Semilinear critical problems with singular nonlinearities on Carnot groups

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Outline of the talk

- The setting of Carnot groups
- Introduction to a class of singular critical subelliptic problems
- The issue of regularity of solutions: representation of higher layer derivatives and their asymptotic behavior at the singularity
- Pohozaev-type identity and non-existence results
- Existence results for a Brezis-Nirenberg type problem

Definition

A **Carnot group** $(\mathbb{G}, \circ)$ *is a connected, simply connected nilpotent Lie group with stratified Lie algebra, i.e.*

$$\mathfrak{g} = \bigoplus_{j=1}^{r} V_j$$

where $[V_1, V_j] = V_{j+1}$ *for* $1 \leq j < r$, $[V_1, V_r] = \{0\}$.

Hence V_1 generates by commutation the whole Lie algebra of $\mathbb{G}$.
The integer r is called the step of the group $\mathbb{G}$.

Definition

Let $X = \{X_1, \ldots, X_{N_1}\}$ *any basis of* V_1. *The second order differential operator*

$$\mathcal{L} = \sum_{j=1}^{N_1} X_j^2$$

is called a **Sub-Laplacian** *on* $\mathbb{G}$.

- A Carnot group $(\mathbb{G}, \circ)$ is canonically isomorphic to a homogeneous Lie group of the form $\mathbb{G} \simeq (\mathbb{R}^N, \circ) = (\mathbb{R}^{N_1} \times \ldots \times \mathbb{R}^{N_r}, \circ)$, with dilations

$$\delta_\lambda(\xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(r)}) = (\lambda\, \xi^{(1)}, \lambda^2\, \xi^{(2)}, \cdots, \lambda^r\, \xi^{(r)})$$

- The number $Q := \sum_{i=1}^r i N_i$ is called the **homogeneous dimension** of $\mathbb{G}$.

- If $Q \geq 3$, there exists a homogeneous norm $d : \mathbb{G} \to [0, +\infty)$ such that

$$\Gamma(\xi) := \frac{C_Q}{d(\xi)^{Q-2}}$$

is a **fundamental solution** of $-\Delta_{\mathbb{G}}$ with pole at 0.

- Sobolev type inequality on $\mathbb{G}$ (Folland-Stein, '75):

$$\|\nabla_{\mathbb{G}} f\|_2^2 \geq C \|f\|_{2^*}^2 \quad \forall\, f \in C_0^\infty(\mathbb{G})$$

where $\nabla_{\mathbb{G}} u = (X_1, \ldots, X_{N_1})$, and $2^* = 2Q/(Q-2)$ is the critical Sobolev exponent.

- **Example: The Heisenberg group $\mathbb{H}^n$**

$\mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ)$, $Q = 2n + 2$
Denoted by $\xi = (z, t) = (x, y, t)$ the points of $\mathbb{H}^n$:

$$\xi \circ \xi' = (x + x', y + y', t + t' + 2(\langle x', y\rangle - \langle x, y'\rangle))$$

$$\delta_\lambda(x, y, t) = \delta_\lambda(z, t) = (\lambda z, \lambda^2 t)$$

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{n}(X_j^2 + Y_j^2),$$

$$X_j = \partial_{x_j} + 2y_j\partial_t \quad Y_j = \partial_{y_j} - 2x_j\partial_t,\ j = \{1, \dots, n\}$$

Easy to check that:

$$[X_j, Y_k] = -4\delta_{jk}\partial_t \quad j, k = 1, \dots, n$$

and all the other commutators are trivial.

Some references on Carnot groups: the classical papers by Folland ('75), Folland-Stein ('82); the monograph by Bonfiglioli-Lanconelli-Uguzzoni, "Stratified Lie groups and potential theory for their Sub-Laplacians", Springer Monographs in Mathematics. Springer, Berlin, (2007).

Singular subelliptic problems

Singular subelliptic problems

We consider the following class of singular Yamabe-type problems

$$\begin{cases} -\Delta_{\mathbb{G}} u = \psi(\xi)^\alpha \dfrac{|u|^{2^*(\alpha)-2}u}{d(\xi)^\alpha} & \text{in } \Omega \subset \mathbb{G},\ 0 \in \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (P_\alpha)$$

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where

- $0 < \alpha < 2$
- $2^*(\alpha) = 2(Q-\alpha)/(Q-2)$ is the critical Hardy-Sobolev exponent
- $d = d(\xi)$ is the *gauge* norm, i.e. the norm associated to the fundamental solution of the Sub-Laplacian $\Delta_{\mathbb{G}}$
- the weight ψ is defined as $\psi := |\nabla_{\mathbb{G}} d|$, ψ homogeneous of degree 0.

