Some remarks on the concentration-compactness principle

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Introduction

In many cases, important special solutions (such as solitary waves, standing waves, stationary solutions) of nonlinear evolution equations are obtained as solutions of a minimization problem of the form

$$(\mathcal{P}_\lambda) \qquad \text{minimize } E(u) \text{ under the constraint } Q(u) = \lambda = \textit{constant},$$

where $E =$ "energy" and $Q =$ "charge," "mass," "momentum," etc.

Moreover, if E and Q are conserved quantities for the evolution equation and any minimizing sequence for the problem $(\mathcal{P}_\lambda)$ has a convergent subsequence, by a well-known result of Cazenave and Lions it follows that the set of solutions of $(\mathcal{P}_\lambda)$ is orbitally stable.

Existence of minimizers

We consider the problem

$$(\mathcal{P}_\lambda) \qquad \begin{array}{l} \text{Minimize } E(u) = \displaystyle\int_{\mathbb{R}^N} F(u(x), \nabla u(x))\, dx \\ \text{under the constraint} \quad Q(u) = \displaystyle\int_{\mathbb{R}^N} G(u(x), \nabla u(x))\, dx = \lambda, \end{array}$$

where $u : \mathbb{R}^N \longrightarrow \mathbb{R}^m$ belongs to some function space $\mathcal{X}$.

Notation: $E_{min}(\lambda) = \inf\{E(u) \mid u \in \mathcal{X}, Q(u) = \lambda\}$.

Aim: Prove the (pre)compactness of minimizing sequences for $(\mathcal{P}_\lambda)$: any sequence $(u_n)_{n\geq 1} \subset \mathcal{X}$ such that $Q(u_n) \longrightarrow \lambda$ and $E(u_n) \longrightarrow E_{min}(\lambda)$ has a convergent subsequence.

Main tool: Concentration-compactness principle (P.-L. Lions, 1984).

Concentration-Compactness Principle

CC Lemma. (P.-L. Lions) Let $(f_n)_{n\geq 1} \subset L^1(\mathbb{R}^N)$ be a sequence of **nonnegative** functions such that

$$\int_{\mathbb{R}^N} f_n \, dx \longrightarrow \alpha_0 > 0 \qquad \text{as } n \longrightarrow \infty.$$

There is a subsequence $(f_{n_k})_{k\geq 1}$ that satisfies one (and only one) of the following properties:

1. Concentration: There is $(y_k)_{k\geq 1} \subset \mathbb{R}^N$ such that for any $\varepsilon > 0$ there are $R_\varepsilon < \infty$ and $k_\varepsilon \in \mathbb{N}$ satisfying

$$\int_{B(y_k, R_\varepsilon)} f_{n_k} \, dx \geq \alpha_0 - \varepsilon, \qquad \forall k \geq k_\varepsilon.$$

Concentration-Compactness Principle

2. Vanishing: For any $R < \infty$ we have

$$\lim_{k\to\infty} \left(\sup_{y\in\mathbb{R}^N} \int_{B(y,R)} f_{n_k}\, dx \right) = 0.$$

3. Dichotomy: There is $\alpha \in (0, \alpha_0)$ and there are nonnegative functions $f_{k,1}, f_{k,2} \in L^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |f_{n_k} - f_{k,1} - f_{k,2}|\, dx \longrightarrow 0,$$

$$\int_{\mathbb{R}^N} f_{k,1}\, dx \longrightarrow \alpha \quad \text{and} \quad \int_{\mathbb{R}^N} f_{k,2}\, dx \longrightarrow \alpha_0 - \alpha,$$

$$\text{dist}(\text{supp}(f_{k,1}), \text{supp}(f_{k,2})) \longrightarrow \infty \quad \text{as } k \longrightarrow \infty.$$

In applications, one wants to eliminate vanishing and dichotomy.

How to rule out vanishing ?

