Existence of positive solutions of fully nonlinear equations

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Joint work with Patricio Felmer (Universidad de Chile)

§1 : Introduction

Consider the existence or nonexistence of solutions to

$$(1) \qquad \begin{cases} -\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'') + V(x)u = |u|^{p-1}u & \text{in } \mathbf{R}, \\ u > 0 & \text{in } \mathbf{R}, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

- $V(x) : \mathbf{R} \to \mathbf{R}$: continuous.
- $1 < p < \infty$.
- $0 < \lambda \leq \Lambda < \infty$: constants.
- $\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'')$: Pucci operators given by

$$\mathcal{M}^{+}_{\lambda,\Lambda}(s) := \begin{cases} \Lambda s & \text{if } s \geq 0, \\ \lambda s & \text{if } s < 0, \end{cases} \qquad \mathcal{M}^{-}_{\lambda,\Lambda}(s) := \begin{cases} \lambda s & \text{if } s \geq 0, \\ \Lambda s & \text{if } s < 0. \end{cases}$$

$$\Lambda = \lambda \quad \Rightarrow \quad \mathcal{M}^{\pm}_{\lambda,\Lambda}(u'') = \Lambda u''.$$

§1 : Introduction

Special case of (1) ($\Lambda = 1 = \lambda$):

(2) $$\begin{cases} -u'' + V(x)u = |u|^{p-1}u \quad \text{in } \mathbf{R}, \quad u > 0 \quad \text{in } \mathbf{R}, \\ u(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases}$$

A sol. of (2) is related to a critical pt. of

$$I(u) := \frac{1}{2}\int_{\mathbf{R}} |u'|^2 + V(x)u^2 \mathrm{d}x - \frac{1}{p+1}\int_{\mathbf{R}} |u|^{p+1}\mathrm{d}x.$$
$$\left(\in C^1(H^1(\mathbf{R}), \mathbf{R}) \quad \text{if } V \in L^\infty(\mathbf{R})\right)$$

Critical Point Theory (Mountain pass Thm.)

+ Concentration Compactness

$\Rightarrow$ **Ex. of sol. of** (2) **under** • $\inf_{\mathbf{R}} V > 0$, • $\sup_{\mathbf{R}} V(x) = \lim_{|x|\to\infty} V(x)$.

(Ambrosetti, Ding, Lions, Ni, Rabinowitz $\cdots$)

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(1) $$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'') + V(x)u = |u|^{p-1}u \text{ in } \mathbf{R},\ u > 0 \text{ in } \mathbf{R},\ \lim_{|x|\to\infty} u(x) = 0.$$

Aim

Replace u'' in (2) by $\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'')$ and observe whether (1) has a solution (find a suitable class of $V(x)$).

- **Known result related to (1)**

Felmer–Quaas ('04) $(V(x) \equiv 1)$

Ex. of pos. radial sol. of

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) + u = |u|^{p-1}u \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega$$

where $\Omega = \mathbf{R}^N$ or $B_R(0) \subset \mathbf{R}^N$ (with $u = 0$ on $\partial\Omega$), $N \geq 3$, $1 < p < p^{\pm}_{*}$.
First they consider the case $\Omega = B_R(0)$. Then take a limit $R \to \infty$.

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§2 : Main results

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'') + V(x)u = |u|^{p-1}u \quad \text{in } \mathbf{R}, \quad 1 < p < \infty, \quad 0 < \lambda \leq \Lambda < \infty.$$

Assumptions on $V(x)$

(V1) $V \in W^{1,\infty}(\mathbf{R}), \quad 0 < \inf_{\mathbf{R}} V, \quad V_\infty := \exists \lim_{|x|\to\infty} V(x) \in (0,\infty).$

(V2) $V'(x) \leq 0 \leq V'(y)$ for $-\infty < x < 0 < y < \infty$.