We look for solutions in the Folland-Stein space $S_0^1(\Omega) = \overline{\mathcal{C}_0^\infty(\Omega)}^{\|\cdot\|_{S_0^1}}$, where $\|u\|_{S_0^1} := \|\nabla_{\mathbb{G}} u\|_2$.

The starting point of the variational formulation of the problem is the Hardy-Sobolev inequality on $\mathbb{G}$

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 \, \mathrm{d}\xi \geq C \left(\int_{\mathbb{G}} \psi^{\alpha} \frac{|u|^{2^*(\alpha)}}{d(\xi)^{\alpha}} \, \mathrm{d}\xi \right)^{\frac{Q-2}{Q-\alpha}}, \quad \forall u \in C_0^{\infty}(\mathbb{G})$$

Limit cases

- $\alpha = 0$: Sobolev-type inequality on groups (Folland-Stein '75)
- $\alpha = 2$: Hardy-type inequality (Lanconelli-Garofalo '90, D'Ambrosio 2005, etc...)

Let us introduce the spaces:

$\Gamma^2(\Omega)$: the space of all continuous functions $u \in C(\Omega)$ such that $X_j u$, $X_i X_j u \in C(\Omega)$, for $i, j = 1, \ldots, m$. Analogously, $\Gamma^{k,\beta}(\overline{\Omega})$ and $\Gamma^{k,\beta}_{loc}(\Omega)$, $0 < \beta < 1$, will denote the Folland-Stein Hölder spaces.

Proposition (L., 2015)

Let $\mathbb{G}$ be a Carnot group and let Ω be an arbitrary open set of $\mathbb{G}$, $0 \in \Omega$. If $u \in S^1_0(\Omega)$ is a weak solution of problem (P_α), then

$$u \in \Gamma^{2,\gamma}_{loc}(\Omega \setminus \{0\}) \cap \Gamma^{\beta}_{loc}(\Omega),$$

for some $\gamma, \beta \in (0,1)$. Moreover, if Ω satisfies the geometric condition

$$\exists\, \delta, r_0 > 0 : |B_d(\xi, r) \setminus \Omega| \geq \delta |B_d(\xi, r)| \quad \forall \xi \in \partial\Omega, \forall r \in (0, r_0),$$

then, u is Hölder continuos up to the boundary of Ω.

The starting point in the study of regularity: the analysis of L_p-regularity performed by the author in the previous paper (L., NoDEA, 2010)

Proposition

A weak solution u to pb. (P_α) satisfies $u \in L^p(\Omega)$, $\forall\, 2^*/2 < p \leq \infty$

So, we know that:

- $|\Delta_{\mathbb{G}} u| \sim \dfrac{C}{d^\alpha}$ at 0
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What about the behavior of higher-layer derivatives of u at 0?

They are first order differential operators, but they have the homogeneity of higher order derivatives.

no good a priori estimates are available.

Then, we follow a **pointwise approach**, suitably adapting a method introduced by Lanconelli and Uguzzoni (2000).

Let $\mathbb{G}$ be a step two Carnot group with Lie algebra $\mathfrak{g} = \mathfrak{G}_1 \oplus \mathfrak{G}_2$, where $\mathfrak{G}_1 = span\{X_1, \ldots, X_m\}$ and $\mathfrak{G}_2 = [\mathfrak{G}_1, \mathfrak{G}_1]$.

Let $u \in S_0^1(\Omega)$ be a solution of (P_α), trivially extended out of Ω. We decompose u as follows:

Define

$$w := \Gamma * f : \mathbb{G} \to \mathbb{R}, \quad w(\xi) = \int_{\mathbb{G}} \Gamma(\xi, \xi') f(\xi') \, d\xi', \tag{1}$$

where Γ is the fundamental solution of $-\Delta_{\mathbb{G}}$ and

$$f = \psi^\alpha \frac{|u|^{2^*(\alpha)-2} u}{d^\alpha}.$$

Moreover, let

$$v := w - u \tag{2}$$

Aim: to estimate the behavior of Yw and Yv at 0, where $Y \in \mathfrak{G}_2$ is any Lie derivative of order two.

1st step: estimate of second layer derivatives of $w = \Gamma * f$.

