A non periodic and asymptotically linear indefinite variational problem in $\mathbb{R}^N$

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Main References

- L.A. Maia, J. C. Oliveira Junior and R. Ruviaro, *A non periodic and asymptotically linear indefinite variational problem in* $\mathbb{R}^N$, IUMJ, online 2015.
- L.A. Maia, J. C. Oliveira Junior and R. Ruviaro, *Nonautonomous and non periodic Schrödinger equation with indefinite linear part*, preprint 2015.

We study the existence of a nontrivial solution of the problem

$$-\Delta u + V(x)u = f(u) \quad \text{in} \quad \mathbb{R}^N ,$$

for $N \geq 3$, V a continuous potential non-periodic and which changes sign and has an asymptotic limit V_∞ as $|x| \to \infty$;
f is an asymptotically linear function as $|u| \to \infty$.

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Model problem

$$f(u) = \frac{u^3}{1 + u^2}.$$

Non autonomous problem

$$-\Delta u + V(x)u = f(x,u) \quad \text{in} \quad \mathbb{R}^N ,$$

Pankov e Pflüger (1998)

- V and f are continuous functions and periodic (in x);
- $F(x,s) := \int_0^s f(x,t)dt$ satisfies Ambrosetti-Rabinowitz condition;
- 0 is in the spectral gap of $-\Delta + V$.

Method: use aproximate problems in N-dimentional cubes of finite volumes which give a sequence of solutions, via a linking theorem, whose limit is a solution of the problem.

Kryszewski e Szulkin (1998)

- same hypotheses of Pankov and Pflüger, 1998.

Method: Degree Theory and a Topological version of the Linking Theorem.

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Jeanjean (1999)

- $V \equiv K$ and f is continuous and periodic (in x);
- $f(x,s)s^{-1} \to a > 0$ as $s \to +\infty$ uniformly in x;
- $K \in (0,a)$;
- If $G(x,s) = \frac{1}{2}f(x,s)s - F(x,s)$, then there exists $D \geq 1$:

$$G(x,s) \leq DG(x,t) \quad 0 \leq s \leq t.$$

Method: monotonicity trick; a sequence of functionals satisfying the Mountain Pass geometry.

Sirakov (2000)

- $V(x)$ may change sign and
  $\lambda_1 := \inf_{u \in H, \|u\|_2 = 1} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 \, dx > 0$
- f satisfies Ambrosetti-Rabinowitz condition.

Method: Mountain Pass Theorem.

Costa and Tehrani (2001)

- $V \equiv \lambda > 0$;
- $s \mapsto f(x,s)/s$ non-decreasing, $s > 0$;
- $f(x,s)s^{-1} \to g(x)$ if $s \to +\infty$;

Method: Mountain Pass Theorem, Nehari manifold, concentration-compactness principle and comparison with a limit problem.

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Szulkin e Weth (2009)

- V and f are continuous and periodic (in x);
- $F(x,s)/s^2 \to +\infty$ uniformly in x as $|s| \to +\infty$;
- $s \mapsto f(x,s)/|s|$ is strictly increasing $(-\infty, 0)$ e $(0, +\infty)$.

Method: Minimization on the generalized Nehari manifold introduced by Pankov in 2005.

Autonomous problem

Jeanjean e Tanaka (2002)

- V is continuous, $V(x) \to V_\infty > 0$ and $\inf V(x) > 0$;
- $f(s)s^{-1} \to a > 0$ if $s \to +\infty$;
- $a < \inf \sigma(-\Delta + V)$;
- F satisfies a NQ condition

Method: concentration-compactness principle of Lions, interaction with a limit problem and Mountain Pass Theorem.

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Azzollini e Pomponio (2009)

- $V \in C^1$ is radially symmetric, $\|(\nabla V(\cdot)(\cdot))^+\|_{N/2} < 2S$, $V \gneqq V_0$ and $V(x) \to V_0 > -m$ as $|x| \to +\infty$;
- f satisfies the general conditions of Berestycki and Lions;

Method: sequence of approximate functionals and Pohozaev manifold.

The potential V satisfies:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$, $-V_0 \leq V(x) \leq V_\infty$ for some constants $V_0, V_\infty > 0$ and $V(x) = V(|x|)$;

(V_2) $\lim_{|x| \to +\infty} V(x) = V_\infty$;

(V_3) $0 \notin \sigma(L)$ and $\inf \sigma(L) < 0$, where $\sigma(L)$ is the spectrum of the operator $L = -\Delta + V$.

