Blowing-up solutions for Yamabe-type problems

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DENIS, THANKS FOR THE INVITATION!

- Let (M, g) be a compact Riemannian manifold without boundary of dimension $m \geq 3$.

Finding metrics of constant scalar curvature in the conformal class of g

The scalar curvature

- The curvature tensor is a map $Riem : \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$ defined by

$$Riem(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - [\nabla_X, \nabla_Y]Z,$$

  where $\mathfrak{X}(M)$ denotes the space of smooth vector fields and $[\cdot, \cdot]$ is the Lie bracket. In a local coordinate system

$$R_{ijkl} = g_{km} R_{ijl}^m \quad \text{and} \quad R_{ijl}^m = Riem\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^l} \otimes \frac{\partial}{\partial x^m}$$

- The Ricci curvature tensor Ric is the contraction of $Riem$, i.e. $R_{ij} = g^{kl} R_{ikjl}$.
- The scalar curvature is the contraction of Ric, i.e. $R = g^{ij} R_{ij}$.
  - For the round metric on $\mathbb{S}^m$, $R_{ij} = (m-1)g_{ij}$ and $R = m(m-1)$.

PDE formulation

- The conformal class of g is $[g] := \{\phi g : \phi \in C^\infty(M), \phi > 0\}$
- if $\tilde{g} = u^{\frac{4}{m-2}} g$, $u > 0$, (u is the *conformal factor*) then

$$R_{\tilde{g}} = -\frac{4(m-1)}{m-2} u^{-\frac{m+2}{m-2}} \underbrace{\left(\Delta_g u - \frac{m-2}{4(m-1)} R_g(u)\right)}_{\mathcal{L}_g := \text{conformal laplacian}}$$

- R_g is the scalar curvature of g and $R_{\tilde{g}}$ is the scalar curvature of $\tilde{g}$

finding a metric $\tilde{g}$ in $[g]$

with constant scalar curvature

$\Leftrightarrow$

finding a positive solution to the PDE

$$\mathcal{L}_g u + \kappa u^{\frac{m+2}{m-2}} = 0 \text{ in } M$$

for some constant κ

The variational structure

- Let us introduce the energy

$$J(u) := \frac{\int\limits_M \left(|\nabla_g u|^2 + \frac{m-2}{4(m-1)} R_g u^2 \right) d\sigma_g}{\left(\int\limits_M |u|^{\frac{2m}{m-2}} d\sigma_g \right)^{\frac{m-2}{m}}}, \quad u \in H^1_g(M)$$

- The Yamabe quotient is

$$Q(M,g) := \inf_{\substack{u \in H^1_g(M) \\ u \neq 0}} J(u)$$

- If the Yamabe quotient $Q(M,g)$ is achieved, the minimum u is a solution to the Yamabe problem, i.e.

$$\boxed{-\Delta_g u + \frac{m-2}{4(m-1)} R_g u = \kappa u^{\frac{m+2}{m-2}}, \ u > 0, \ \text{in } M}$$

for some $\kappa \in \mathbb{R}$

The problem

Is the Yamabe quotient $Q(M, g)$ achieved?

The Yamabe problem on the round sphere $(\mathbb{S}^m, g_0)$ Obata (1972)

$$S_{g_0} = m(m-1)$$

$$\Downarrow$$

$$-\Delta_{g_0} u + \frac{m(m-2)}{4} u = u^{\frac{m+2}{m-2}}, \ u > 0, \ \text{in } (\mathbb{S}^m, g_0)$$

$$\Downarrow$$

which is equivalent (via the stereographic projection) to

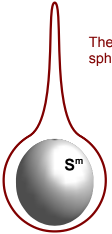
$$-\Delta U = U^{\frac{m+2}{m-2}}, \ U > 0, \ \text{in } \mathbb{R}^m$$

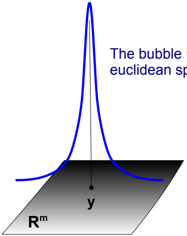
$$\Downarrow$$

The solutions are the standard bubbles

$$U_{\delta,y}(x) := \delta^{-\frac{m-2}{2}} U\left(\frac{x-y}{\delta}\right), \ x, y \in \mathbb{R}^m, \ \delta > 0,$$

- $U(x) := \alpha_m \dfrac{1}{(1+|x|^2)^{\frac{m-2}{2}}}$
- y is the center of the bubble
- δ is the weight of the bubble





A minimization result

Theorem Aubin (1976)

$$Q(M,g) < Q(\mathbb{S}^m, g_0)$$

$$\Downarrow$$

$Q(M,g)$ is achieved!

