Qualitative properties of solutions for nonlinear Schrödinger equations with nonlinear boundary conditions on the half-line

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Nonlinear model

Nonlinear Schrödinger equation with nonlinear boundary condition:

$$\begin{cases} i\partial_t u + \partial_x^2 u + k|u|^p u = 0, & x \in \mathbb{R}_+,\ t \in (0, T), \\ u(x,0) = u_0(x), \\ \partial_x u(0,t) + \lambda |u(0,t)|^r u(0,t) = 0. \end{cases}$$

- $\lambda = 0$: Homogeneous Neumann b.c.
- $r = 0$: Homogeneous Robin b.c.
- $\lambda \neq 0, r \neq 0$: Nonlinear Robin b.c.

Function spaces

- $u_0 \in H^s(\mathbb{R}_+)$ with $s \in \left(\frac{1}{2}, \frac{7}{2}\right) - \left\{\frac{3}{2}\right\}$.
- Compatibility condition: $u_0'(0) = -\lambda |u_0(0)|^r u_0(0)$ for $s > \frac{3}{2}$.
- $X_T^s \equiv \{u \in C([0,T]; H^s(\mathbb{R}_+)) \cap C(\mathbb{R}_+^x; H^{\frac{2s+1}{4}}(0,T))$ s.t. $\|u\|_{X_T^s} < \infty\}$ where

$$\|u\|_{X_T^s} := \sup_{t\in[0,T]} \|u(\cdot,t)\|_{H^s(\mathbb{R}_+)} + \sup_{x\in\mathbb{R}_+} \|u(x,\cdot)\|_{H^{\frac{2s+1}{4}}(0,T)}.$$

What is known?

Ackleh-Deng (2004, Differential and Integral Equations) studied the following model

$$\begin{cases} i\partial_t u + \partial_x^2 u = 0, & x \in \mathbb{R}_+,\ t \in (0,T), \\ u(x,0) = u_0(x), & \\ \partial_x u(0,t) + |u(0,t)|^r u(0,t) = 0. & \end{cases} \tag{1}$$

Theorem (Ackleh-Deng, 2004)

If $u_0 \in H^3(\mathbb{R}_+)$, then there is $T_0 > 0$ such that (1) possesses a unique local solution $u \in C([0,T_0); H^1(\mathbb{R}_+))$. Moreover, (large) solutions with negative initial energy blow-up in H^1 if $r \geq 2$ and are global otherwise.

Critical exponent: $r = 2$

Related studies

Lasiecka-Triggiani (2006, J. Evolution Equations) studied

$$\begin{cases} i\partial_t u + \Delta u = 0, & x \in \Omega,\ t \in (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \Gamma_0,\ t \in (0,T), \\ \frac{\partial u}{\partial \nu} - i|u(x,t)|^r u(x,t) = 0, & x \in \Gamma_1,\ t \in (0,T). \end{cases}$$

where Ω is an open, bounded domain whose (smooth) boundary is partitioned into Γ_0 and Γ_1.

Theorem (Lasiecka-Triggiani, 2006)

- *(Well-posedness): If $u_0 \in L^2(\Omega)$, then there is a unique (mild) solution $u \in C([0,T]; L^2(\Omega))$.*
- *(Regularity): If $u_0 \in H^2(\Omega)$ satisfying the necessary compatibility conditions, then the solution u belongs to $C([0,T]; H^1_{\Gamma_0}(\Omega))$.*

Orientation

Step 1 Study the linear Schrödinger equation with inhomogeneous terms both in the main equation and on the boundary:

$$\begin{cases} i\partial_t u + \partial_x^2 u + f(x,t) = 0, \quad x \in \mathbb{R}_+,\ t \in (0,T), \\ u(x,0) = u_0(x), \\ \partial_x u(0,t) = h(t). \end{cases} \tag{2}$$

In Step 1 we obtain the optimal (sharp) regularity theory for (2) and nice space-time (Strichartz) estimates for the corresponding (boundary) evolution operator.

