Ground states of time-harmonic semilinear Maxwell equations

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Overview

Physical motivation

Well-known results and cylindrical symmetry

Nonsymmetric problem

Critical point theory and analysis of (PS)-sequences

Maxwell equations

$$\begin{aligned}
&\nabla \times H = J + \tfrac{\partial}{\partial t} D \quad \text{(Ampere's law)}\\
&\operatorname{div}(D) = \rho\\
&\tfrac{\partial}{\partial t} B + \nabla \times E = 0 \quad \text{(Faraday's law)}\\
&\operatorname{div}(B) = 0,
\end{aligned}$$

$E, B, D, H : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}^3$, E - electric field, B - magnetic field, D - electric displacement field, H - magnetizing field, J - electric current intensity, ρ - electric charge density.

$$\begin{aligned}
D &= \varepsilon E + P\\
H &= \tfrac{1}{\mu} B - M
\end{aligned}$$

$P, M : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}^3$ polarization field and magnetization field, $\varepsilon, \mu : \mathbb{R}^3 \to \mathbb{R}$ permittivity and permeability of the material.

Assumption

$J = \rho = M = 0$

$$\nabla \times \left(\frac{1}{\mu}\nabla \times E\right) + \varepsilon \frac{\partial^2}{\partial t^2} E = -\frac{\partial^2}{\partial t^2} P. \tag{1}$$

If $\mathrm{div}\,(E) = 0$ then $\nabla \times (\nabla \times E) = \nabla(\mathrm{div}\,(E)) - \Delta E = -\Delta E$, wave equation.

TE, TM modes, $P(E) = W(|E|^2)E$, $\lim_{s\to\infty} W(s) < \infty$

- C.A. Stuart, H.S. Zhou, a series of papers 1991–2010, ODE formulation of (1)

$E(x,t) = \Re\{E(x)e^{i\omega t}\}$, $P(x,t) = \Re\{P(x)e^{i\omega t}\}$.

The time-harmonic Semilinear Maxwell equation

$$\nabla \times (\nabla \times E) + V(x)E = f(x,E) \text{ in } \mathbb{R}^3,$$

where $E : \mathbb{R}^3 \to \mathbb{R}^3$, $V(x) = -\mu\omega^2\varepsilon(x) \le 0$ and $f(x,E) = \mu\omega^2 P(E)$. $f(x,E) = \partial_E F(x,E)$. In Kerr-like media $F(x,E) \sim |E|^4$.

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In general, $F(x, E) \sim |E|^p$, for $|E| \geq 1$, $2 < p < 6 = 2^*$.

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Born-Infeld theory: $V(x) = 0$, $F(x,E) = W(|E|^2)$

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$A = E$ gauge potential related to magnetic field $H = \nabla \times A$.

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$E \in \mathcal{D}^{1,2}(\mathbb{R}^3,\mathbb{R}^3)$, $\mathcal{G}$ acts on $\mathcal{D}^{1,2}(\mathbb{R}^3,\mathbb{R}^3)$,

$$\begin{aligned} \mathrm{Fix}\mathcal{G} &\subset \mathcal{D}^{1,2}(\mathbb{R}^3,\mathbb{R}^3) \cap \{\mathrm{div}\,(E) = 0\}, \\ (\nabla \times (\nabla \times E) =) - \Delta E &= 2W'(|E|^2)E \text{ in } \mathbb{R}^3 \qquad (2) \end{aligned}$$

and standard variational methods apply.

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If $V = V(r, x_3)$, $F = F((r, x_3), |E|)$ with $r = \sqrt{x_1^2 + x_2^2}$ are cylindrically symmetric, then we can solve $(\mathcal{P})$.

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Time-harmonic EM fields, Ω is a bounded domain of $\mathbb{R}^3$

$$\begin{cases} \nabla \times (\nabla \times E) + \lambda E = f(x,E) & \text{in } \Omega, \\ \nu \times E = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\nu : \partial\Omega \to \mathbb{R}^3$ is the exterior normal.