We recognize that

- $f \in L^1(\mathbb{G}) \cap L^{Q/\alpha,\infty}(\mathbb{G})$
- $f \in \Gamma^{\gamma}_{loc}(\Omega \setminus \{0\})$
- w solves the equation $-\Delta_{\mathbb{G}} w = f$ weakly in $\mathbb{G}$

Proposition (Representation formula for 2nd layer derivatives of w)

Let $\mathbb{G}$ be a Carnot group of step 2. Then, $w \in \Gamma^{2,\gamma}_{loc}(\Omega \setminus \{0\})$ and $\forall Y \in \mathfrak{G}_2$, it holds

$$Yw(\xi_0) = \int_{B_d(\xi_0,r)} Y\Gamma(\xi_0,\xi')(f(\xi')-f(\xi_0))\mathrm{d}\xi' + \int_{\mathbb{G}\setminus B_d(\xi_0,r)} Y\Gamma(\xi_0,\xi')f(\xi')\mathrm{d}\xi',$$

for any $\xi_0 \in \Omega \setminus \{0\}$ and $r > 0$ such that $B_d(\xi_0,r) \Subset \Omega \setminus \{0\}$.

Key point: $|Y\Gamma| \leq \dfrac{C}{d^Q}$

By using the representation formula for Yw, we get the following sharp asymptotic estimate at 0:

Proposition

Let $\mathbb{G}$ be a Carnot group of step two. If w is the function defined in (1) *and $Y \in \mathfrak{G}_2$ is a Lie derivative of order two, then*

$$|Yw(\xi)| = \mathcal{O}(d(\xi)^{-\alpha}) \quad \textit{as} \quad d(\xi) \to 0.$$

2nd step: estimate of second layer derivatives of v.

Recall that v is $\Delta_{\mathbb{G}}$-harmonic in $\Omega \setminus \{0\}$:

Lemma

Let $\mathbb{G}$ be a Carnot group and let $Y \in \mathfrak{G}_k$ be a Lie derivative of order k. If U is an arbitrary open set of $\mathbb{G}$ and v is a $\Delta_{\mathbb{G}}$-harmonic function on U, then there exists a positive constant c such that

$$|Yv(\xi)| \leq cr^{-k} \sup_{B_d(\xi,r)} |v|,$$

for every $B_d(\xi, r) \Subset U$. The constant c only depends on Y (and the structure of $\mathbb{G}$) and not on v, $r > 0$ or $\xi \in U$.

Proposition

Let $\mathbb{G}$ be a Carnot group and let $Y \in \mathfrak{G}_2$ be a Lie derivative of order two. Let v be the function defined in (2). Then,

$$|Yv(\xi)| = \mathcal{O}(d(\xi)^{-2}), \quad \textit{as} \quad d(\xi) \to 0.$$

Combining the previous estimates on w and v, we get

Theorem

Let $\mathbb{G}$ be a Carnot group of step two. Suppose that $\Omega \subset \mathbb{G}$ is an open set, $0 \in \Omega$, and let $u \in S_0^1(\Omega)$ be a weak solution of pb. (P_α). Then, for every $Y \in \mathfrak{G}_2$, we have

$$|Yu(\xi)| = \mathcal{O}(d(\xi)^{-2}) \quad \textit{as } d(\xi) \to 0.$$

The above estimate ensures $Yu \in L^2_{loc}$ at least for $Q > 4$.

The estimate is, however, sufficient to let us implement Pohozaev-type identities in the present singular Carnot setting.

Pohozaev-type identities

We start from the following Rellich-Pohozaev identity on groups

Theorem (Garofalo-Lanconelli, '92)

Let $\mathbb{G}$ be a Carnot group and let $\Omega \subset \mathbb{G}$ be a C^1 bounded open set with outer normal ν. For $u \in C^2(\overline{\Omega})$, it holds

$$\int_\Omega (-\Delta_{\mathbb{G}} u) Y u \, \mathrm{d}\xi = -\frac{1}{2} \int_\Omega divY \, |\nabla_{\mathbb{G}} u|^2 \, \mathrm{d}\xi + \sum_{i=1}^m \int_\Omega X_i u [X_i, Y] u \, \mathrm{d}\xi$$

$$+ \frac{1}{2} \int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^2 < Y, \nu > \mathrm{d}\sigma - \int_{\partial\Omega} \sum_{i=1}^m X_i u < X_i, \nu > Y u \mathrm{d}\sigma$$

where Y is any smooth vector field.

