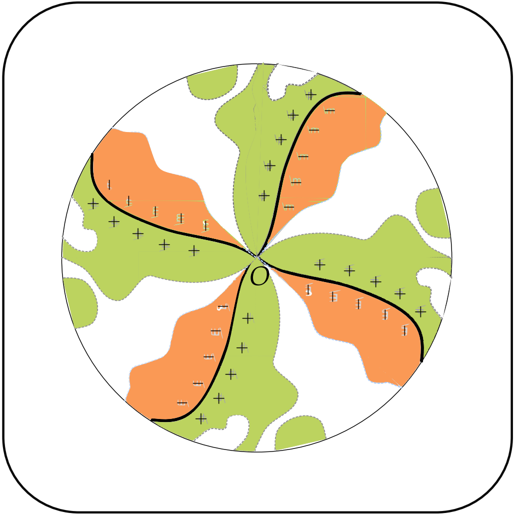
C. $NL_p \cap \partial\Omega = \emptyset$



2 3 4

Proof. Geometrical arguments (......... in $\mathbb{R}^2$)

A. $O \notin NL_p$



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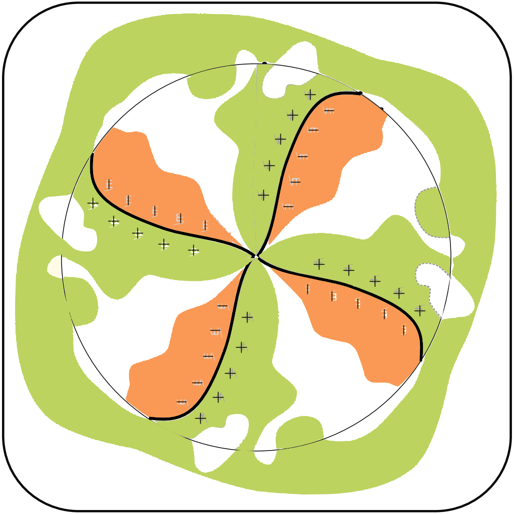
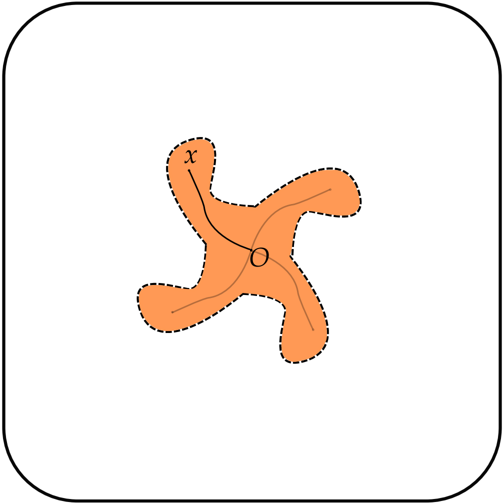


B. each nodal region is G-symmetric



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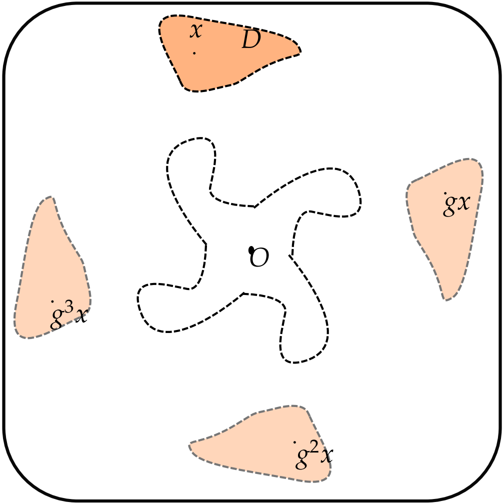
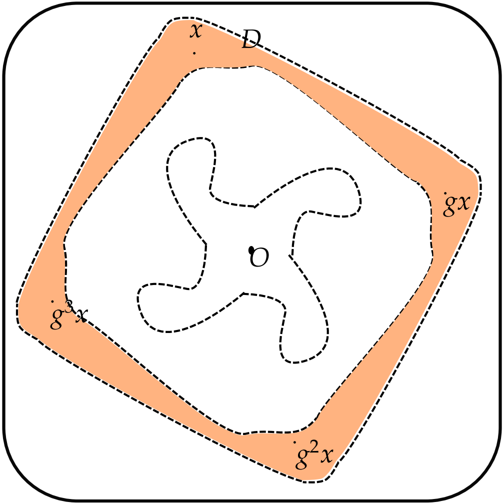
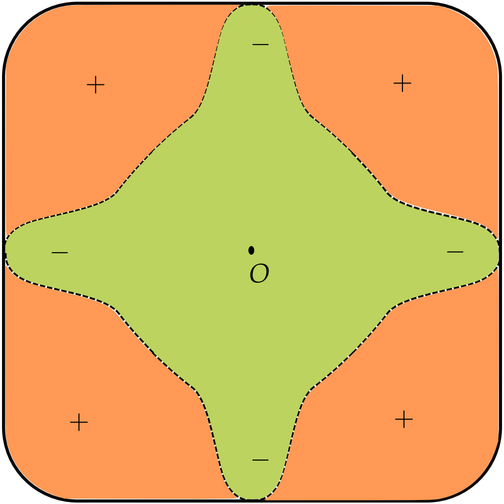