(V3) There exist $C_0, \xi_0 > 0$ such that for all $x \in \mathbf{R}$,

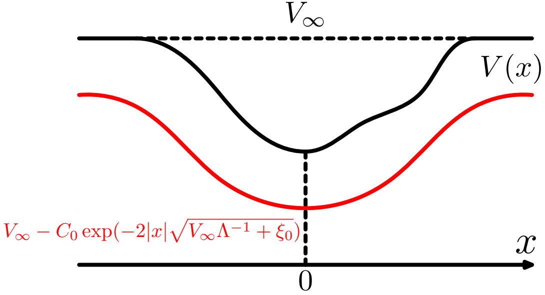
$$\text{(for } \mathcal{M}^{+}_{\lambda,\Lambda}) \quad V_\infty - C_0 \exp\left(-2|x|\sqrt{V_\infty \Lambda^{-1} + \xi_0}\right) \leq V(x)\,(\leq V_\infty),$$

$$\text{(for } \mathcal{M}^{-}_{\lambda,\Lambda}) \quad V_\infty - C_0 \exp\left(-2|x|\sqrt{V_\infty \lambda^{-1} + \xi_0}\right) \leq V(x)\,(\leq V_\infty).$$

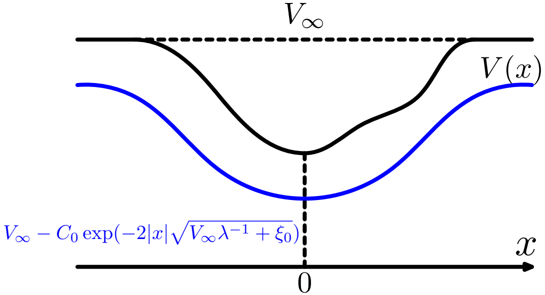
NOTE: $-\exp\left(-2|x|\sqrt{V_\infty \Lambda^{-1} + \xi_0}\right) \leq -\exp\left(-2|x|\sqrt{V_\infty \lambda^{-1} + \xi_0}\right)$.

Remark: $V(x)$ needs NOT be symmetric.

§2 : Main results



Example of V for $\mathcal{M}^{+}_{\lambda,\Lambda}$



Example of V of $\mathcal{M}^{-}_{\lambda,\Lambda}$

§2 : Main results

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Theorem (Existence result)

Under (V1)–(V3), (1) *has a solution.*

Remark

Instead of $|u|^{p-1}u$, we may treat more general nonlinearity $f(u)$.

§2 : Main results

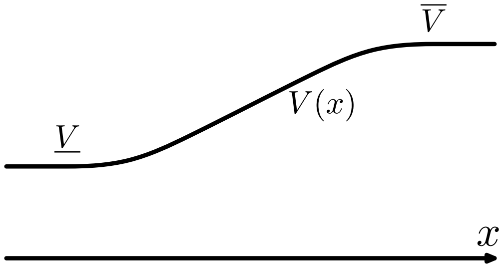
Theorem (Nonexistence result)

Assume $V \in W^{1,\infty}(\mathbf{R})$ *satisfies*

$$V'(x) \geq 0 \quad \text{in } \mathbf{R}, \quad 0 < \underline{V} := \lim_{x \to -\infty} V(x) < \lim_{x \to \infty} V(x) =: \overline{V}.$$

Then (1) *has no solution.*

The same statement holds when $V'(x) \leq 0$ *in* $\mathbf{R}$.



Example of $V(x)$

§3 : Difficulties (for Existence)

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'') + V(x)u = f(u) \quad \text{in } \mathbf{R}, \quad 0 < \lambda \le \Lambda < \infty.$$

- Critical Point Theory may not be applied.
- Difficult to apply the sub-supersolution method (Comparison principle or the order property).

We use the degree theoretical approach (Benjamin('71), Nussbaum('73), de Figueiredo–Lions–Nussbaum('82), Felmer–Quaas('04)).

- Find a function space X (Banach sp.) and show

  (i) $T_{\pm}(v) := \left(-\mathcal{M}^{\pm}_{\lambda,\Lambda} + V(x)\right)^{-1}(f(v)) : X \to X$ is *compact*.

  ($u = T_{\pm}(v) \in X$ is a sol. of $-\mathcal{M}^{\pm}_{\lambda,\Lambda}(u'') + V(x)u = f(v)$ in $\mathbf{R}$)

  (ii) $\exists R \gg 1$ s.t. $\deg_X(\mathrm{id} - T_{\pm}, B_R(0), 0) = 0$.

  NOTE: $\deg_X(\mathrm{id} - T_{\pm}, B_r(0), 0) = 1$ for $r \ll 1$.

  (Critical Point Theory $\rightarrow$ $X = H^1(\mathbf{R})$).

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