Observe that, when $\mathbb{G}$ has step 2, it holds for $u \in \Gamma^2(\overline{\Omega})$

Now, choose $Y = Z$, where Z denotes the infinitesimal generator of the one-parameter group of dilations δ_λ, i.e. $\left[\frac{d}{d\lambda} u(\delta_\lambda(\xi))\right]_{\lambda=1} = Zu$

Properties of the infinitesimal generator Z of the anisotropic dilations δ_λ

Euclidean case: $Z = x \cdot \nabla$

Stratified case: $Z = \sum_{i=1}^{r} \sum_{j=1}^{N_i} i\, x_j^{(i)} \dfrac{\partial}{\partial x_j^{(i)}}$.

Example: If $\mathbb{G} = \mathbb{R}^m \times \mathbb{R}^n$ is a Carnot group of step two, then

$$Z = \sum_{i=1}^{m} x_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} 2y_j \frac{\partial}{\partial y_j}$$

By choosing $X = Z$, the Rellich-Pohozaev identity becomes, for $u \in \Gamma^2(\overline{\Omega})$

$$\int_\Omega (-\Delta_{\mathbb{G}} u) Zu \, \mathrm{d}\xi = -\frac{Q-2}{2} \int_\Omega |\nabla_{\mathbb{G}} u|^2 \mathrm{d}\xi$$

$$+ \frac{1}{2} \int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^2 < Z, \nu > \mathrm{d}\sigma - \int_{\partial\Omega} \sum_{i=1}^{m} X_i u < X_i, \nu > Zu \, \mathrm{d}\sigma.$$

(we have used that $divZ = Q$ and $[X_i, Z] = X_i$, $\forall i = 1, \ldots, m$)

Nonexistence results on δ_λ-starshaped domains

Definition

Let $\Omega \subset \mathbb{G}$ be a C^1 connected open set, $0 \in \Omega$. We say that Ω is δ_λ-starshaped with respect to the origin if

$$< Z, \nu > (\xi) \geq 0 \quad \forall \xi \in \partial\Omega.$$

Theorem (L., 2015)

Let $\mathbb{G}$ be a Carnot group of step two. Let $\Omega \subset \mathbb{G}$ be a smooth connected bounded domain, δ_λ-starshaped about the origin. Then, the problem

$$-\Delta_{\mathbb{G}} u = \psi^\alpha \frac{u^{2^*(\alpha)-1}}{d(\xi)^\alpha} \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

has no nontrivial nonnegative weak solutions $u \in S_0^1(\Omega) \cap \Gamma^2(\overline{\Omega} \setminus \{0\})$.

Examples of domains to which the non-existence result applies.

The non-existence result apply, for instance, to the gauge balls in the Heisenberg group $\mathbb{H}^n$, i.e.

$$B_R(0) = \{(z,t) \in \mathbb{H}^n \mid d(\xi) = (|z|^4 + |t|^2)^{1/4} < R\}.$$

Here the Γ^2-regularity up to the boundary of solutions is ensured by Jerison's result ('81) and so the previous Pohozaev's arguments apply.

Future developments:

To obtain non-existence results for unbounded domains. In this case, we shall need to estimate the decay at ∞ of higher-layer derivatives of solutions.

Existence results

We consider a Brezis-Nirenberg type problem on $\mathbb{G}$:

Theorem (L., 2015)

Let $\mathbb{G}$ be a Carnot group of homogeneous dimension $Q > 3$ and let $\Omega \subset \mathbb{G}$ be a bounded domain, $0 \in \Omega$. Then, problem

$$\begin{cases} -\Delta_{\mathbb{G}} u = \psi(\xi)^\alpha \dfrac{|u|^{2^*(\alpha)-2}u}{d(\xi)^\alpha} + \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (P_{\alpha,\lambda})$$

admits at least one positive solution $u \in S_0^1(\Omega)$ for any $0 < \lambda < \lambda_1$.

- Euclidean case: Jannelli-Solimini ('96, '99), Caldiroli-Malchiodi (2002), Felli-Schneider (2005), Jannelli-L. (2014), etc...

Following the Euclidean scheme (Brezis-Nirenberg, '93): a sufficient condition for the existence of a positive solution to $(P_{\alpha,\lambda})$ when $0 < \lambda < \lambda_1$ is that

$$S_{\lambda,\alpha} := \inf_{u \in S_0^1(\Omega)} Q_\lambda(u) = \inf_{u \in S_0^1(\Omega)} \frac{\int_\Omega (|\nabla_{\mathbb{G}} u|^2 - \lambda u^2)\, \mathrm{d}\xi}{\left(\int_\Omega \psi^\alpha \frac{|u|^{2^*(\alpha)}}{d(\xi)^\alpha}\, \mathrm{d}\xi\right)^{2/2^*(\alpha)}} < S_\alpha,$$

(where S_α is the best Hardy-Sobolev constant), since this ensures that $S_{\lambda,\alpha}$ is achieved.

