L^1 estimates for elliptic (pseudo)-differential operators

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In the last decade, a series of interesting results have been proved in the context of div-curl estimates.

Theorem (Bourgain and Brezis, JEMS - 2007)

Assume $N \geq 3$ and $0 \leq k \leq N$ for $k \neq 1, N-1$. Then

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C\left(\|du\|_{L^1} + \|d^*u\|_{L^1}\right), \quad u \in C_c^\infty(\mathbb{R}^N; \Lambda^k\mathbb{R}^N) \qquad (1)$$

where d is the exterior differential and d^* its dual operator.

When $k = 0$ the estimate (1) becomes the Gagliardo-Nirenberg inequality

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C\|\nabla\, u\|_{L^1}.$$

This estimate can not be proved by taking the limit in the standard estimate obtained by combining the Calderón-Zygmund theory with Sobolev's inequalities

$$\|u\|_{L^{p^*}} \leq C_p\,(\|du\|_{L^p} + \|d^*u\|_{L^p}), \quad u \in C_c^\infty(\mathbb{R}^N; \Lambda^k\mathbb{R}^N), \qquad (2)$$

for $1 < p < N$ and $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{N}$ since $C_p \to \infty$ as $p \searrow 1$.

The a priori estimate (1) was also proved by Lanzani and Stein using an elementary lemma of Van Schaftingen. Moreover they showed that

$$k = 1 \Rightarrow \|u\|_{L^{\frac{N}{N-1}}} \leq C\,(\|du\|_{L^1} + \|d^*u\|_{H^1})\,.$$

$$k = N-1 \Rightarrow \|u\|_{L^{\frac{N}{N-1}}} \leq C\,(\|du\|_{H^1} + \|d^*u\|_{L^1})\,.$$

References

- **2002-2007** - *Bourgain and Brezis*: div-curl estimates.
- **2004-2015** - *Van Schaftingen*: L^1 estimates for constant vector fields.
- **2005** - *Lanzani and Stein*: L^1 estimates for complexes.
- **2007** - *V. Maz'ya*: sharp constant of L^1 estimates.
- **2009** - *Chanillo and Van Schaftingen*: subelliptic estimates.
- **2009** - *Mitrea and Mitrea* : regularity of the div-curl system.
- **2010-2012** - *Po-Lam*: CR complexes.
- **2011** - *Chanillo and Po-Lam*: improving Strichartz estimates for the Schrodinger equation.
- **2013** - *Baldi and Franchi*: Heisenberg Groups.
- **2014** - *Lanzani and Raich*: Higher order L^1 estimates.
- **2011-2015** - *Hounie and Picon*: L^1 estimate for (pseudo) complex associated to system of complex vector fields.

Very recently, Van Schaftingen characterized the previous L^1 estimate for homogeneous linear differential operators of order m on $\mathbb{R}^N$ from E to F (vector spaces) with constant coefficients.

Theorem (Van Schaftingen, JEMS - 2013)

Let $A(D)$ as before. The estimate

$$\|D^{m-1}u\|_{L^{\frac{N}{N-1}}} \leq C\|A(D)u\|_{L^1}, \tag{3}$$

holds for every $u \in C_c^\infty(\mathbb{R}^N; E)$ if and only if $A(D)$ is elliptic and canceling.

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A homogeneous linear differential operator $A(D)$ as before is **canceling** if

$$\bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} A(\xi)[E] = \{0\} \qquad (4)$$

Theorem (Calderón and Zygmund -1952)

Let $1 < p < N$ and $A(D)$ as before. Then the estimate

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holds for every $u \in C_c^\infty(\mathbb{R}^N; E)$ if only if $A(D)$ is elliptic.

Remark 1: A(D) is elliptic if for every $\xi \in \mathbb{R}^N \setminus \{0\}$, $A(\xi) : E \to F$ is injective.

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Remark 1: A(D) is elliptic if for every $\xi \in \mathbb{R}^N \setminus \{0\}$, $A(\xi) : E \to F$ is injective.

Remark 2: the restriction $p > 1$ is essential (Ornstein - 1962)

First-order canceling operators

Example 1: $A(D) = \nabla$.

Clearly $A(D)$ is elliptic since $A(\xi) = \xi$.