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The nonlinearity f satisfies:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ and $\lim_{s \to 0} \dfrac{f(s)}{s} = 0$;

(f_2) $\lim_{|s| \to +\infty} \dfrac{|f(s)|}{|s|} = m > V_\infty$ and $\dfrac{|f(s)|}{|s|} < m$ for all $s \in \mathbb{R} \setminus \{0\}$;

(f_3) Setting $F(s) := \int_0^s f(t)dt$ and $Q(s) := \frac{1}{2} f(s)s - F(s)$, then for all $s \in \mathbb{R} \setminus \{0\}$,

$$F(s) \geq 0, \quad Q(s) > 0 \quad \text{and} \quad \lim_{s \to \infty} Q(s) = +\infty.$$

Main result

Theorem

Under assumptions $(V_1)-(V_3)$ *and* $(f_1)-(f_3)$ *equation* (P) *has a nontrivial radially symmetric weak solution* $u \in H^1(\mathbb{R}^N)$.

$E := H^1(\mathbb{R}^N)$, $I : E \to \mathbb{R}$:

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 + V(x)u^2\right)dx - \int_{\mathbb{R}^N} F(u)dx.$$

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Conditions (V_2) and (V_3) imply that the eigenvalue problem

$$-\Delta u + V(x)u = \lambda u, \quad u \in L^2(\mathbb{R}^N)$$

has a sequence of eigenvalues $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k < 0$: ϕ_i the eigenfunction corresponding to λ_i, $i \in \{1, 2, \cdots, k\}$, in $H^1(\mathbb{R}^N)$.

$$E^- := \text{span}\{\phi_i,\ i = 1, 2, \cdots, k\} \quad \text{and} \quad E^+ := \left(E^-\right)^\perp,$$

then $E = E^+ \oplus E^-$.
The essential spectrum of $-\Delta + V$ is the interval $[V_\infty, +\infty)$, and this implies that dim $E^- < \infty$.

Inner product $\langle \cdot, \cdot \rangle$ in E:

$$\langle u, v \rangle = \begin{cases} \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x) uv) dx & \text{if } u, v \in E^+, \\ -\int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x) uv) dx & \text{if } u, v \in E^-, \\ 0 & \text{if } u \in E^+ \text{ and } v \in E^-, \end{cases}$$

such that the corresponding norm $\| \cdot \|$ is equivalent to $\| \cdot \|_E$, the usual norm in $E = H^1(\mathbb{R}^N)$.
I may be written as

$$I(u) = \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 - \int_{\mathbb{R}^N} F(u) dx$$

for every function $u = u^+ + u^- \in E$.

Lemma

Let $(u_n) \subset E$ be a $(Ce)_c$ sequence, i.e. $I(u_n) \to c > 0$ and $\|I'(u_n)\|_{E^*}(1 + \|u_n\|) \to 0$ as $n \to +\infty$. Then, (u_n) is bounded, up to a subsequence.

Theorem

(Linking Theorem under the $(Ce)_c$ condition) Let $E = E^+ \oplus E^-$ be a Banach space with dim $E^- < \infty$. Let $R > \rho > 0$ and let $u \in E^+$ be a fixed element such that $\|u\| = \rho$. Define

$$\begin{aligned} M &:= \{w = tu + v^- : \|w\| \le R,\ t \ge 0,\ v^- \in E^-\}, \\ M_0 &:= \{w = tu + v^- : v^- \in E^-, \|w\| = R,\ t \ge 0 \text{ or } \\ &\qquad\qquad \|w\| \le R,\ t = 0\}, \\ N_\rho &:= \{w \in E^+ : \|w\| = \rho\}. \end{aligned}$$

Let $I \in C^1(E, \mathbb{R})$ be such that

$$b := \inf_{N_\rho} I > a := \max_{M_0} I.$$

Then $c \ge b$ and there exists a $(Ce)_c$sequence at level c for the functional I, where

$$c := \inf_{\gamma \in \Gamma} \max_{w \in M} I(\gamma(w)), \quad \Gamma := \{\gamma \in C(M, E) : \gamma|_{M_0} = Id\}.$$

- The limit problem in $\mathbb{R}^N$

$$-\Delta w + V_\infty w = f(w) \qquad (P_\infty)$$

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with $w \in E$;

- $I_\infty(w) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla w|^2 + V_\infty |w|^2)dx - \int_{\mathbb{R}^N} F(w)dx;$
- $u_0 \in E$ is the radial positive ground state solution of (P_∞).