Step 2 Replace the non-homogeneous source term f with $k|u|^p u$ and the non-homogeneous boundary term $h(t)$ with $-\lambda|u(0,t)|^r u(0,t)$. Use a contraction argument on the space X_T^s.

1. First, we consider

$$\begin{cases} i\partial_t u + \partial_x^2 u = 0, x \in \mathbb{R}_+, t \in (0,T), \\ u(x,0) = 0, \partial_x u(0,t) = h(t). \end{cases} \quad (3)$$

Lemma (Extension)

Let $s \in \left(\frac{1}{2}, \frac{7}{2}\right) - \left\{\frac{3}{2}\right\}$, $h \in H^{\frac{2s-1}{4}}(0,T)$ with $h(0) = 0$ if $s > \frac{3}{2}$. Then, there exists $h_e \in H^{\frac{2s-1}{4}}$ with compact support in $[0, 2T+1)$ which extends h.

2. $$\begin{cases} i\partial_t u_e + \partial_x^2 u_e = 0, x \in \mathbb{R}_+, t > 0, \\ u_e(x,0) = 0, \partial_x u_e(0,t) = h_e(t) \end{cases} \quad (4)$$

Boundary evolution operator

- Laplace (in time) - Fourier (in space) transform gives

$$u_e = \frac{1}{i\pi}\int_0^\infty e^{(-i\beta^2 t + i\beta x)}\tilde{h}_e(-i\beta^2)d\beta - \frac{1}{\pi}\int_0^\infty e^{(i\beta^2 t - \beta x)}\tilde{h}_e(i\beta^2)d\beta.$$

- We set $\phi_{h_e} = F^{-1}\{\nu_1\}$ and $\psi_{h_e} = F^{-1}\{\nu_2\}$ where $\nu_1(\beta) := \frac{1}{i\pi}\tilde{h}_e(-i\beta^2)$ for $\beta \geq 0$ and zero otherwise, and $\nu_2(\beta) := -\frac{1}{\pi}\tilde{h}_e(i\beta^2)$ for $\beta \geq 0$ and zero otherwise.
- Then,

$$u_e(x,t) = [W_b(t)h_e](x) := [W_{b,1}(t)h_e](x) + [W_{b,2}(t)h_e](x)$$

where

$$[W_{b,1}(t)h_e](x) := \int_{-\infty}^{\infty} \exp(-i\beta^2 t + i\beta x)\hat{\phi}_{h_e}(\beta)d\beta$$

and

$$[W_{b,2}(t)h_e](x) := \int_{-\infty}^{\infty} \exp(i\beta^2 t - \beta x)\hat{\psi}_{h_e}(\beta)d\beta.$$

Analysis of $W_{b,1}$ and $W_{b,2}$

- We can extend $W_{b,1}(t)h_e$ to $\mathbb{R}$ without changing its definition.

Lemma

$u(x,t) = [W_{b,1}(t)h_e](x)$ *solves the initial value problem*

$$i\partial_t u + \partial_x^2 u = 0, u(x,0) = \phi_{h_e}(x), x \in \mathbb{R}, t \in \mathbb{R}_+.$$

- Such an observation does not apply to $W_{b,2}$.

Space time estimates on $\mathbb{R}$

Lemma (Cazenave, 2003)

Let $s \in \mathbb{R}$, $T > 0$, $\phi \in H^s$, *and* $u := W_{\mathbb{R}}\phi$*. Then, there exists* $C = C(s)$ *such that*

$$\sup_{t\in[0,T]} \|u(\cdot,t)\|_{H^s} + \sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{H^{\frac{2s+1}{4}}(0,T)} \leq C\|\phi\|_{H^s}. \tag{5}$$

Lemma (Cazenave, 2003)

Let $T > 0$, $f \in L^1(0,T;H^s)$, *and* $u := \int_0^t W_{\mathbb{R}}(t-\tau)f(\tau)d\tau$*. Then, for any* $s \in \mathbb{R}$*, there exists a constant* $C = C(s) > 0$ *such that*

$$\sup_{t\in[0,T]} \|u(\cdot,t)\|_{H^s} + \sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{H^{\frac{2s+1}{4}}(0,T)} \leq C\|f\|_{L^1(0,T;H^s)}. \tag{6}$$

