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The phase diagram of the Caffarelli-Kohn-Nirenberg inequalities

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Joint work with JEAN DOLBEAULT and MARIA ESTEBAN

Caffarelli-Kohn-Nirenberg inequalities (1984)

$$\int_{\mathbb{R}^d} \frac{|\nabla v(x)|^2}{|x|^{2a}} dx \geq C(d,a,b) \left(\int_{\mathbb{R}^d} \frac{|v(x)|^p}{|x|^{pb}} dx \right)^{2/p}$$

$$p = \frac{2d}{d-2+2(b-a)}$$

$C(d,a,b)$ **is the sharp constant**

$$a \leq b \leq a+1 \ , \ d \geq 3$$
$$a < b \leq a+1 \ , \ d = 2$$

Functional for the sharp constant.

Rotationally invariant!

$a = b = 0$, **Sobolev's inequality**

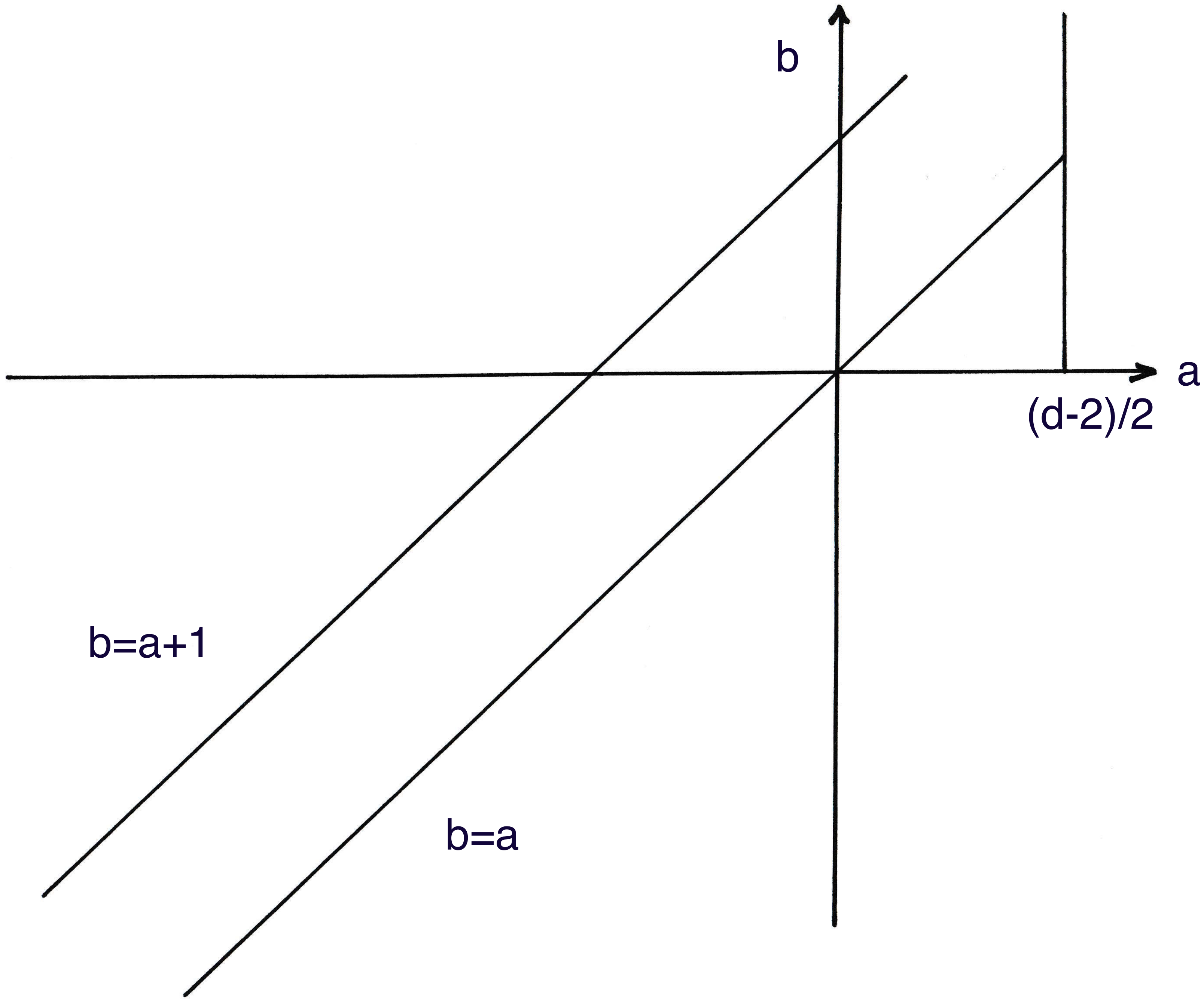
$a = 0, b = 1$, **Hardy's inequality**

$a = 0, 0 \le b < 1$, **Glaser, Martin, Grosse, Thirring (1976)**

In each case the sharp constant $C(d, a, b)$ **is known**

Existence of optimizers in the entire open strip

F. Catrina and Z.-Q. Wang (2001)


b
a
(d-2)/2
b=a+1
b=a

F. Catrina and Z.-Q. Wang (2001)

Rotational symmetry of the optimizer can be broken!