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Time-harmonic EM fields

- T. Bartsch, J.M.: ARMA 2015, arXiv:1509.01994: Ω bounded
- T. Bartsch, T. Dohnal, M. Plum, W. Reichel: arXiv:1411.7153: cylindrical symmetry and $0 \notin \partial\sigma$
- J.M.: ARMA 2015

Time-harmonic Semilinear Maxwell equation

In general, $F(x,E) \sim |E|^p$, for $|E| \geq 1$, $2 < p < 6 = 2^*$,
$F(x,E) \sim 0$, for $|E| << 1$

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Example

$$f(x,E) = \Gamma(x) \min\{|E|^{p-2}, |E|^{q-2}\}E, \quad 2 < p \leq q,$$

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Theorem (J.M. (2015))

If $V = 0$ *and* $E = u + \nabla w$ *is a classical solution such that* $\mathrm{div}\,(u) = 0$,

$$u \in \mathcal{C}^2(\mathbb{R}^3, \mathbb{R}^3),\ w \in \mathcal{C}^2(\mathbb{R}^3) \qquad (2)$$

and

$$F(E),\ \langle f(E), \nabla w\rangle \text{ and } |f(E)||w| \in L^1(\mathbb{R}^3), \qquad (3)$$

then

$$\int_{\mathbb{R}^3} |\nabla \times E|^2\, dx = 6 \int_{\mathbb{R}^3} F(E)\, dx. \qquad (4)$$

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Corollary

Suppose that Γ *is constant and positive.*
(a) If $V = 0$, *and* $2 < p \leq q < 6$ *or* $6 < p \leq q$, *then there is no classical solution .*
(b) If V *is constant and negative,* $2 < p \leq q \leq 6$, *then there is no classical solution.*

Our assumptions: $2 < p < 6 < q$, $V(x) \leq 0$ is vanishing.

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Energy functional

$$\mathcal{J}(E) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \times E|^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|E|^2 \, dx - \int_{\mathbb{R}^3} F(x, E) \, dx,$$

and critical points correspond to solutions to (2).

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First difficulties

- $\nabla \times (\nabla \varphi) = 0$ for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$, $\mathcal{J}$ is strongly indefinite and its critical points have infinite Morse index.

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Our goals:

- Find a variational setting.
- The geometry and the regularity of $\mathcal{J}$.
- Find a ground state solution.

Assumptions: 2<p<6<q

(V) $V \in L^{\frac{p}{p-2}}(\mathbb{R}^3) \cap L^{\frac{q}{q-2}}(\mathbb{R}^3)$, $V \leq 0$ a.e. on $\mathbb{R}^3$ and $|V|_{\frac{3}{2}} < S$, where

$$S := \inf_{u \in \mathcal{D}^{1,2} \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \, dx}{|u|_6^2}$$

is the classical best Sobolev constant.

In applications: $V(x) = -\mu\omega^2\varepsilon(x) \leq 0$, ENZ media.

(F1) $F = F(x,u) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is differentiable with respect to u, and $f = \partial_u F : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is a Carathéodory function. Moreover f is $\mathbb{Z}^3$-periodic in x.

(F2) If $V < 0$ a.e. on $\mathbb{R}^3$ then F is convex in $u \in \mathbb{R}^3$, otherwise F is uniformly strictly convex with respect to $u \in \mathbb{R}^3$, i.e. for any compact $A \subset (\mathbb{R}^3 \times \mathbb{R}^3) \setminus \{(u,u) : \ u \in \mathbb{R}^3\}$

$$\inf_{\substack{x \in \mathbb{R}^3 \\ (u_1,u_2) \in A}} \left(\frac{1}{2}(F(x,u_1) + F(x,u_2)) - F\left(x, \frac{u_1+u_2}{2}\right)\right) > 0.$$

(F3) There are $2 < p < 2^* = 6 < q$ and constants $c_1, c_2 > 0$ such that

$$F(x,u) \geq c_1 \min(|u|^p, |u|^q)$$

and

$$|f(x,u)| \leq c_2 \min(|u|^{p-1}, |u|^{q-1})$$

for all $x, u \in \mathbb{R}^3$.