Estimates on $W_{b,2}$

Lemma (Space Traces)

Let $s \geq \frac{1}{2}$ *and* $T > 0$. *Then, there exists* $C > 0$ *(independent of* T*) such that*

$$\sup_{t\in[0,T]} \|W_{b,2}(\cdot)h_e\|_{H^s} \leq C(1+T)\|h\|_{H^{\frac{2s-1}{4}}(0,T)} \tag{7}$$

for any $h \in H^{\frac{2s-1}{4}}(0,T)$ *with* $h(0) = 0$ *if* $s > \frac{3}{2}$.

Lemma (Time traces)

Let $s \geq \frac{1}{2}$ *and* $T > 0$. *Then, there exists* $C > 0$ *(independent of* T*) such that*

$$\sup_{x\in\mathbb{R}_+} \|W_{b,2}h_e\|_{H^{\frac{2s+1}{4}}(0,T)} \leq C(1+T)\|h\|_{H^{\frac{2s-1}{4}}(0,T)} \tag{8}$$

for any $h \in H^{\frac{2s-1}{4}}(0,T)$ *with* $h(0) = 0$ *if* $s > \frac{3}{2}$.

Representation formula

$$u_e(x,t) = W_{\mathbb{R}}(t)u_0^* - i\int_0^t W_{\mathbb{R}}(t-\tau)f^*(\tau)d\tau + W_b([h-g-p]_e(t))$$

with $g(t) = \partial_x W_{\mathbb{R}}(t)u_0^*|_{x=0}$ and $p(t) = -i\partial_x \int_0^t W_{\mathbb{R}}(t-\tau)f^*(\tau)d\tau|_{x=0}$, then $u = u_e|_{[0,T)}$ will solve

$$\begin{cases} i\partial_t u + \partial_x^2 u = f, t \in (0,T),\ x \in \mathbb{R}_+, \\ u(x,0) = u_0, \partial_x u(0,t) = h(t). \end{cases} \tag{9}$$

Theorem

Let $T > 0$, $s \in \left(\frac{1}{2}, \frac{7}{2}\right) - \{3/2\}$, $h \in H^{\frac{2s-1}{4}}(0, T)$, $f \in L^1(0, T; H^s(\mathbb{R}_+))$, $u_0 \in H^s(\mathbb{R}_+)$, *and if* $s \in \left(\frac{3}{2}, \frac{7}{2}\right)$, $u_0'(0) = h(0)$. *Then there exists* $C > 0$ *(independent of* T*) such that the solution* u *satisfies*

$$\sup_{t \in [0,T]} \|u(x,\cdot)\|_{H^s(\mathbb{R}_+)} + \sup_{x \in \mathbb{R}_+} \|u(x,\cdot)\|_{H^{\frac{2s+1}{4}}(0,T)}$$
$$\leq C \left(\|u_0\|_{H^s(\mathbb{R}_+)} + (1+T)\|h\|_{H^{\frac{2s-1}{4}}(0,T)} + \|f\|_{L^1(0,T;H^s(\mathbb{R}_+))} \right).$$

Nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u - \partial_x^2 u + f(u) = 0, & x \in \mathbb{R}_+,\ t \in (0, T), \\ u(x,0) = u_0, \\ \partial_x u(0,t) = h(u(0,t)), \end{cases} \tag{10}$$

where $f(u) = k|u|^p u$, $p > 0$.

- Closed-loop problem: $h(u(0,t)) = -\lambda|u(0,t)|^r u(0,t)$.

$$[\Psi(u)](t) := W_{\mathbb{R}}(t)u_0^* - i\int_0^t W_{\mathbb{R}}(t-\tau)f(u^*(\tau))d\tau + W_b(t)([h(u(0,\cdot)) - g - p(u^*)]_e) \quad (11)$$

with

$$g(t) = \partial_x W_{\mathbb{R}}(t)u_0^*|_{x=0}$$

and

$$[p(u^*)](t) = -i\partial_x \int_0^t W_{\mathbb{R}}(t-\tau)f(u^*(\tau))d\tau|_{x=0}.$$

Goal: Show that Ψ maps an appropriately chosen closed ball $\bar{B}_R(0)$ of $X^s_{T_0}$ onto itself for sufficiently small T_0.