V. Felli and M. Schneider (2003)

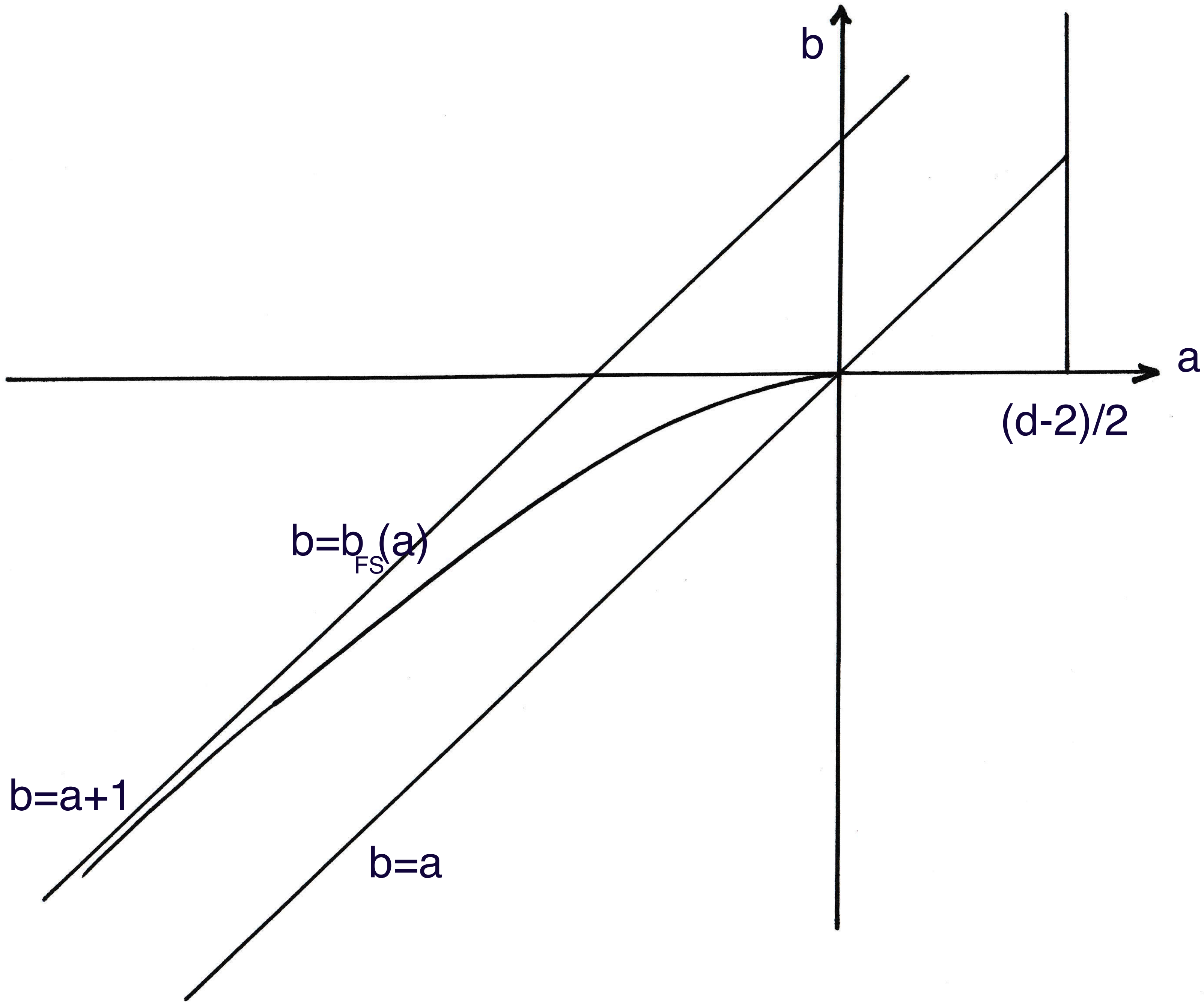
If

$$b < \frac{d(\frac{d-2}{2} - a)}{2\sqrt{(\frac{d-2}{2} - a)^2 + d - 1}} + a - \frac{d-2}{2} =: b_{\mathrm{FS}}(a)$$

then the optimizing function is not radial!

Follows from analyzing the second variation around

optimizing function in the radial class.



b
a
(d-2)/2
b=b_FS(a)
b=a+1
b=a

Symmetry results:

For $a, b \geq 0$ result follows using the symmetric decreasing rearrangement

For $b \geq 0$, the result follows by a rearrangement inequality due to Betta, Brock, Mercaldo and Posteraro (1999)