The above inequality is proved by calculating the ratio Q_λ in the one-parameter family

$$u_\epsilon = \phi U_\epsilon,$$

where U_ϵ are the Hardy-Sobolev extremals, and ϕ is a suitable cut-off function, and letting $\epsilon \to 0$.

In the Carnot setting, we do not know the expression of such extremals.

How to proceed?

Existence and asymptotic behavior of H-S. extremals

Key observation: the explicit expression is not necessary, some qualitative properties of H-S extremals are sufficient.

See: L., 2010, 2015 (Carnot case)
Jannelli, L. 2014 (polyharmonic Euclidean case)

Qualitative properties of HS extremals

- **Existence**: by Lions' concentration-compactness arguments in the stratified Lie context;
- **Asymptotic behavior at ∞**: HS extremals are positive entire solutions of

$$-\Delta_{\mathbb{G}} u = \psi^{\alpha} \frac{u^{2^*(\alpha)-1}}{d(\xi)^{\alpha}}, \quad u \in S^1(\mathbb{G}).$$

$\Rightarrow \quad u(\xi) \simeq d(\xi)^{2-Q}$ as $d(\xi) \to \infty$.

Let $U > 0$ be a fixed HS minimizer and consider, for $\varepsilon > 0$, the abstract family of *HS concentrating functions*

$$U_\varepsilon(\xi) = \varepsilon^{\frac{2-Q}{2}} U(\delta_{\frac{1}{\varepsilon}}(\xi)).$$

Let $R > 0$ be such that $B_d(0, R) \subset \Omega$ and let $\varphi \in C_0^\infty(B_d(0, R))$, $0 \le \varphi \le 1$, $\varphi \equiv 1$ in $B_d(0, R/2)$. We define

$$u_\varepsilon(\xi) = \varphi(\xi) U_\varepsilon(\xi).$$

The functions u_ε satisfy the following estimates, as $\varepsilon \to 0$:

$$\int_\Omega |\nabla_{\mathbb{G}} u_\varepsilon|^2 \, \mathrm{d}\xi = S_\alpha^{\frac{Q-\alpha}{2-\alpha}} + O(\varepsilon^{Q-2})$$

$$\int_\Omega \psi^\alpha \frac{u_\varepsilon^{2^*(\alpha)}}{d(\xi)^\alpha} \, \mathrm{d}\xi = S_\alpha^{\frac{Q-\alpha}{2-\alpha}} + O(\varepsilon^{Q-\alpha})$$

$$\int_\Omega u_\varepsilon^2 \, \mathrm{d}\xi = \begin{cases} c\,\varepsilon^2 + O(\varepsilon^{Q-2}) & \text{if } Q > 4 \\ c\,\varepsilon^2 |\ln \varepsilon| + O(\varepsilon^2) & \text{if } Q = 4. \end{cases}$$

By using the above expansions, we can conclude.

References

- A.L., *Critical growth problems with singular nonlinearities on Carnot groups*, Nonlinear Analysis **126** (2015)
- A.L., *L^p-weak regularity and asymptotic behavior of solutions for critical equations with singular potentials on Carnot groups*, NoDEA **17** (2010).
- E. Jannelli, A.L., *Critical polyharmonic problems with singular nonlinearities*, Nonlinear Analysis **110** (2014)
- E. Jannelli, S. Solimini, *Concentration estimates for critical problems*, Ricerche Mat. 48 (1999)
- A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, Stratified Lie groups and potential theory for their Sub-Laplacians, Springer Monographs in Mathematics. Springer, Berlin, 2007.

- N. Garofalo, E. Lanconelli, *Existence and nonexistence results for semilinear equations on the Heisenberg group*, Indiana Univ. Math. J. 41(1992), 71-98.
- N. Garofalo, D. Vassilev, *Regularity near the characteristic set in the non-linear Dirichlet problem and conformal geometry of sub-Laplacians on Carnot Groups*, Math. Ann. 318 (2000).
- E. Lanconelli, F. Uguzzoni, *Asymptotic behaviour and non existence theorems for semilinear Dirichlet problems involving critical exponent on unbounded domains of the Heisenberg group*, Boll. UMI (1998).
- E. Lanconelli, F. Uguzzoni, *Non-existence results for semilinear Kohn-Laplace equations in unbounded domains*, Comm. PDE's **25** (2000).