Estimates

Lemma (Nonlinearity)

Let $f(u) = |u|^p u$ *and* $s > \frac{1}{2}$*. Moreover, let* (p, s) *satisfy one of the following assumptions:*

(a1) *If* s *is integer, then assume that* $p \geq s$ *if* p *is an odd integer and* $[p] \geq s - 1$ *if* p *is non-integer.*

(a2) *If* s *is non-integer, then assume that* $p > s$ *if* p *is an odd integer and* $[p] \geq [s]$ *if* p *is non-integer.*

If $u, v \in H^s$*, then*

$$\|f(u)\|_{H^s} \lesssim \|u\|_{H^s}^{p+1},$$

$$\|f(u) - f(v)\|_{H^s} \lesssim (\|u\|_{H^s}^p + \|v\|_{H^s}^p)\|u - v\|_{H^s}.$$

Lemma

Let $h \in H^{\sigma+\epsilon}(0, T)$*,* $\sigma, \epsilon > 0$*. Then* $\|h\|_{H^\sigma(0,T)} \leq T^{\frac{\epsilon}{1+\sigma+\epsilon}} \|h\|_{H^{\sigma+\epsilon}(0,T)}$*.*

Theorem (Local well-posedness)

Let $T > 0$, $s \in \left(\frac{1}{2}, \frac{7}{2}\right) - \left\{\frac{3}{2}\right\}$, $p, r > 0$, $k, \lambda \in \mathbb{R} - \{0\}$, $u_0 \in H^s(\mathbb{R}_+)$ *together with* $u_0'(0) = -\lambda |u_0(0)|^r u_0(0)$ *whenever* $s > \frac{3}{2}$*. Assume one of the following:*

(A1) *If* s *is integer, then* $p \geq s$ *if* p *is an odd integer and* $[p] \geq s - 1$ *if* p *is non-integer.*

(A2) *If* s *is non-integer, then* $p > s$ *if* p *is an odd integer and* $[p] \geq [s]$ *if* p *is non-integer.*

(A3) $r > \frac{2s-1}{4}$ *if* r *is an odd integer and* $[r] \geq \left[\frac{2s-1}{4}\right]$ *if* r *is non-integer.*

Then, the following hold true.

(i) *Local Existence and Uniqueness: There exists a unique local solution* $u \in X^s_{T_0}$ *for some* $T_0 = T_0\left(\|u_0\|_{H^s(\mathbb{R}_+)}\right) \in (0, T]$.

(ii) *Continuous Dependence: If* B *is a bounded subset of* $H^s(\mathbb{R}_+)$*, then there is* $T_0 > 0$ *(depends on the diameter of* B*) such that the flow* $u_0 \to u$ *is Lipschitz continuous from* B *into* $X^s_{T_0}$.

(iii) *Blow-up Alternative: If* S *is the set of all* $T_0 \in (0, T]$ *such that there exists a unique local solution in* $X^s_{T_0}$*, then whenever* $T_{max} := \sup_{T_0 \in S} T_0 < T$*, it must be true that* $\lim_{t \uparrow T_{max}} \|u(t)\|_{H^s(\mathbb{R}_+)} = \infty$.

Damped Equation

$$\begin{cases} i\partial_t u - u_{xx} + k|u|^p u + iau = 0, & t > 0,\ x \in I = (0,\infty), \\ u(x,0) = u_0(x), & x > 0, \\ u_x(0,t) = -\lambda |u(0,t)|^r u(0,t), & t > 0, \end{cases}$$

where $\lambda, p, k, r > 0$ and $a \geq 0$.

Goal

To understand the competition between the bad term (nonlinear Robin boundary condition of focusing type) and the good terms (defocusing nonlinearity and damping).