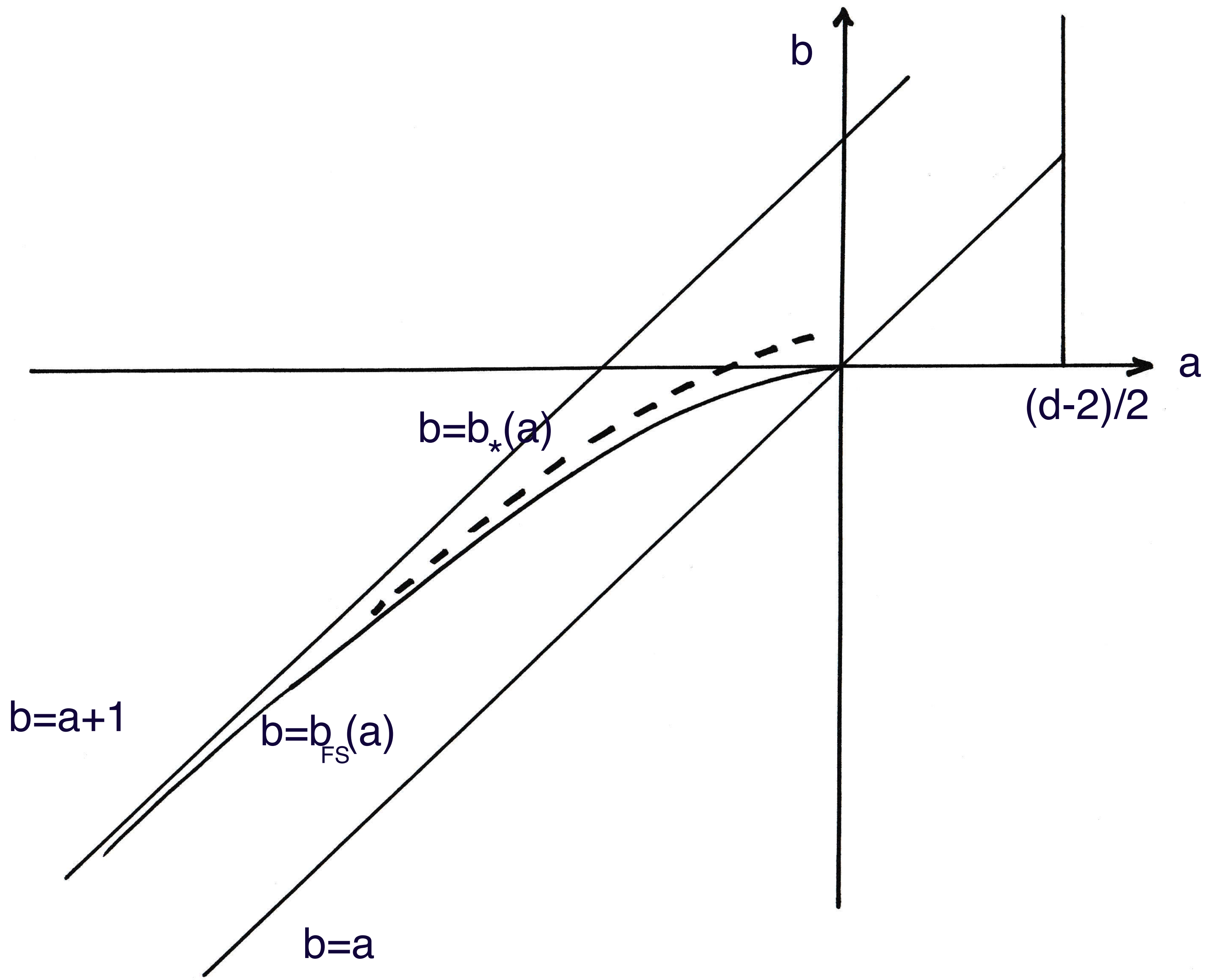
There is symmetry in the region

$$b \geq b_\star(a)$$

where

$$b_\star(a) = \frac{d(d-1) + 4d(a - \frac{d-2}{2})^2}{6(d-1) + 8(a - \frac{d-2}{2})^2} + a - \frac{d-2}{2}$$

Dolbeault, Esteban, Loss (2012)


b
a
(d-2)/2
b=b*(a)
b=a+1
b=bFS(a)
b=a

Theorem: Dolbeault-Esteban-Loss (2015)

If $d \geq 2$ **and**

$$b \geq b_{FS}(a) \ , \ a < 0$$

then the optimizers for the Caffarelli-Kohn-Nirenberg

inequalities are radial functions and are given by

$$\left(\frac{1}{A + B|x|^{2\alpha}}\right)^{\frac{n-2}{2}}$$

$$\alpha = \frac{(1 + a - b)(\frac{d-2}{2} - a)}{\frac{d-2}{2} - a + b} \ , \ n = \frac{2p}{p-2} = \frac{d}{1 + a - b}$$

A, B **are positive constants**

In the case of radial symmetry the sharp constant is given by

$$C(d,a,b) = |\mathbb{S}^{d-1}|^{\frac{p-2}{p}} \left[\frac{(a-a_c)^2\,(p-2)^2}{p+2}\right]^{\frac{p-2}{2\,p}} \left[\frac{p+2}{2\,p\,(a-a_c)^2}\right] \left[\frac{4}{p+2}\right]^{\frac{6-p}{2\,p}} \left[\frac{\Gamma\left(\frac{2}{p-2}+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2}{p-2}\right)}\right]^{\frac{p-2}{p}}$$

$$a_c = \frac{d-2}{2}$$

Note: For $a = b = 0$ the optimizers are not necessarily radial.