Lieb's Lemma. Let $u \in L^1_{loc}(\mathbb{R}^N)$ be such that $\nabla u \in L^p(\mathbb{R}^N)$ and $\|\nabla u\|_{L^p} \leq C$. Assume that $meas(\{x \mid |u(x)| \geq \varepsilon\}) \geq \delta > 0$. There is a constant $\alpha = \alpha(N, p, C, \delta, \varepsilon)$ and there is $x_0 \in \mathbb{R}^N$ such that

$$meas(\{x \in B(x_0, 1) \mid |u(x)| \geq \varepsilon/2\}) \geq \alpha.$$

Lions' Lemma. Let $1 < p \leq \infty$, $1 \leq q < \infty$, $p^* = \frac{pN}{N-p}$ if $p < N$, $p^* = \infty$ if $p \geq N$. If $p < N$, we also assume $q \neq p^*$.
Assume that $(u_n)_{n\geq 1}$ is bounded in $L^q(\mathbb{R}^N)$ and $(|\nabla u_n|)_{n\geq 1}$ is bounded in $L^p(\mathbb{R}^N)$. If there is $R > 0$ such that

$$\lim_{n\to\infty} \left(\sup_{y\in\mathbb{R}^N} \int_{B(y,R)} |u_n|^q \, dx \right) = 0$$

then $u_n \longrightarrow 0$ in $L^r(\mathbb{R}^N)$ for any $r \in (\min(q, p^*), \max(q, p^*))$.

How to rule out vanishing ?

Remark. In the particular case when $E(u) = \int_{\mathbb{R}^N} |\nabla u|^2 + F(u)\,dx$ and $Q(u) = \int_{\mathbb{R}^N} G(u)\,dx$, O. Lopes ('97) gave necessary and sufficient conditions for the existence of vanishing minimizing sequences.

What about dichotomy ?

Proposition (M). Assume that dichotomy occurs for a sequence of **nonnegative** functions $(f_k)_{k\geqslant 1} \subset L^1(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} f_k \, dx \longrightarrow \alpha_0 > 0.$$

Fix an increasing function $\Phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\Phi(t) \geqslant 2t$.

Then there exist a subsequence of $(f_k)_{k\geqslant 1}$ (still denoted the same), $\alpha \in (0, \alpha_0)$, and sequences $(y_k)_{k\geqslant 1} \subset \mathbb{R}^N$ and $R_k \longrightarrow \infty$ such that

- $\displaystyle\int_{B(y_k, R_k)} f_k \, dx \longrightarrow \alpha$ and $f_k \mathbf{1}_{B(y_k,R_k)}$ "concentrates around $(y_k)_{k\geqslant 1}$."
- $\displaystyle\int_{B(y_k, \Phi(R_k)) \setminus B(y_k, R_k)} f_k \, dx \longrightarrow 0.$

Iterating this argument leads to a "profile decomposition" result.

What about dichotomy ?

Corollary (profile decomposition). Consider a sequence of **nonnegative** functions $(f_k)_{k\geqslant 1}$ which is bounded in $L^1(\mathbb{R}^N)$.

Fix an increasing function $\Phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying $\Phi(t) \geqslant 2t$.

There is a finite or countable family of sequences $(y_k^i)_{k\geqslant 1} \subset \mathbb{R}^N$ and $R_k^i \longrightarrow \infty$ as $k \longrightarrow \infty$ such that

- $\displaystyle\int_{B(y_k^i,R_k^i)} f_k\,dx \to \alpha_i > 0$ and $f_k\big|_{B(y_k^i,R_k^i)}$ "concentrates around $(y_k^i)_{k\geqslant 1}$."
- $\displaystyle\int_{B(y_k^i,\Phi(R_k^i))\setminus B(y_k^i,R_k^i)} f_k\,dx \leqslant \frac{1}{2^{k+i}}$.
- $B(y_k^i,\Phi(R_k^i)) \cap B(y_k^j,\Phi(R_k^j)) = \emptyset$ if $i \neq j$.
- $f_k = \displaystyle\sum_{i\geqslant 1} f_k \mathbf{1}_{B(y_k^i,R_k^i)} + g_k$, where $(g_k)_{k\geqslant 1}$ is a "vanishing" sequence.

Remark. The same result is valid for bounded sequences of measures on arbitrary metric spaces.

How to rule out dichotomy?

Come back to the problem

$$(\mathcal{P}_\lambda) \quad \begin{array}{l} \text{Minimize } E(u) = \displaystyle\int_{\mathbb{R}^N} F(u(x), \nabla u(x))\, dx \\ \text{under the constraint} \quad Q(u) = \displaystyle\int_{\mathbb{R}^N} G(u(x), \nabla u(x))\, dx = \lambda. \end{array}$$

Recall that $E_{min}(\lambda) = \inf\{E(u) \mid u \in \mathcal{X}, Q(u) = \lambda\}$.