For every $\xi \in \mathbb{R}^N$ we have $A(\xi)[\mathbb{R}] = \mathbb{R}\xi$ and then

$$\bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} A(\xi)[\mathbb{R}] = \begin{cases} 0, & \text{if } N \geq 2 \\ \mathbb{R}, & \text{if } N = 1 \end{cases}.$$

Then

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C \|\nabla u\|_{L^1}, \quad u \in C_c^\infty(\mathbb{R}^N).$$

Example 2: $A(D) = (d, d^*)$. Then for $\xi \in \mathbb{R}^N$ and $v \in \Lambda^k \mathbb{R}^N$ follow

$$A(\xi)v = (\xi \wedge v, *(\xi \wedge *v)).$$

Ellipticity follows from the Lagrangian identity

$$|v|^2 |\xi|^2 = |\xi \wedge v|^2 + |*(\xi \wedge *v)|^2.$$

For the cancellation if

$$(f, g) \in \bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} A(\xi)[E] \Longrightarrow \begin{cases} \xi \wedge f & = 0 \\ \xi \wedge *g & = 0 \end{cases}.$$

Since $2 \leq k \leq N-2$ this implies that $f = 0$ and $g = 0$. Then

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C \left(\|du\|_{L^1} + \|d^* u\|_{L^1} \right), \quad u \in C_c^\infty(\mathbb{R}^N; \Lambda^k \mathbb{R}^N)$$

for $k \neq 1, N-1$.

Example 3: $A(D) = \frac{1}{2}(Du + D^* u)$. Then

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C\|A(D)u\|_{L^1}, \quad u \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N).$$

is the Korn-Sobolev-Strauss inequality.

Higher-order canceling operators

Example 4: $A(D) = (S_m, S_m^*)$ for $m = 1, 2, 3..$ with

$$S_m = d(d^* d)^m \text{ and } S_m^* = (d^* d)^m d^*.$$

Then

$$\|D^{2m}u\|_{L^{\frac{N}{N-1}}} \leq C\left(\|S_m u\|_{L^1} + \|S_m^* u\|_{L^1}\right), \quad u \in C_c^\infty(\mathbb{R}^N; \Lambda^k \mathbb{R}^N).$$

is the L^1 Higher order (odd) div-curl estimate.

Question: is it possible to obtain an analogous characterization problem for differential operators with variable coefficients ?

Previous result: first order operator

Consider $L_1, ..., L_n$ the system of complex vector fields with smooth coefficients defined on an open set $\Omega \subseteq \mathbb{R}^N$ and define

$$\nabla_{\mathcal{L}} u = (L_1 u, .., L_n u), \quad u \in C^\infty(\Omega).$$

Theorem (Hounie and P. , Math. Res. Lett. - 2011)

If the system of vector fields $L_1, ..., L_n$, $n \geq 2$, is linearly independent and elliptic then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$ such that for some $C > 0$

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C \left(\sum_{j=1}^{n} \|L_j u\|_{L^1} \right), \quad \forall\, u \in C_c^\infty(U),$$

holds. Conversely, if the estimate holds, the system must be elliptic on U.

Let $A(x, D)$ is a linear differential operator of order m with smooth complex coefficients in $\Omega \subset \mathbb{R}^N$ from a complex vector space E to a complex vector space F,

$$A(x, D) = \sum_{|\alpha| \leq k} A_\alpha(x) \partial^\alpha : C^\infty(\Omega; E) \to C^\infty(\Omega; E).$$

Main result

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Theorem (Hounie and P. – 2015)

Let $A(x, D)$ as before. The following properties are equivalent:

(i) *A(x,D) is elliptic and canceling*

(ii) *every point $x_0 \in \Omega$ is contained in a ball $B = B(x_0, r) \subset \Omega$ such that the a priori estimate*

$$\|D^{m-1}u\|_{L^{\frac{N}{N-1}}} \leq C\|A(x, D)u\|_{L^1}, \quad \forall\, u \in C_c^\infty(B; E), \tag{5}$$

holds for some $C > 0$.

Recall that the ellipticity of $A(x, D)$ at $x_0 \in \Omega$ means that for every $\xi \in \mathbb{R}^N \setminus \{0\}$ the map $a_\nu(x_0, \xi) : E \to F$ is injective, where

$$a_\nu(x_0, \xi) \doteq \sum_{|\alpha|=m} A_\alpha(x)\xi^\alpha$$

is the the principal symbol.