(F4) For any $x \in \mathbb{R}^3$ and $u \in \mathbb{R}^3$, $u \neq 0$ $\langle f(x,u), u\rangle > 2F(x,u)$.

(F5) If $\langle f(x,u), v\rangle = \langle f(x,v), u\rangle \neq 0$ then

$$F(x,u) - F(x,v) \leq \frac{\langle f(x,u), u\rangle^2 - \langle f(x,u), v\rangle^2}{2\langle f(x,u), u\rangle}.$$

If in addition $F(x,u) \neq F(x,v)$ then the strict inequality holds.

$F(x,u) \sim |u|^q$ for small $|u|$, $F(x,u) \sim |u|^p$ for large $|u|$.

Example

$F(x,u) = \Gamma(x)\big((1 + |Mu|^q)^{\frac{p}{q}} - 1\big)$, *where* $\Gamma \in L^\infty(\mathbb{R}^3)$ *is* $\mathbb{Z}^3$ *periodic, positive and bounded away from* 0, $M \in GL(3)$.

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Variational setting

$$L^{p,q} = L^p(\mathbb{R}^3, \mathbb{R}^3) + L^q(\mathbb{R}^3, \mathbb{R}^3)$$

with

$$|E|_{p,q} := \inf\{|E_1|_p + |E_2|_q \quad | \quad E = E_1 + E_2, \\ E_1 \in L^p(\mathbb{R}^3, \mathbb{R}^3),\ E_2 \in L^q(\mathbb{R}^3, \mathbb{R}^3)\}$$

Energy functional

$\mathcal{J} : \mathcal{D}(\mathrm{curl}; p, q) \to \mathbb{R}$,

$$\mathcal{J}(E) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \times E|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|E|^2\, dx - \int_{\mathbb{R}^3} F(x, E)\, dx,$$

where $\mathcal{D}(\mathrm{curl}; p, q)$ is the completion of $\mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ with respect to the norm

$$\|E\|_{\mathrm{curl};p,q} := \left(|\nabla \times E|_2^2 + |E|_{p,q}^2\right)^{1/2}.$$

(V) ($V \in L^{\frac{p}{p-2}}(\mathbb{R}^3) \cap L^{\frac{q}{q-2}}(\mathbb{R}^3)$), (F1) and (F3) imply $\mathcal{J} \in \mathcal{C}^1$.

Variational setting

$$\mathcal{U} = \left\{ E \in \mathcal{D}(\mathrm{curl}; p, q) \middle| \int_{\mathbb{R}^3} \langle E, \nabla\varphi \rangle \, dx = 0 \text{ for any } \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3) \right\}$$

$$= \{ E \in \mathcal{D}(\mathrm{curl}; p, q) | \; \mathrm{div}\; E = 0 \}$$

since $|\nabla \times E|_2 = |\nabla E|_2$

$$= \{ E \in \mathcal{D}^{1,2}(\mathbb{R}^3, \mathbb{R}^3) | \; \mathrm{div}\; E = 0 \} \subset L^6(\mathbb{R}^3, \mathbb{R}^3) \subset L^{p,q}$$

Let $\mathcal{W}$ be the completion of $\mathcal{C}_0^\infty(\mathbb{R}^3)$ with respect to

$$|\nabla w|_{p,q}.$$

$$\nabla \mathcal{W} = \{ \nabla w | \; w \in \mathcal{W} \} \subset L^{p,q}.$$

Helmholtz's decomposition

$$\begin{aligned} \mathcal{D}(\mathrm{curl}; p, q) &= \mathcal{U} \oplus \nabla\mathcal{W} \subset L^{p,q}, \\ \|u + \nabla w\| &= (|\nabla u|_2^2 + |\nabla w|_{p,q}^2)^{1/2} \sim \|u + \nabla w\|_{\mathrm{curl}; p,q} \end{aligned}$$

$$\mathcal{J}(u + \nabla w) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)|u + \nabla w|^2 \, dx - \int_{\mathbb{R}^3} F(x, u + \nabla w) \, dx.$$