Remark 1 (P.-L. Lions) The function E_{min} is **subadditive:**

$$E_{min}(a+b) \leq E_{min}(a) + E_{min}(b).$$

Proof. Let $\varepsilon > 0$. Consider u and v with compact support such that $Q(u) = a$, $E(u) < E_{min}(a) + \frac{\varepsilon}{2}$, $Q(v) = b$, $E(v) < E_{min}(b) + \frac{\varepsilon}{2}$. Let x be such that the supports of u and $v(x + \cdot)$ are disjoint. Then $Q(u + v(x+\cdot)) = a + b$ and $E(u + v(x + \cdot)) \leq E_{min}(a) + E_{min}(b) + \varepsilon$.

How to rule out dichotomy?

Remark 2 (P.-L. Lions) Dichotomy cannot occur for any minimizing sequence of $(\mathcal{P}_\lambda)$ **iff** E_{min} is **strictly subadditive** at level λ:

$$\forall \alpha < \lambda, \quad E_{min}(\lambda) < E_{min}(\alpha) + E_{min}(\lambda - \alpha).$$

Indeed, assume that for a minimizing sequence u_n we have, after passing to a subsequence, $u_n \sim u_{n,1} + u_{n,2}$, with $Q(u_{n,1}) \longrightarrow \alpha$, $Q(u_{n,2}) \longrightarrow \lambda - \alpha$. Then

$$E(u_n) \sim E(u_{n,1}) + E(u_{n,2})$$

and in the limit $n \longrightarrow \infty$ we get $E_{min}(\lambda) \geq E_{min}(\alpha) + E_{min}(\lambda - \alpha)$.

How to rule out dichotomy?

Except some cases where the functionals are homogeneous, it is difficult to compute E_{min} or to prove directly the strict subadditivity (cf. P.-L. Lions, *On some minimization problems in Mathematical Physics: how to check strict subadditivity conditions, 1989*).

How to rule out dichotomy?

Theorem (M.) Assume that:

- $\lim_{\lambda \to 0} E_{min}(\lambda) = E_{min}(0) = 0$.
- Functions in $\mathcal{X}$ tend to a constant at infinity.
- F and G have a common one-directional symmetry, for instance $F(u, \xi_1, \ldots, -\xi_N) = F(u, \xi_1, \ldots, \xi_N)$ and $G(u, \xi_1, \ldots, -\xi_N) = G(u, \xi_1, \ldots, \xi_N)$.

Then:

a) E_{min} is concave.

b) If there is $\lambda \in (0, \lambda_0)$ such that $E_{min}(\lambda) + E_{min}(\lambda_0 - \lambda) = E_{min}(\lambda_0)$ then E_{min} is linear on $[0, \lambda_0]$ and there is a *vanishing* minimizing sequence for $(\mathcal{P}_{\lambda_0})$.

Corollary. No vanishing for $(\mathcal{P}_{\lambda_0}) \Rightarrow$ no dichotomy for $(\mathcal{P}_{\lambda_0})$!

How to rule out dichotomy?

Part b) is trivial. The concavity of E_{min} implies that

$E_{min}(\lambda) \geq \frac{\lambda}{\lambda_0} E_{min}(\lambda_0)$ and $E_{min}(\lambda_0 - \lambda) \geq \frac{\lambda_0 - \lambda}{\lambda_0} E_{min}(\lambda_0)$ for all $\lambda \in]0, \lambda_0[$.

We may have equality if and only if E_{min} is linear on $[0, \lambda_0]$.

Fix $n \in \mathbb{N}^*$. Let $v_n \in \mathcal{X}$ be a function with compact support such that $Q(v_n) = \frac{\lambda_0}{n}$ and $E(v_n) \leq E_{min}(\frac{\lambda_0}{n}) + \frac{1}{n^2} = \frac{1}{n} E_{min}(\lambda_0) + \frac{1}{n^2}$.

Choose $x_1, \dots, x_n \in \mathbb{R}^N$ "far away from each other," such that the supports of $v_n(\cdot + x_j)$ et $v_n(\cdot + x_k)$ are disjoint for $j \neq k$. Let

$u_n = v_n(\cdot + x_1) + \cdots + v_n(\cdot + x_n)$.

Then $Q(u_n) = nQ(v_n) = \lambda_0$, $E(u_n) = nE(v_n) \leq E_{min}(\lambda_0) + \frac{1}{n}$ and $(u_n)_{n \geq 1}$ is a vanishing minimizing sequence. QED

Nonexistence of minimizers

Proposition (M.) We consider the same assumptions as in the previous theorem.

a) Assume that any minimizer for $(\mathcal{P}_\lambda)$ is a C^1 function. If E_{min} is linear on $[0, \lambda_0]$, then for all $\lambda \in]0, \lambda_0]$, problem $(\mathcal{P}_\lambda)$ cannot have minimizers.

b) Suppose, moreover, that F and G have two common directions of symmetry. If E_{min} is affine on the interval $]a, b[$, where $a > 0$, then for any $\lambda \in]a, b[$, $(\mathcal{P}_\lambda)$ does not admit minimizers.