- $Q(M,g) \leq 0$ Trudinger (1968)
- $Q(M,g) > 0$ and (M,g) is not locally conformally flat and $m \geq 6$ Aubin (1976)
- $Q(M,g) > 0$ and either (M,g) is locally conformally flat or $3 \leq m \leq 5$

Schoen (1984) ("the positive mass theorem"), Bahri (1990) ("critical points at infinity")

Uniqueness

- $Q(M,g) < 0 \quad \Rightarrow$ The solution of negative scalar curvature is unique
- $Q(M,g) = 0 \quad \Rightarrow$ The solution of zero scalar curvature is unique (up to a constant factor)
- $Q(M,g) > 0 \quad \Rightarrow$ The uniqueness fails in general!
  - Example: if $M = \mathbb{S}^1 \times \mathbb{S}^{n-1}$ there are a large number of high energy solutions with high Morse index Schoen (1989)

Question

What can be said about the full set of solutions to the Yamabe problem

when the Yamabe quotient is positive?

Compactness conjecture Schoen (1988)

The set $\{\tilde{g} \in [g] \; : \; R_{\tilde{g}} = 1\}$ of solutions to the Yamabe problem
in the positive Yamabe quotient case
is $\boxed{\text{compact}}$ (in any C^k topology)
(unless the manifold is conformally equivalent to the round sphere).

- In the case of the round sphere $(\mathbb{S}^m, g_0)$ the set of solutions is not compact! Obata (1972)

- By Harnack inequality and standard elliptic estimate the compactness issue is equivalent to establish a priori estimates for solutions to the PDE, i.e.

The set $\{\tilde{g} \in [g] \ : \ R_{\tilde{g}} = 1\}$ in the positive Yamabe quotient case

is $\boxed{\text{compact}}$

$\Updownarrow$

The set

$$\left\{u \in C^\infty(M) \ : \ \mathcal{L}_g u + u^{\frac{m+2}{m-2}} = 0, \ u > 0 \text{ in } M\right\}$$

is $\boxed{\text{compact}}$

- If the first eigenvalue of $-\mathcal{L}_g$ is positive, for each $\xi \in M$ there exists a unique smooth function $G_g(\cdot, \xi)$ on $M \setminus \{\xi\}$ such that

$$-\mathcal{L}_g G_g(\cdot, \xi) = \delta_\xi \text{ in the distribution sense}$$

  where δ_ξ is the Dirac measure at ξ. $G_{g_\xi}(\cdot, \xi)$ is Green's function

- Up to a conformal change of metric g_ξ, Green's function has the following expansion

$$G_{g_\xi}(x, \xi) \sim \beta_m \frac{1}{\left(d_{g_\xi}(x, \xi)\right)^{m-2}} + \mathcal{H}(\xi) \quad \text{if} \quad x \sim \xi$$

  where β_m is a positive constant and the function $\xi \to \mathcal{H}(\xi)$ is smooth

- $\mathcal{H}$ can be identified with the mass of a stereographic projection of the manifold with respect to ξ
- $\mathcal{H}(\xi) > 0$ (Positive Mass Conjecture)
  - $3 \leq m \leq 7$ Schoen - Yau (1979)
  - $m \geq 6$ and (M, g) is locally conformally flat Schoen - Yau (1979,1988)
  - (M, g) is a spin manifold (i.e. $\mathcal{H}^2(M, \mathbb{Z}_2) = 0$) Witten (1981)

A compactness result

Theorem Khuri - Marques - Schoen (2009)

Assume

- $3 \leq m \leq 24$
- Positive Mass Conjecture holds

Then there exists a constant $c > 0$ depending only on g such that

$$c^{-1} \leq u \leq c \quad \text{in } M \qquad \text{and} \qquad \|u\|_{C^{2,\alpha}(M)} \leq c$$

where $\alpha \in (0,1)$ for any solutions to

$$\mathcal{L}_g u + u^{\frac{m+2}{m-2}} = 0, \ \ u > 0 \text{ in } M.$$

Previous results

- (M, g) is locally conformally flat Schoen (1991)
- $m = 3$ Schoen - Zhang (1996), Li - Zhu (1999)
- $m = 4, 5$ Druet (2004)