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$$\mathcal{J}(u+\nabla w) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + V(x) |u+\nabla w|^2 \, dx - \int_{\mathbb{R}^3} F(x, u+\nabla w) \, dx.$$

Geometry of $\mathcal{J}$

Take $u + \nabla w \in \mathcal{U} \oplus \nabla \mathcal{W}$. In view of (V) ($|V|_{\frac{3}{2}} < S$, $V \leq 0$)

$$\begin{aligned}
\mathcal{J}(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)|u|^2\,dx - \int_{\mathbb{R}^3} F(x,u)\,dx,\\
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\end{aligned}$$

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Take $F(x,E) = \frac{1}{p}((1+|E|^q)^{\frac{p}{q}} - 1)$, $\nabla w_n \rightharpoonup \nabla w$ in $L^{p,q}$, then

$$(1+|\nabla w_n|^q)^{\frac{p}{q}-1}|\nabla w_n|^{q-2}(\nabla w_n) \not\rightharpoonup (1+|\nabla w|^q)^{\frac{p}{q}-1}|\nabla w|^{q-2}(\nabla w)$$

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Nehari manifold technique

$$\mathcal{J} : X^+ \oplus X^0 \oplus X^- \to \mathbb{R},\ \dim(X^0) < \infty,\ u = u^+ + u^0 + u^-$$

$$\mathcal{J}(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - N(u)$$

- Z. Nehari: (1960), (1961)
- A. Pankov: (2005)
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$$X' = X^0 \oplus X^-$$

$$I(u) = \frac{1}{2}\|u^-\|^2 + N(u) \geq 0$$

Abstract setting

$(X, \|\cdot\|)$ is a reflexive Banach space, $X = X^+ \oplus X'$, X^+, X' are closed subspaces of X and $X^+ \cap X' = \{0\}$.
If $u \in X$ then $u = u^+ + u'$ where $u^+ \in X^+$ and $u' \in X'$, $\|u\|^2 = \|u^+\|^2 + \|u'\|^2$.
X^+ is a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$ such that $\langle u, u \rangle = \|u\|^2$ for any $u \in X^+$.

Nehari-Pankov manifold

$\mathcal{N} := \{u \in X \setminus X' \,|\, \mathcal{J}'(u)(u) = 0,\ \mathcal{J}'(u)(h') = 0 \text{ for any } h' \in X'\}$.

$u_n \xrightarrow{\mathcal{T}} u$ provided that $u_n^+ \to u^+$ and $u_n' \rightharpoonup u'$.

Definition

We say that $\mathcal{J}$ satisfies the $(PS)_c^{\mathcal{T}}$-condition in $\mathcal{N}$ if every $(PS)_c$-sequence in $\mathcal{N}$ has a subsequence which converges in $\mathcal{T}$:

$$u_n \in \mathcal{N},\ \mathcal{J}'(u_n) \to 0,\ \mathcal{J}(u_n) \to c \Rightarrow u_n \xrightarrow{\mathcal{T}} u \in X \ \text{ along a subsequence.}$$

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Geometry of $\mathcal{J}(u) = \frac{1}{2}\|u^+\|^2 - I(u)$

Assumptions

$(J1)$ $0 < \inf_{u \in X^+,\ \|u\|=r} \mathcal{J}(u)$ for some $r > 0$.
$(J2)$ $\|u^+\| + I(u) \to \infty$ as $\|u\| \to \infty$.
$(J3)$ $I(t_n u_n)/t_n^2 \to \infty$ if $u_n^+ \to u_0^+$ for some $u_0^+ \neq 0$ and $t_n \to \infty$ as $n \to \infty$.