Ideas of the proof:

Reformulate CKN as a Sobolev inequality in n dimension

where n is not necessarily and integer.

$$\int_{\mathbb{R}_+\times\mathbb{S}^{d-1}} |\mathsf{D}w(r,\omega)|^2 r^{n-1}\,dr\,d\omega \geq$$

$$C(d,a,b)\,\alpha^{1-2/p}\left(\int_{\mathbb{R}_+\times\mathbb{S}^{d-1}} |w(r,\omega)|^p r^{n-1}\,dr\,d\omega\right)^{2/p}$$

$$\mathsf{D}w(r,\omega) = \left(\alpha\frac{dw}{dr},\frac{\nabla w}{r}\right)\ ,\ |\mathsf{D}w(r,\omega)|^2 = \alpha^2\left|\frac{dw}{dr}\right|^2 + \frac{|\nabla w|^2}{r^2}$$

$$n = \frac{2p}{p-2}\ \ ,\ \int_{\mathbb{S}^{d-1}} d\omega = 1$$

Follows from CKN by setting

$$w(r, \omega) = v(r^{1/\alpha}, \omega) \ .$$

$$\alpha = \frac{(1+a-b)(\frac{d-2}{2}-a)}{\frac{d-2}{2}-a+b} \ , n = \frac{2p}{p-2} = \frac{d}{1+a-b}$$

In these new variables the Felli-Schneider region is given by

$$\alpha > \sqrt{\frac{d-1}{n-1}} \,,$$

We always have

$$n \geq d \text{ , i.e. } \sqrt{\frac{d-1}{n-1}} \leq 1 \text{ , } p = \frac{2n}{n-2} \,.$$

Our Theorem can be rephrased:

If $d \geq 2$ **and** $\alpha \leq \sqrt{\frac{d-1}{n-1}}$ **and** $n > d$**, then the minimizer of**

$$\frac{\int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} |\mathsf{D}w(r,\omega)|^2 d\mu}{\left(\int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} |w(r,\omega)|^p d\mu\right)^{2/p}} \text{ , } d\mu = r^{n-1} dr\, d\omega$$

is a radial function

$$u = w^p \ , w \geq 0 \ , \ \mathsf{p} = \frac{m}{1-m} u^{m-1} \ , \ m = 1 - \frac{1}{n}$$

Total mass

$$\int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} w(r,\omega)^p d\mu = \int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} u(r,\omega) d\mu$$

Generalized Fisher information

$$\mathcal{I}(u) := \int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} u |\mathsf{D}\mathsf{p}|^2 d\mu$$

Lemma:

$$\frac{1}{4} \left(\frac{n-2}{n-1} \right)^2 \mathcal{I}(u) = \int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} |\mathsf{D} w(r,\omega)|^2 d\mu$$

Fast diffusion flow

$$\frac{\partial u}{\partial t} = \mathcal{L} u^m \ , m = 1 - \frac{1}{n}$$

$$\mathcal{L} = -\mathsf{D}^*\mathsf{D} = \alpha^2(\partial_r^2 + \frac{n-1}{r}\partial_r) + \frac{\Delta}{r^2}$$

Δ is the Laplace-Beltrami operator on the sphere.

Self similar solutions of Barenblatt type

$$u_\star(t, r, \omega) = t^{-n}\left(c_\star + \frac{r^2}{2(n-1)\alpha^2 t^2}\right)^{-n}$$

Mass is preserved and $c_\star$ chosen such that

$$\int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} u_\star(t, r, \omega) d\mu = 1 \ .$$

Lemma:

$$\frac{d}{dt}\int_{\mathbb{R}_+\times\mathbb{S}^{d-1}} u(t,r,\omega)d\mu = 0 \ ,$$

and

$$\frac{d}{dt}\mathcal{I}(u(t)) =$$

$$-2(n-1)^{n-1}\int_{\mathbb{R}_+\times\mathbb{S}^{d-1}}\left[\frac{1}{2}\mathcal{L}|\mathsf{Dp}|^2 - \mathsf{Dp}\cdot\mathsf{D}\mathcal{L}\mathsf{p} - \frac{1}{n}(\mathcal{L}\mathsf{p})^2\right]\mathsf{p}^{1-n}d\mu$$

reminiscent of the Bochner-Weitzenböck formula.