Theorem Brendle (2008), Brendle-Marques (2009)

Assume

- $m \geq 25$

Then there exists a smooth Riemannian metric g on $\mathbb{S}^m$ and a sequence of positive functions $u_n \in C^\infty(\mathbb{S}^m)$ such that

- g is not locally conformally flat
- u_n is a solution to the Yamabe equation

$$\mathcal{L}_g u_n + u_n^{\frac{m+2}{m-2}} = 0, \; u_n > 0, \; \text{in } \mathbb{S}^m.$$

- u_n blows-up as $n \to \infty$, i.e. $\lim_n \max_{\mathbb{S}^m} u_n = +\infty$

Previous results

- C^k-examples Ambrosetti - Malchiodi (1999), Berti - Malchiodi (2001)

- The proof of compactness strongly relies on proving sharp pointwise estimates at a blow-up point of the solution.
  - If a sequence of solutions u_n blows-up then it must concentrate at some points of the manifold. These are the blow-up points.
- If $3 \le m \le 24$ all the possible blow-up points are isolated and simple,

  i.e. around each blow-up point ξ_0 the solution can be approximated by a standard bubble

$$u_n(x) \sim \alpha_m \frac{\mu_n^{\frac{m-2}{2}}}{\left(\mu_n^2 + (d_g(x, \xi_n))^2\right)^{\frac{m-2}{2}}}$$

  for some $\xi_n \to \xi_0$ and $\mu_n \to 0$.

Blow-up points Schoen 1988

- $\xi_0 \in M$ is a **blow-up point** for u_n if there exists $\xi_n \in M$ such that

$$\xi_n \to \xi_0 \text{ and } u_n(\xi_n) \to +\infty$$

- $\xi \in M$ is an **isolated blow-up point** for u_n if there exists $\xi_n \in M$ such that
  - ξ_n is a local maximum of u_n
  - $\xi_n \to \xi_0$
  - $u_n(\xi_n) \to +\infty$
  - there exist $c > 0$ and $R > 0$ such that

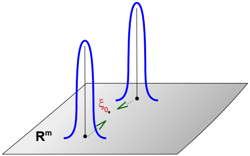
$$u_n(x) \le c \frac{1}{d_g(x, \xi_n)^{\frac{m-2}{2}}} \text{ for any } x \in B(\xi_0, R)$$

- $\xi \in M$ is an **isolated simple blow-up point** for u_n if the function

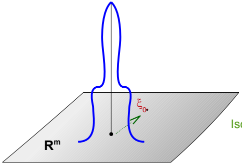
$$\hat{u}_n(r) := r^{\frac{m-2}{2}} \frac{1}{|\partial B(\xi_n, r)|_g} \int_{\partial B(\xi_n, r)} u_n d\sigma_g$$

has a exactly one critical point in $(0, R)$.

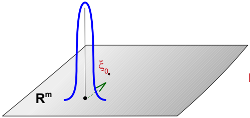
Blow-up points Schoen 1988



Not isolated
blow-up point
(clustering)



Isolated but not simple
blow-up point
(towering)



Isolated and simple
blow-up point

Our problem

A linear perturbation of the Yamabe problem

$$(E_\epsilon) \quad -\mathcal{L}_g u + \epsilon u = u^{p-1},\ u > 0,\ \text{in } (M, g)$$

- The first eigenvalue of $-\mathcal{L}_g(u) := -\Delta_g u + \frac{m-2}{4(m-1)} R_g u$ is positive
- ϵ is a small parameter

Questions

- Do there exist solutions to (E_ϵ) which blow-up as $\epsilon \to 0$?
- Do there exist solutions to (E_ϵ) with non-isolated blow-up points (clustering)?
- Do there exist solutions to (E_ϵ) with non-simple blow-up points (towering)?

Questions

- Do there exist solutions to (E_ϵ) which blow-up as $\epsilon \to 0$?
- Do there exist solutions to (E_ϵ) with non-isolated blow-up points (clustering)?
- Do there exist solutions to (E_ϵ) with non-simple blow-up points (towering)?