Nonexistence of minimizers

Corollary. In addition to the assumptions of the previous Proposition, part (b), suppose that "concentration $\Rightarrow$ compactness." Let

$$\Lambda_0 = \sup\{\lambda > 0 \mid E_{min} \text{ is linear on } [0, \lambda]\}.$$

Then:

• Problem $(\mathcal{P}_\lambda)$ does not admit minimizers for any $\lambda \in [0, \Lambda_0]$ and all minimizing sequences have the vanishing property.

• E_{min} is strictly concave on $[\Lambda_0, \infty)$, problem $(\mathcal{P}_\lambda)$ has minimizers for $\lambda \in (\Lambda_0, \infty)$ and, moreover, all minimizing sequences have convergent subsequences.

Some nonlocal functionals

The previous results can be extended to some minimization problems involving nonlocal functionals (but the constraint has to be **local**, $Q(u) = \int_{\mathbb{R}^N} G(u(x), \nabla u(x))\, dx$):

1. Fractional powers of the Laplacian

$$E_1(u) = \int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{u}(\xi)|^2 \, d\xi + \int_{\mathbb{R}^N} F(u(x), \nabla u(x))\, dx, \ s \in (0,1) \qquad \text{or}$$

$$E_2(u) = \int_{\mathbb{R}^N} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 \, d\xi + \int_{\mathbb{R}^N} F(u(x), \nabla u(x))\, dx, \ s \in (0,1).$$

2. Generalized Choquard functional:

$$E_3(u) = \int_{\mathbb{R}^N} F(u(x), \nabla u(x))\, dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{H(u(x))H(u(y))}{|x-y|^{N-2}}\, dx\, dy, \qquad N \geq 3.$$

"Classical" Choquard functional:

$$E_C(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \, dx - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\, dx\, dy.$$

Some nonlocal functionals

3. Davey-Stewartson functional:

$$E_4(u) = \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} F(u(x), \nabla u(x))\, dx - \int_{\mathbb{R}^3} \frac{\xi_1^2}{|\xi|^2} |\widehat{u^2}(\xi)|^2 \, d\xi.$$

4. Functionals arising in the theory of solitary water waves (cf. Buffoni & Toland):

$$E_5(u) = \int_{\mathbb{R}} \xi \coth(\xi) |\widehat{u}(\xi)|^2 \, d\xi + \int_{\mathbb{R}} F(u(x), u'(x))\, dx \quad \text{or}$$

$$\tilde{E}_5(u) = \int_{\mathbb{R}^N} \frac{\xi_1^2}{|\xi|} \coth(\xi_1) |\widehat{u}(\xi)|^2 \, d\xi + \int_{\mathbb{R}^N} F(u(x), \nabla u(x))\, dx \quad .$$

Some nonlocal functionals

As before, the main step is to prove the concavity of the function

$$E^i_{min}(\lambda) = \inf\{E_i(u) \mid u \in \mathcal{X}, Q(u) = \lambda\}.$$

We need two more ingredients:

1. The fact that, if dichotomy occurs, i.e. a minimizing sequence $(u_n)_{n\geq 1}$ "splits" into two parts $(u_{n_1})_{n\geq 1}$ and $(u_{n_2})_{n\geq 1}$, then they are sufficiently "far away" from each other. This follows from the "profile decomposition" result for bounded sequences of measures.

2. A commutator estimate: if the nonlocal term in the functional can be written as $\|Lu\|^2_{L^2}$, where L is a pseudodifferential operator, we need to estimate the commutator $[L, \chi]u = (L\chi - \chi L)u$ in L^2. This can be done thanks to the work of Coifman & Meyer (1978), Bona, Albert & Saut (1997), M. Taylor (2001)...

Lagrange multipliers

If u is a solution of $(\mathcal{P}_\lambda)$, E and Q are differentiable at u and $Q'(u) \neq 0$, there exists $\beta = \beta(u)$ such that

$$E'(u) = \beta(u) Q'(u).$$

Question: Is it possible to choose λ in order to get a specific value for β? Can we reach all β in some interval?