B. each nodal region is G-symmetric

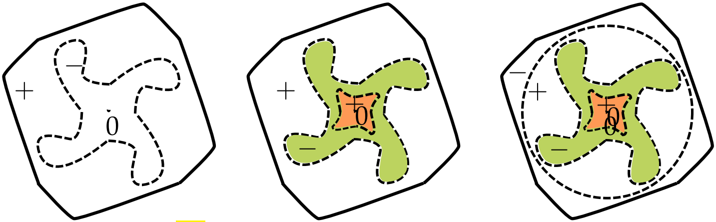


C. $NL_p \cap \partial\Omega = \emptyset$



Exactly two nodal regions by iterating the procedure

u_p $\qquad$ $pE_p(u_p) \leq 4.97 \cdot 4\pi e$



NOD. REG. $\qquad$ 2 $\qquad$ 3 $\qquad$ 4

In the 2nd and 3rd case define a new G-symmetric *initial condition* $\widetilde{u}_0$ (fig.) and iterate all the procedure:

$$pE_p(\widetilde{u}_0) \leq \quad (4.97-1)\cdot 4\pi e \quad - - - - \quad (4.97-2)\cdot 4\pi e$$

$\downarrow$ $\qquad$ $\downarrow$

$\widetilde{u}_p$ with at most 3 n.r. $\qquad$ $\widetilde{u}_p$ with 2 nodal regions

(new iteration) $\downarrow$

$\widehat{u}_p$ with 2 nodal regions

Theorem (De Marchis - I. - Pacella)

Let G be a cyclic group of rotations of the plane about the origin with $|G| \geq 4$. Assume that $\Omega \subset \mathbb{R}^2$ is simply connected and invariant under the action of G.

Then for p sufficiently large there exists an initial condition $u_0 = u_{0,p} \in H^1_0(\Omega)$ such that the solution $v(t, u_0)$ to $(\mathcal{P}_p)$ is global, sign-changing $\forall t \geq 0$, $\omega(u_0) \neq \emptyset$.

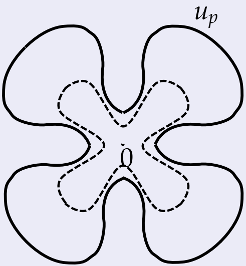
Moreover any $u_p \in \omega(u_0)$ is a sign changing G-symmetric solution to $(\mathcal{E}_p)$ which has the following properties:

a. $p \int_\Omega |\nabla u_p|^2 \leq \alpha \cdot 8\pi e$, for some $\alpha < 5$

b. 2 nodal domains

c. $NL_p \cap \partial\Omega = \emptyset$

d. $O \notin NL_p$



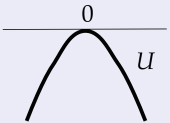
$|G|$ is the order of the group G, $\omega(u_0)$ is the ω-limit set

Asymptotic analysis of u_p as $p \to \infty$

Theorem [De Marchis, I., Pacella]

Let $(u_p)_p$ be the solutions to $(\mathcal{E}_p)$ given by the previous theorem. Then

- CONCENTRATION: $0 < \delta \leq \|u_p^{\pm}\|_\infty \leq C < \infty$ & $u_p \to 0$ in $C^1_{loc}(\bar{\Omega} \setminus \{0\})$
- $x_p^+ \to 0$, $x_p^- \to 0$ & NL_p shrinks to 0
- 0 is a NON SIMPLE concentration point: TOWER OF BUBBLES
- positive and negative parts $u_p^{\pm}$ concentrate at 0 with DIFFERENT LIMIT PROBLEMS

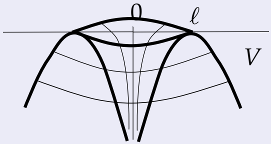


REGULAR BUBBLE

scaling properly u_p^+ around x_p^+

$$U(x) := \log \left(\frac{1}{(1 + \frac{1}{8}|x|^2)^2} \right)$$

$$\begin{cases} -\Delta U = e^U & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^U = 8\pi, \ U(0) = 0, \ U \leq 0 \end{cases}$$



SINGULAR BUBBLE

scaling properly u_p^- around x_p^-

$$V(x) := \log \left(\frac{2\alpha^2 \beta^\alpha |x|^{\alpha-2}}{(\beta^\alpha + |x|^\alpha)^2} \right)$$

$$\begin{cases} -\Delta V = e^V - 4\pi\eta\delta_0 & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^V = 8\pi(1+\eta), \ V(\ell) = 0, \ V \leq 0 \end{cases}$$

M. Grossi, C. Grumiau & F. Pacella, Journal Math. Pure and Appl. 2014 - radial case

Back to problem $(\mathcal{P}_p)$

Theorem [De Marchis, I.]

Let $(u_p)_p$ be the sign-changing solution of $(\mathcal{E}_p)$ that we found previously and let

$$u_0(x) = \lambda\, u_p(x), \quad \text{for } \lambda > 0.$$

Then for p sufficiently large there exists $\epsilon(p) > 0$ such that if

$$0 < |1 - \lambda| < \epsilon(p)$$

then the solution $v_p(t, u_0)$ of $(\mathcal{P}_p)$ blows-up in finite time.

F. Dickstein, F. Pacella & B. Sciunzi, Journal of Evolution Equations 3014 - radial case

CONSEQUENCE: the set of initial data for which the corresponding solution is global is not star-shaped about the origin

Thank you!