Answers

- NO if $\boxed{m = 3, 4, 5}$ and $\boxed{\epsilon \leq 0,}$ Druet (2004)
- YES if $\boxed{m \geq 4}$ and $\boxed{\epsilon > 0,}$ Esposito-Pistoia-Vétois (2014)
- YES Pistoia-Vaira (2015)
- YES Pistoia-Vaira (2015)

The Weyl tensor

- The Schouten curvature tensor is $A_{ij} = \frac{1}{(m-2)}\left(R_{ij} - \frac{1}{2(m-1)}Rg_{ij}\right)$.
- The Weyl curvature tensor is $W_{ijkl} = R_{ijkl} - g_{ik}A_{jl} + g_{il}A_{jk} + g_{jk}A_{il} - g_{jl}A_{ik}$.
  - The Weyl tensor is conformally invariant!
  - If $m \geq 4$, (M, g) is locally conformally flat (i.e. it is locally conformal to Euclidean space) iff the Weyl curvature tensor vanishes identically!

Existence of blowing-up solutions

Theorem Esposito-Pistoia-Vétois (2014)

Assume either

- $m \geq 6$ and (M, g) is not locally conformally flat
- $\xi_0 \in M$ is a C^1-stable critical point of $\xi \to |W_g(\xi)|_g^2$ with $|W_g(\xi_0)|_g \neq 0$

or

- $m = 4, 5$ or (M, g) is locally conformally flat
- $\xi_0 \in M$ is a C^1-stable critical point of $\xi \to \mathcal{H}(\xi)$

$$\Downarrow$$

problem (E_ϵ) has a solution which blows-up at ξ_0 as $\epsilon \to 0$

About towering and clustering ...

Theorem Pistoia-Vaira (2015)

Assume $m \geq 7$ and (M, g) is not locally conformally flat

- If $\xi_0 \in M$ is a non-degenerate maximum point of $\xi \to |W_g(\xi)|_g^2$ with $|W_g(\xi_0)| \neq 0$, then ξ_0 is a clustering blow-up point
- If $\xi_0 \in M$ is a non-degenerate critical point of $\xi \to |W_g(\xi)|_g^2$ with $|W_g(\xi_0)| \neq 0$, then ξ_0 is a towering blow-up point

A finite dimensional Ljapunov-Schmidt reduction

- Step 1: find a good approximation of the solution
- Step 2: reduce the problem to a finite dimensional one
- Step 3: find a solution of the reduced problem

The approximated solution

We look for a solution as

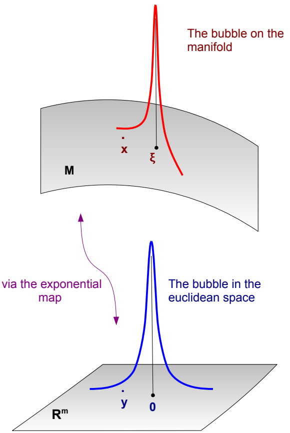
$$u_\epsilon(x) = W_\epsilon(x) + \phi(x), \; x \in M$$

- the main term

$$W_\varepsilon(x) = \begin{cases} \delta^{-\frac{m-2}{2}} U\left(\dfrac{d_{g_\xi}(x,\xi)}{\delta}\right) & \text{if } x \text{ is close to } \xi \\ \beta_m \delta^{\frac{m-2}{2}} G_{g_\xi}(x,\xi) & \text{otherwise} \end{cases}$$

- U is the standard bubble
- $\xi \in M$ is the centre of the bubble
- $\delta \in \mathbb{R}^+$ is the weight of the bubble
- $G_{g_\xi}(\cdot,\xi)$ is a Green's function with pole at ξ
- ϕ is a remainder term