Lemma:

Set

$$k[\mathsf{p}] := \frac{1}{2}\mathcal{L}|\mathsf{Dp}|^2 - \mathsf{Dp}\cdot\mathsf{D}\mathcal{L}\mathsf{p} - \frac{1}{n}(\mathcal{L}\mathsf{p})^2$$

Then

$$k[\mathsf{p}] = \alpha^4(1-\frac{1}{n})\left[\mathsf{p}'' - \frac{\mathsf{p}'}{r} - \frac{\Delta\mathsf{p}}{\alpha^2(n-1)r^2}\right]^2 + 2\alpha^2\frac{1}{r^2}\left|\nabla\mathsf{p}' - \frac{\nabla\mathsf{p}}{r}\right|^2$$
$$+\frac{1}{r^4}\left[\frac{1}{2}\Delta|\nabla\mathsf{p}|^2 - \nabla\mathsf{p}\cdot\nabla\Delta\mathsf{p} - \frac{1}{n-1}(\Delta\mathsf{p})^2 - (n-2)\alpha^2|\nabla\mathsf{p}|^2\right]$$

Lemma:

Assume that $d \geq 3$ and that p is a positive function in $C^3(\mathbb{S}^{d-1})$. Then

$$\int_{\mathbb{S}^{d-1}} \left[\frac{1}{2}\Delta|\nabla \mathsf{p}|^2 - \nabla \mathsf{p} \cdot \nabla\Delta \mathsf{p} - \frac{1}{n-1}(\Delta \mathsf{p})^2 - (n-2)\alpha^2|\nabla \mathsf{p}|^2\right] \mathsf{p}^{1-n} d\omega$$

$$= \left[\frac{(n-2)(d-1)}{(d-2)(n-1)}\right] \int_{\mathbb{S}^{d-1}} \|L\mathsf{p} - \frac{n-1}{2}\frac{3(n-d)}{(d+1)(n-2)}M\mathsf{p}\|^2 \mathsf{p}^{1-n} d\omega$$

$$+(n-d)\left[\frac{1}{2(d+1)}\right]\left[\frac{n+3}{2} + \frac{3(n-1)(n+1)(d-2)}{2(d+1)(n-2)}\right] \int_{\mathbb{S}^{d-1}} \frac{|\nabla \mathsf{p}|^4}{\mathsf{p}^2} \mathsf{p}^{1-n} d\omega$$

$$+(n-2)\left[\frac{(d-1)}{(n-1)} - \alpha^2\right] \int_{\mathbb{S}^{d-1}} |\nabla \mathsf{p}|^2 \mathsf{p}^{1-n} d\omega$$

where

$$\mathrm{L}\mathsf{p} := \mathrm{H}\mathsf{p} - \frac{1}{d-1}\,(\Delta\mathsf{p})\,g\ ,$$

the trace free Hessian and

$$\mathrm{M}f := \frac{\nabla\mathsf{p}\otimes\nabla\mathsf{p}}{\mathsf{p}} - \frac{1}{d-1}\,\frac{|\nabla\mathsf{p}|^2}{\mathsf{p}}\,g\,.$$

Note that

$$\|M\mathsf{p}\|^2 = \frac{d-2}{d-1}\frac{|\nabla\mathsf{p}|^4}{\mathsf{p}^2}$$

Suppose that

$$\alpha \leq \sqrt{\frac{d-1}{n-1}}\ , n > d\ , \text{and}\ \left.\frac{\mathcal{I}(u(t))}{dt}\right|_{t=0} = 0$$

then

$$\mathsf{p}(r,\omega) = \mathsf{p}(r)$$

and

$$\mathsf{p}'' - \frac{\mathsf{p}'}{r} = r\left(\frac{\mathsf{p}'}{r}\right)' = 0 \rightarrow \mathsf{p} = A + Br^2\ .$$