Energy

$$E(t) \equiv \|u_x(t)\|_{L^2(I)}^2 - \frac{2\lambda}{r+2}|u(0,t)|^{r+2} + \frac{2k}{p+2}\|u(t)\|_{L^{p+2}(I)}^{p+2}$$

Energy identities

L^2-rate

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2(I)}^2 = -a\|u(t)\|_{L^2(I)}^2$$

H^1-rate

$$E'(t) = -2aE(t) - \frac{2akp}{p+2}\|u(t)\|_{L^{p+2}(I)}^{p+2} + \frac{2a\lambda r}{r+2}|u(0,t)|^{r+2}.$$

How to get blow-up solutions?

Theorem (H^1-Blow-up)

Suppose $r > \max\{2, p-2\}$, $E(0) \leq 0$, *and*

$$\frac{(a-b)}{2}\int_0^\infty x^2|u_0(x)|^2dx < Im\int_0^\infty xu_0(x)'\bar{u}_0(x)dx$$

where $b = \frac{a(r+2)(4-M)}{4(r+2)-2M} \leq 0$, $M = \max\{8, 2p\}$. *Then, there exists* $T > 0$ *such that the corresponding local solution* u *satisfies*

$$\lim_{t \to T^-} \|u_x(t)\|_{L^2(I)} = \infty.$$

How to stabilize solutions?

Proposition (Stabilization I)

Let $a > 0, r < 2$ and u be a local solution. Then u is global and decays to zero exponentially fast in the following sense:

$$\|u(t)\|_{H^1(I)}^2 \leq Ce^{-(2a-\epsilon)t}, t \geq 0$$

where $\epsilon > 0$ is fixed and can be chosen arbitrarily small.

Proposition (Stabilization II)

Let $a > 0, 2 \leq r < \frac{p}{2}$ and u be a local solution. Then u is global and decays to zero exponentially fast in the following sense:

$$\|u(t)\|_{H^1(I)}^2 \leq Ce^{-(a\mu-\epsilon)t}, t \geq 0,$$

where $\mu = \frac{(p+2)(p-2r)}{p(p+2)-2r} > 0$, and $\epsilon > 0$ is fixed and can be chosen arbitrarily small.

Stabilization for small data

Proposition (Stabilization III)

Let $a > 0, r = 2, p \leq 4$ *and* u *be a local solution such that* u_0 *is sufficiently small in* L^2 *sense. Then* u *is global and moreover* u *decays to zero exponentially fast in the following sense:*

$$\|u(t)\|_{H^1(I)}^2 \leq Ce^{-2at}, t \geq 0.$$

Proposition (Stabilization IV)

Let $a > 0, r > 2, r \geq \frac{p}{2}$ *and* u *be a local solution such that* u_0 *is sufficiently small in* $H^1 \cap L^{p+2}$ *sense. Then* u *is global and moreover* u *decays to zero exponentially fast in the following sense:*

$$\|u(t)\|_{H^1(I)}^2 \leq Ce^{-2at}, t \geq 0.$$

Nonlinear Powers	Blow-up ($a \geq 0$)	Local$\Rightarrow$Global ($a \geq 0$)	Exp. Stabilization ($a > 0$)
r<2	NO	YES	YES Decay rate $\sim O(e^{-(2a-\epsilon)t})$
$2 \leq r < \frac{p}{2}$	NO	YES	YES Decay rate $\sim O(e^{-(a\mu-\epsilon)t})$
$r = 2, p \leq 4$	OPEN	Small Sol. Large Sol: OPEN	Small Sol. Decay rate $\sim O(e^{-2at})$ Large Sol: OPEN
$r > 2, p - 2 \geq r \geq \frac{p}{2}$	OPEN	Small Sol. Large Sol: OPEN	Small Sol. Decay rate $\sim O(e^{-2at})$ Large Sol:OPEN
$r > 2, r > p - 2$	YES	ONLY Small Sol.	ONLY Small Sol. Decay rate $\sim O(e^{-2at})$

Table: Blow-up, local/global solutions, stabilization

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