$(J1)$ - $(J3)$ implies that for $u_0 \in X^+ \setminus \{0\}$ there is $R > r$

$$\sup_{u \in \partial M} \mathcal{J}(u) \leq 0 = \mathcal{J}(0) < \inf_{u \in X^+, \|u\|=r} \mathcal{J}(u)$$

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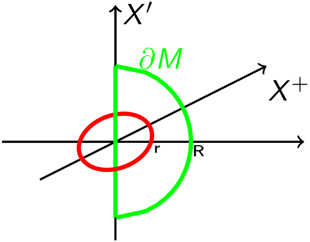

Theorem (T. Bartsch, J.M. (2015))

Let $\mathcal{J} \in \mathcal{C}^1(X, \mathbb{R})$ *be a map of the form*

$$\mathcal{J}(u) = \frac{1}{2}\|u^+\|^2 - I(u)$$

for any $u = u^+ + u' \in X^+ \oplus X'$ *such that* $(J1) - (J3)$ *hold and* $(J4)$ $I(u) \geq I(0) = 0$ *for any* $u \in X$ *and,* I *is* $\mathcal{T}$*-sequentially lower semicontinuous, i.e. if* $u_n \xrightarrow{\mathcal{T}} u_0$ *then* $\liminf_{n\to\infty} I(u_n) \geq I(u_0)$.
$(J5)$ *If* $u_n \xrightarrow{\mathcal{T}} u_0$ *and* $I(u_n) \to I(u_0)$ *then* $u_n \to u_0$.
$(J6)$ *If* $u \in \mathcal{N}$ *then* $\mathcal{J}(u) > \mathcal{J}(tu + h')$ *for any* $t \geq 0$, $h' \in X'$ *such that* $tu + h' \neq u$.
Then $c := \inf_{\mathcal{N}} \mathcal{J} > 0$ *and*
(a) $\mathcal{J}$ *has a* $(PS)_c$*-sequence* $(u_n)_{n\in\mathbb{N}} \subset \mathcal{N}$*, i.e.* $\mathcal{J}(u_n) \to c$ *and* $\mathcal{J}'(u_n) \to 0$ *as* $n \to \infty$.
(b) If $\mathcal{J}$ *satisfies the* $(PS)_c^{\mathcal{T}}$*-condition in* $\mathcal{N}$ *then* c *is achieved by a critical point of* $\mathcal{J}$.

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Lemma

$(J1)$ *there is* $r > 0$ *such that*

$$0 < \inf_{u \in \mathcal{U}, \|u\| = r} \mathcal{J}(u).$$

$(J2)$ *If* $\|u_n + \nabla w_n\| \to \infty$ *then*

$$\|u_n\| + I(u_n + \nabla w_n) \to \infty.$$

$(J3)$ *if* $u_n \to u_0 \neq 0$ *and* $t_n \to \infty$ *as* $n \to \infty$, *then*

$$I(t_n(u_n + \nabla w_n))/t_n^2 \to \infty.$$

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(J5) If $u_n + \nabla w_n \xrightarrow{\mathcal{T}} u_0 + \nabla w_0$ and $I(u_n + \nabla w_n) \to I(u_0 + \nabla w_0)$ then

$$u_n + \nabla w_n \to u_0 + \nabla w_0.$$

(J6) If $u + \nabla w \in \mathcal{N}$ then $\mathcal{J}(u+\nabla w) > \mathcal{J}(tu + \nabla h')$ for any $t \geq 0$, $h' \in \mathcal{W}$ such that $u + \nabla w \neq tu + \nabla h'$.