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Definition

Let $x_0 \in \Omega$. The operator $A(x, D)$ is said to be canceling at x_0 when

$$\bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} a_\nu(x_0, \xi)[E] = \{0\} . \tag{6}$$

If (6) holds for every $x_0 \in \Omega$ we say that $A(x, D)$ is canceling.

Example: $A(x,D) = \nabla_{\mathcal{L}}$ gradient associated to the system $\mathcal{L} = \{L_1, \dots, L_n\}$. Let $\ell(x_0, \xi)$ the principal symbol of $\nabla_{\mathcal{L}}$ at x_0.

Canceling property.

The following properties are equivalent:

(i) $\nabla_{\mathcal{L}}$ is canceling at x_0;

(ii) the range of the map $\xi \mapsto \ell(x_0, \xi) \in \mathbb{C}^n$ has dimension ≥ 2;

(iii) there exist two vector fields $L_{\nu_1}, L_{\nu_2} \in \mathcal{L}$ that are linearly independent at x_0.

Since the system is elliptic follows $\nabla_{\mathcal{L}}$ is elliptic and as $n \geq 2$ then $\nabla_{\mathcal{L}}$ is canceling. Thus,

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C \|\nabla_{\mathcal{L}} u\|_{L^1}, \quad u \in C_c^\infty(B).$$

Main step in the proof: to prove the estimate

$$\|D^{m-1}u\|_{L^{\frac{N}{N-1}}} \leq C\|A(x,D)u\|_{L^1}, \quad \forall\, u \in C_c^\infty(B;E),$$

it is enough (using tools from pseudodifferential operators) show there exist $B = B(x_0, r)$ and $C = C(B) > 0$ such that

$$\left|\int A(x,D)u \cdot \varphi\right| \leq C\left(\|A(x,D)u\|_{L^1} + \|D^{m-1}u\|_{L^1}\right)\|\nabla\varphi\|_{L^N}$$

for every $u \in C_c^\infty(B;E)$. □

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for every $u \in C_c^\infty(B;E)$. □

We say that a closed subspace $\mathfrak{X} \subset \mathcal{D}'(\Omega;F)$ has the **canceling property** if for every $K \subset\subset \Omega$ there exists a constant $C = C(K)$ such that

$$\left|\int f \cdot \varphi\right| \leq C\,\|f\|_{L^1}\|\nabla\varphi\|_{L^N}, \quad f \in \mathfrak{X} \cap C_c^\infty(K;F). \tag{7}$$

Example: Let $L(x, D) : C^\infty(\Omega, F) \to C^\infty(\Omega, G)$ be a linear differential operator.

$L(x, D)$ is co-canceling at $x_0 \in \Omega$ if for every $f \in F$ the polinomial with coefficients in G defined by

$$p(\xi) = \ell_\nu(x_0, \xi) f : \mathbb{R}^N \setminus \{0\} \to G$$

does not vanish identically.

Theorem

If $L(x, D)$ is co-canceling then $\mathfrak{X} = Ker\ L(x, D)$ has the canceling property.

Application: L^1 pseudoddiferential operators

Consider a properly supported elliptic pseudodifferential

$$a(x, D) : C_c^\infty(\Omega; E) \to C_c^\infty(\Omega; F) \in OpS^m_{1,0}(\Omega)$$

and set $\mathfrak{X} = a(x, D)\, C_c^\infty(\Omega; E)$.

Theorem

If $\mathfrak{X} = a(x, D)\, C_c^\infty(\Omega; E)$ has canceling property then every point $x_0 \in \Omega$ is contained in a ball $B = B(x_0, r) \subset \Omega$ such that the a priori estimate

$$\|D^{m-1} u\|_{L^{\frac{N}{N-1}}} \leq C \,\|a(x, D)u\|_{L^1}, \quad u \in C_c^\infty(B; E), \qquad (8)$$

holds for some $C > 0$. □

Let $a^t(x, D) : \mathcal{D}'(\Omega; F) \to \mathcal{D}'(\Omega; E)$ be the transpose operator. A standard duality argument gives

Corollary

Every point $x_0 \in \Omega$ is contained in a neighborhood $U \subset \Omega$ such that for any $f \in L^N(\mathbb{R}^N; E)$ the equation

$$a^t(x, D)u = f$$

admits a bounded solution in U.

Obrigado!