Let $w \in L^p(\mathbb{R}_+ \times \mathbb{S}^{d-1}, d\mu)$ **be a positive solution of**

$$-\mathcal{L}w = w^{p-1}$$

$$u \to u_\varepsilon = u + \varepsilon \mathcal{L}u^m \ , \quad w \to w_\varepsilon = w + \varepsilon w^{-p+1}\mathcal{L}w^{pm}$$

$$-\int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} \mathcal{L}w[w^{-p+1}\mathcal{L}w^{pm}]d\mu = \int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} w^{p-1}[w^{-p+1}\mathcal{L}w^{pm}]d\mu = 0$$

$$-\int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} \mathcal{L}w[w^{-p+1}\mathcal{L}w^{pm}]d\mu = \frac{d}{dt}\Big|_{t=0}\mathcal{I}(u(t)) = 0$$

The boundary term in the integration by parts

$$\alpha^2\, r^{n-1} \int_{\mathbb{S}^{d-1}} \left(\frac{n}{2}\, u^m \left(\frac{|\mathsf{Dp}|^2}{\mathsf{p}} \right)' + \frac{2\,(n-1)^2}{n-2}\,(u^m)' \right) \Bigg|_r^R$$

requires a detailed regularity analysis

of the positive solutions of

$$-\mathcal{L} w = w^{p-1}$$

$$n > d!$$

Note that we have classified all

positive solution in L^p of the equation

$$-\mathcal{L}w = w^{p-1} \ .$$

Extensions of this result to

$$\mathbb{R}_+ \times \mathbb{S}^{d-1} \to \mathbb{R}_+ \times \mathcal{M}$$

where $\mathcal{M}$ is any compact smooth Riemannian manifold.

The result is then expressed in terms of the Ricci curvature.

Similar results also hold for $d = 2$.

Bidaut-Véron- Véron, 1991: $\mathcal{N}$ a compact Riemannian Manifold.
Any positive solution of

$$-\Delta_{\mathcal{N}} w + \lambda w = w^{p-1}$$

is constant under suitable assumptions on λ and the Ricci curvature.

The conditions are sharp if $\mathcal{N} = \mathbb{S}^d$.

In particular the inequality

$$\|\nabla w\|_2^2 \geq \frac{d}{p-2} \left(\|w\|_p^2 - \|w\|_2^2 \right)$$

is best possible.

$$\mathcal{N} = \mathbb{R} \times \mathcal{M}$$

Using logarithmic variables, our result implies

that any positive solution of

$$-\Delta_{\mathcal{N}}\phi + \lambda\phi = -\phi_{ss} - \Delta_{\mathcal{M}}\phi + \lambda\phi = \phi^{p-1} \; , \phi \in L^p(\mathcal{N})$$

is of the form

$$\phi(s,\omega) = \phi(s) \; ,$$

under suitable assumptions on λ and on the Ricci curvature of $\mathcal{M}$.

The conditions are sharp if $\mathcal{M} = \mathbb{S}^{d-1}$.

If $\Lambda \le 4\,\frac{d-1}{p^2-4}$**, the inequality**

$$\|\phi\|^2_{L^p(\mathbb{R}\times\mathbb{S}^{d-1})} \le C(d,a,b)\int_{\mathbb{R}\times\mathbb{S}^{d-1}}\left(|\partial_s\phi|^2+|\nabla\phi|^2+\Lambda\phi^2\right)ds\,d\omega$$

is best possible. There is equality if and only if up to translations in s **and multiplication by a constant,**

$$\phi(s,\omega)=\phi_*(s):=\frac{1}{\left[\cosh(B\,s)\right]^{\frac{2}{p-2}}}\quad\forall\,s\in\mathbb{R}$$

with $B=\frac{1}{2}\sqrt{\Lambda}\,(p-2)$**.**

For details see the paper on http://arxiv.org/abs/1506.03664

THANK YOU!