(J5)

$$u_n + \nabla w_n \xrightarrow{\mathcal{T}} u_0 + \nabla w_0,\ I(u_n + \nabla w_n) \to I(u_0 + \nabla w_0)$$

$$I(u + \nabla w) = -\frac{1}{2}\int_{\mathbb{R}^3} V(x)|u + \nabla w|^2\,dx + \int_{\mathbb{R}^3} F(x, u + \nabla w)\,dx$$

then

$$\int_{\mathbb{R}^3} F(x, u_n + \nabla w_n)\,dx \to \int_{\mathbb{R}^3} F(x, u_0 + \nabla w_0)\,dx.$$

Lemma

$\int_{\mathbb{R}^3}(F(x, u_n + \nabla w_n) - F(x, (u_n + \nabla w_n) - (u_0 + \nabla w_0)))\,dx \to$
$\int_{\mathbb{R}^3} F(x, u_0 + \nabla w_0)\,dx$

Therefore

$$\int_{\mathbb{R}^3} F(x, (u_n + \nabla w_n) - (u_0 + \nabla w_0)))\,dx \to 0$$

and by **(F4)** $(F(x,u) \geq c_1 \min\{|u|^p, |u|^q\})$

$$\|(u_n + \nabla w_n) - (u_0 + \nabla w_0)\|_{p,q} \to 0.$$

Hence $u_n + \nabla w_n \to u_0 + \nabla w_0$.

PS-sequence on the Nehari-Pankov manifold

In view of Theorem, $\mathcal{J}$ has a $(PS)_c$-sequence $(E_n = u_n + \nabla w_n)_{n\in\mathbb{N}} \subset \mathcal{N}$, i.e. $\mathcal{J}(E_n) \to c$ and $\mathcal{J}'(E_n) \to 0$ as $n \to \infty$.

$$\begin{aligned}\mathcal{N} &= \{E \in \mathcal{D}(\mathrm{curl}; p, q) \ \mid \ \mathcal{J}'(E)(E) = 0,\\ &\qquad\qquad\qquad\qquad\qquad \mathcal{J}'(E)(\nabla\psi) = 0 \text{ for any } \psi \in \mathcal{C}^\infty(\mathbb{R}^3)\}\\ &= \{E \in \mathcal{D}(\mathrm{curl}; p, q) \ \mid \ \mathcal{J}'(E) = 0 \text{ on } \mathbb{R}E \oplus \nabla\mathcal{W}\}\end{aligned}$$

However $(PS)_c^{\mathcal{T}}$-condition is not satisfied, since there is no compact embedding

$$X^+ = \mathcal{U} \not\subset\subset L^{p,q}.$$

Lemma

$\mathcal{J}$ is coercive on $\mathcal{N}$, thus $(E_n = u_n + \nabla w_n)_{n\in\mathbb{N}}$ is bounded.

Theorem (J.M. (2015))

If $(E_n)_{n=0}^{\infty} \subset \mathcal{N}$ is bounded then, up to a subsequence, there is $N \in \mathbb{N} \cup \{\infty\}$, $\bar{E}_0 \in \mathcal{D}(\mathrm{curl}, p, q)$ and there are sequences $(\bar{E}_i)_{i=1}^{N} \subset \mathcal{N}_0$ and $(x_n^i)_{n \geq i} \subset \mathbb{Z}^3$ with $x_n^0 = 0$ such that the following conditions hold as $n \to \infty$

$$E_n(\cdot + x_n^i) \rightharpoonup \bar{E}_i \text{ in } \mathcal{D}(\mathrm{curl}, p, q), \tag{2}$$
$$E_n(\cdot + x_n^i) \to \bar{E}_i \text{ a.e. in } \mathbb{R}^3, \tag{3}$$

for any $0 \leq i < N+1$, and

$$E_n - \sum_{i=0}^{\min\{n,N\}} \bar{E}_i(\cdot - x_n^i) \to 0 \text{ in } L^{p,q}. \tag{4}$$

Moreover

$$\lim_{n\to\infty} I(E_n) = I(\bar{E}_0) + \sum_{i=1}^{N} I_0(\bar{E}_i) < \infty, \tag{5}$$

where $\mathcal{N}_0$ and I_0 are given under assumption $V = 0$.