We know: $\beta \in [d_r E_{min}(\lambda), d_\ell E_{min}(\lambda)]$. Moreover, if minimizing sequences are compact, then there exist 2 minimizers u_1 and u_2 such that $\beta(u_1) = d_r E_{min}(\lambda)$ and $\beta(u_2) = d_\ell E_{min}(\lambda)$.

Affirmative answer if: (a) E_{min} is differentiable throughout, or (b) for each λ, the set of minimizers is (in some sense) connected. Both conditions seem very hard to check.

Lagrange multipliers

Example.

Minimize $\int_{\mathbb{R}^N} |\nabla u|^p + F(u)\, dx$ under the constraint $\int_{\mathbb{R}^N} G(u)\, dx = \lambda$.

Minimizers satisfy the Pohozaev identity:

$(N-p)\int_{\mathbb{R}^N} |\nabla u|^p\, dx + N\int_{\mathbb{R}^N} F(u)\, dx - N\beta(u)\int_{\mathbb{R}^N} G(u)\, dx = 0.$

If u and v are solutions (with the same λ) and $\int_{\mathbb{R}^N} |\nabla u|^p\, dx = \int_{\mathbb{R}^N} |\nabla v|^p\, dx$, the Pohozaev identities imply $\beta(u) = \beta(v)$.

Question: If u and v are minimizers and $\int_{\mathbb{R}^N} |\nabla u|^p\, dx < \int_{\mathbb{R}^N} |\nabla v|^p\, dx$, can we construct another minimizer w such that

$$\int_{\mathbb{R}^N} |\nabla u|^p\, dx < \int_{\mathbb{R}^N} |\nabla w|^p\, dx < \int_{\mathbb{R}^N} |\nabla v|^p\, dx?$$

Lagrange multipliers

Example. Let $p \in (2, 2^*)$. Let

$$I_p(\lambda) = \inf\Big\{ \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 dx \mid u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^p\, dx = \lambda \Big\}.$$

It is easy to see that $I_p(\lambda) = \lambda^{\frac{2}{p}} I_p(1)$ and minimizers exist.

The Lagrange multiplier at level λ is $\frac{2}{p}\lambda^{\frac{2}{p}-1} I_p(1)$.

Let $G(s) = \begin{cases} s^p \text{ if } s \geq 0, \\ |s|^q \text{ if } s < 0. \end{cases}$ Then G is C^2, convex. Problem $(\mathcal{P}_\lambda)$:

$$\text{minimize } \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 dx \text{ under the constraint } \int_{\mathbb{R}^N} G(u)\, dx = \lambda.$$

Take λ_* such that $\lambda_*^{\frac{2}{p}} I_p(1) = \lambda_*^{\frac{2}{q}} I_q(1)$. If u is a solution of $(\mathcal{P}_{\lambda_*})$, then:

- either $u = u_+$ minimizes $I_p(\lambda_*)$, Lagrange multiplier$= \frac{2}{p}\lambda_*^{\frac{2}{p}-1} I_p(1)$,
- or $u = u_-$ minimizes $I_q(\lambda_*)$ and Lagrange multiplier$= \frac{2}{q}\lambda_*^{\frac{2}{q}-1} I_p(1)$.

Symmetry of minimizers

Question: Assume that the functionals E and Q are symmetric.
Is it true that the minimizers of $(\mathcal{P}_\lambda)$ are also symmetric?

Symmetry of minimizers

Consider the minimization problem

$$(\mathcal{P}_\lambda) \qquad \begin{array}{l} \text{Minimize } E(u) = \displaystyle\int_{\mathbb{R}^N} F(u(x), |\nabla u(x)|)\, dx \text{ in } \mathcal{X} \\ \text{under the constraint} \quad Q(u) = \displaystyle\int_{\mathbb{R}^N} G(u(x), |\nabla u(x)|)\, dx = \lambda. \end{array}$$

Theorem (M., ARMA 2009) Assume that:

- Functions in $\mathcal{X}$ tend to a constant at infinity.
- $(\mathcal{P}_\lambda)$ admits minimizers and *any* minimizer is C^1.

Then, after a translation, any minimizer is radially symmetric.

Remark 1. This result is valid for vector-valued minimizers, without any assumption on the sign of the components.

Remark 2. There are several generalizations: multiple constraints, dependence on $|x|$, partial symmetry...

Proof in the case $N = 2$

Lemma. Let u be a minimizer. Assume in addition that any straight line Π containing the origin O satisfies:

$$(1) \quad \int_{\Pi^+} G(u(x), |\nabla u(x)|)\, dx = \int_{\Pi^-} G(u(x), |\nabla u(x)|)\, dx = \frac{\lambda}{2}.$$

Then u radially symmetric with respect to O.