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$$M = \{w = tu_0^+(\cdot - y) + v^-;\ \|w\| \leq R,\ t \geq 0,\ v^- \in E^-\}$$

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$$M_0 = \{w = tu_0^+(\cdot - y) + v^-;\ v^- \in E^-, \|w\| = R,\ t \geq 0 \\ \text{or } \|w\| \leq R,\ t = 0\}.$$

Lemma

There exist $R > 0$ and $y \in \mathbb{R}^N$ with $R, |y|$ sufficiently large such that

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Idea of the proof:

- $M_0 = M_1 \cup M_2$ of

$$M_1 = \{w = tu_0^+(\cdot - y) + v^-;\ v^- \in E^-, \|w\| \leq R,\ t = 0\}$$

and

$$M_2 = \{w = tu_0^+(\cdot - y) + v^-;\ v^- \in E^-, \|w\| = R,\ t \geq 0\}.$$

Proof of main result

I restricted to $H^1_{rad}(\mathbb{R}^N) = \tilde{E}^- \oplus \tilde{E}^+$. If $u \in \tilde{E}^+$ with $\|u\| = \rho$,

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(u)dx \geq \frac{1}{2}\rho^2 - \varepsilon\|u\|^2_{L^2(\mathbb{R}^N)} - C_\varepsilon\|u\|^p_{L^p(\mathbb{R}^N)}.$$

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By the Sobolev embeddings and the equivalence of the norms, there exist constants $C_3, C_4 > 0$ such that

$$I(u) \geq \frac{1}{2}\rho^2 - \varepsilon C_3\|u\|^2 - C_4\|u\|^p = \left(\frac{1}{2} - \varepsilon C_3\right)\rho^2 - C_4\rho^p.$$

So there exists $\rho_0 > 0$ satisfying

$$I(u) \geq \rho_0 > 0$$

for all $u \in \tilde{E}^+$ with $\|u\| = \rho$.

$$b := \inf_{N_\rho} I(u) \geq \rho_0 > 0 = a = \max_{M_0} I(u).$$

- The Linking Theorem ensures the existence of a Cerami sequence $(u_n) \subset H^1_{rad}(\mathbb{R}^N)$ at level $c > 0$ for I.
- Sequence (u_n) has a bounded subsequence, still denoted by (u_n), and there exists $u \in H^1_{rad}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ up to a subsequence.

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- $$I'(u_n - u) \to 0 \ \text{ if } \ n \to \infty.$$
- Since (u_n) is bounded, it follows that

$$\begin{aligned} o_n(1) = I'(u_n - u)(u_n - u) \;&=\; \|u_n^+ - u^+\|^2 - \|u_n^- - u^-\|^2 \\ &\quad - \int_{\mathbb{R}^N} f(x, u_n - u)(u_n - u)dx. \end{aligned}$$

- $\|u_n^+ - u^+\|^2 = o_n(1)$.
- Since $\|u_n^- - u^-\|^2 = o_n(1)$, then $u_n \to u$ in $H^1_{rad}(\mathbb{R}^N)$.

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$$o_n(1) = I'(u_n - u)(u_n - u) = \|u_n^+ - u^+\|^2 - \|u_n^- - u^-\|^2 - \int_{\mathbb{R}^N} f(x, u_n - u)(u_n - u)dx.$$

- $\|u_n^+ - u^+\|^2 = o_n(1)$.
- Since $\|u_n^- - u^-\|^2 = o_n(1)$, then $u_n \to u$ in $H^1_{rad}(\mathbb{R}^N)$.
- $I(u_n) \to c$ and $I'(u_n) \to 0$, imply $I(u) = c$ and $I'(u) = 0$.
- $c > 0$, $u \in H^1_{rad}(\mathbb{R}^N)$ is a nontrivial radial weak solution. By $V(x) = V(|x|)$ and the Principle of Symmetric Criticality, u is also a nontrivial radial weak solution of (P).

THANK YOU !