Consequences

Corollary (Weak-to-weak* continuity on $\mathcal{N} \cup \{0\}$)

If $E_n = u_n + \nabla w_n \in \mathcal{N}$ *and* $E_n \rightharpoonup E_0$, *then*

$$\mathcal{J}'(E_n)(\phi + \nabla\psi) \to \mathcal{J}'(E_0)(\phi + \nabla\psi)$$

for any $\phi + \nabla\psi \in \mathcal{D}(\mathrm{curl}; p, q)$.

Theorem

If $(E_n)_{n=0}^{\infty} \subset \mathcal{N}$ *is a* $(PS)_c$*-sequence at level* $c > 0$, *then, up to a subsequence, there is* $\bar{E}_0 \in \mathcal{D}(\mathrm{curl}, p, q)$ *and a finite sequence* $(\bar{E}_i)_{i=1}^{N} \subset \mathcal{N}_0$ *of critical points of* $\mathcal{J}_0$ *such that (2)-(4) hold and*

$$c = \mathcal{J}(\bar{E}_0) + \sum_{i=1}^{N} \mathcal{J}_0(\bar{E}_i), \tag{6}$$

where $\mathcal{J}_0$ *is the energy functional under assumption* $V = 0$.

- If $V = 0$ then $N = 0$, $\mathcal{J}_0(\bar{E}_0) = c$ and $\mathcal{J}_0'(\bar{E}_0) = 0$.
- If $0 < c < \inf_{\mathcal{N}_0} \mathcal{J}_0$ then $N = 0$, $\mathcal{J}(\bar{E}_0) = c$ and $\mathcal{J}'(\bar{E}_0) = 0$.

Consequences

Corollary (Weak-to-weak* continuity on $\mathcal{N} \cup \{0\}$)

If $E_n = u_n + \nabla w_n \in \mathcal{N}$ *and* $E_n \rightharpoonup E_0$, *then*

$$\mathcal{J}'(E_n)(\phi + \nabla\psi) \to \mathcal{J}'(E_0)(\phi + \nabla\psi)$$

for any $\phi + \nabla\psi \in \mathcal{D}(\mathrm{curl}; p, q)$.

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Existence of solutions

$$\nabla \times (\nabla \times E) + V(x)E = f(x, E) \text{ in } \mathbb{R}^3. \qquad (\mathcal{P})$$

Theorem

Assume that (F1)-(F5) and (V) hold. Then there is a solution to $(\mathcal{P})$*. If* $V < 0$ *a.e. on* $\mathbb{R}^3$ *or* $V = 0$ *then* $(\mathcal{P})$ *has a ground state solution, i.e. there is a critical point* $E \in \mathcal{N}$ *such that*

$$\mathcal{J}(E) = \inf_{\mathcal{N}} \mathcal{J} > 0,$$

Questions

$$\nabla \times (\nabla \times E) + V(x)E = f(x, E) \text{ in } \mathbb{R}^3. \qquad (\mathcal{P})$$

Question

One shows that if $0 < |V|_{\frac{p}{p-2}}, |V|_{\frac{q}{q-2}} < \lambda_0$ *for small* $\lambda_0 > 0$*,* $V(x) \geq 0$ *a.e. in* $\mathbb{R}^3$*, then* $\mathcal{J}$ *has the linking geometry.*
Is there any solution of $(\mathcal{P})$ *of the form* $E = u + \nabla w$ *with* $u \neq 0$*?*

Question

Let $V(x) \geq V_0 > 0$*. If* $V(x) = V(x_3, r)$ *is cylindrically symmetric, then* $(\mathcal{P})$ *has a solution for* $f(x, E) = |E|^{p-2}E$*,* $2 < p < 6$*. Is there any ground state solution? Is there any solution of* $(\mathcal{P})$ *for nonsymmetric* V*?*