Proof: Let Π be any straight line containing O.

Choose a coordinate system such that $\Pi = Oy$. Let

$$v_1(x,y) = \begin{cases} u(x,y) \text{ if } x \leq 0 \\ u(-x,y) \text{ if } x > 0, \end{cases} \qquad v_2(x,y) = \begin{cases} u(-x,y) \text{ if } x < 0 \\ u(x,y) \text{ if } x \geq 0. \end{cases}$$

Proof for $N = 2$

Then $v_1, v_2 \in \mathcal{X}$ and

- $Q(v_1) = Q(v_2) = \lambda \quad \Rightarrow \quad E(v_1) \geq E(u), \;\; E(v_2) \geq E(u)$
- $E(v_1) + E(v_2) = 2E(u)$

$\Rightarrow \; v_1, \; v_2$ are also minimizers $\Rightarrow \; v_1, \; v_2 \in C^1(\mathbb{R}^2)$.

The symmetry with respect to x implies $\dfrac{\partial v_1}{\partial x}(0, y) = \dfrac{\partial v_2}{\partial x}(0, y) = 0$ for all y. Since $u = v_1$ for $x < 0$,

$$\frac{\partial u}{\partial x}(0, y) = \lim_{s \uparrow 0} \frac{\partial u}{\partial x}(s, y) = \lim_{s \uparrow 0} \frac{\partial v_1}{\partial x}(s, y) = \frac{\partial v_1}{\partial x}(0, y) = 0.$$

We proved that for any line Π containing O,

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Pi, \text{ where } n \text{ is the normal to } \Pi.$$

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ we have $\dfrac{\partial u}{\partial \theta} = 0$
$\Rightarrow \; u$ does not depend on θ, i.e. u is radial. QED

Proof for $N = 2$

Let u be a minimizer for $(\mathcal{P})$. After translation, we may assume that

$$\int_{\{x<0\}} G(u, |\nabla u|)\, dx\, dy = \int_{\{x>0\}} G(u, |\nabla u|)\, dx\, dy = \frac{\lambda}{2}.$$

$u_1, u_2 =$ the two functions obtained from u by mirror symmetry with respect to Oy. Then u_1, u_2 are minimizers and are symmetric in x.

After translation in the y direction, we may assume that

$$\int_{\{y<0\}} G(u_1, |\nabla u_1|)\, dx\, dy = \int_{\{y>0\}} G(u_1, |\nabla u_1|)\, dx\, dy = \frac{\lambda}{2}.$$

$u_{1,1}, u_{1,2} =$ functions obtained from u_1 by mirror symmetry / Ox.

Then $u_{1,1}, u_{1,2}$ are minimizers and are even w.r.t. x **and** w.r.t. y.

Lemma $\Rightarrow u_{1,1}$ and $u_{1,2}$ are radial with respect to O.

Since $u_{1,1}(x,0) = u_1(x,0) = u_{1,2}(x,0)$ for all x

$\Rightarrow u_{1,1} = u_{1,2} = u_1 \quad \Rightarrow u_1$ is radial.

Proof for $N = 2$

Similarly there is $k \in \mathbb{R}$ such that

$$\int_{\{y<k\}} G(u_2, |\nabla u_2|)\, dx\, dy = \int_{\{y>k\}} G(u_2, |\nabla u_2|)\, dx\, dy = \frac{\lambda}{2}.$$

Same argument $\Rightarrow$ u_2 is radial with respect to $(0, k)$.

If $k \neq 0$: $u(0, \cdot) = u_1(0, \cdot) = u_2(0, \cdot)$ is symmetric with respect to 0 and k $\Rightarrow$ $u(0, \cdot)$ is $2|k|-$periodic. Then $G(u_1, |\nabla u_1|)$ is a radial function and its profile is $2|k|-$periodic. If $\int_{\mathbb{R}^2} G(u_1, |\nabla u_1|)\, dx\, dy$ converges, its value is necessarily 0 $\Rightarrow$ $\lambda = 0$, absurd.

Conclusion: We have $k = 0$ $\Rightarrow$ u_2 is radial with respect to O.

Since $u_1(0, \cdot) = u(0, \cdot) = u_2(0, \cdot)$

$\Rightarrow$ $u_1 = u_2 = u$ $\Rightarrow$ u is radial. QED

Thank you very much for your attention!