The problem is finding $(\delta, \xi) \in \mathbb{R}^+ \times M$



Reduction to a finite dimensional problem

$u_\epsilon = W_\epsilon + \phi$ is a solution to (E_ϵ)

$$\Updownarrow$$

$u_\epsilon \in H_g^1(M)$ is a critical point of the energy

$$I_\epsilon(u) := \frac{1}{2}\int_M \left(|\nabla_g u|_g^2 + \frac{m-2}{4(m-1)} R_g u^2 \right) d\sigma_g + \frac{\epsilon}{2}\int_M u^2 d\sigma_g - \frac{m-2}{2m}\int_M |u|^{\frac{2m}{m-2}} d\sigma_g$$

$$\Updownarrow$$

$(\delta, \xi) \in \mathbb{R}^+ \times M$ is a critical point of the reduced energy

$$\widetilde{I}_\epsilon(\delta, \xi) := I_\epsilon(W_\epsilon + \phi)$$

The reduced energy is

$$\widetilde{I}_\epsilon(\delta,\xi) := S + F(\delta,\xi) + h.o.t.$$

where

$$F(\delta,\xi) := \begin{cases} -\mathcal{H}(\xi)\delta^2 + \epsilon\delta^2\ln(1/\delta) & \text{if } m=4, \\ -\mathcal{H}(\xi)\delta^{m-2} + \epsilon\delta^2 & \text{if } m=5 \text{ or } m\geq 6 \text{ and } (M,g) \text{ is l.c.f.} \\ \\ -|W_g(\xi)|_g^2\delta^4\ln(1/\delta) + \epsilon\delta^2 & \text{if } m=6 \text{ and } (M,g) \text{ is not l.c.f.} \\ -|W_g(\xi)|_g^2\delta^4 + \epsilon\delta^2 & \text{if } m\geq 7 \text{ and } (M,g) \text{ is not l.c.f.} \end{cases}$$

The final step of the proof!

If (δ_0,ξ_0) is a C^1-stable critical point of F $\Rightarrow$

there exists $(\delta_\epsilon,\xi_\epsilon)$ critical point of $\tilde{I}_\epsilon$ such that $(\delta_\epsilon,\xi_\epsilon)\to(\delta_0,\xi_0)$ as $\epsilon\to 0$ $\Rightarrow$

there exists a solution u_ϵ blowing-up at ξ_0 as $\epsilon\to 0$ ■

Example: $m \geq 7$ and (M, g) is not l.c.f.

- The function F is

$$F(\delta,\xi) = \underbrace{-|W_g(\xi)|_g^2\delta^4 + \epsilon\delta^2}_{\delta:=d\sqrt{\epsilon}} = \epsilon^2 \underbrace{\left(-|W_g(\xi)|_g^2 d^4 + d^2\right)}_{\varphi(d,\xi)}$$

- The reduced energy is

$$\widetilde{I}_\epsilon(\delta,\xi) := S + \epsilon^2\varphi(d,\xi) + o\left(\epsilon^2\right)$$

- ξ_0 is a non-degenerate critical point of $\xi \to |W_g(\xi)|_g^2$ with $|W_g(\xi_0)|_g \neq 0$ $\Rightarrow$ (d_0,ξ_0), $d_0 := \frac{1}{\sqrt{2}|W_g(\xi_0)|_g}$ is a non-degenerate critical point of φ $\Rightarrow$ there exists $(d_\epsilon,\xi_\epsilon)$ critical point of $\widetilde{I}_\epsilon$ such that $(d_\epsilon,\xi_\epsilon) \to (d_0,\xi_0)$ as $\epsilon \to 0$ ■

The clustering: approximated solution

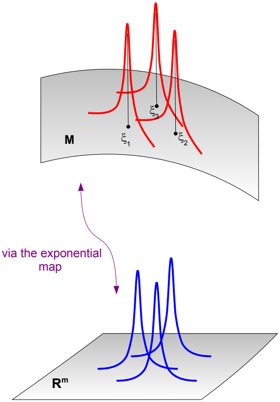
For any integer $k \geq 1$, we look for a solution as

$$u_\epsilon(x) = \sum_{i=1}^{k} W_i(x) + \phi(x), \ x \in M, \text{ where}$$

$$W_i(x) = \begin{cases} \delta_i^{-\frac{m-2}{2}} U\left(\frac{d_{g_{\xi_i}}(x, \xi_i)}{\delta}\right) & \text{if } x \text{ is close to } \xi_i \\ 0 & \text{otherwise} \end{cases}$$

- $\xi_i = \xi_0 + \epsilon^{\frac{m-6}{2m}} q_i, \ q_i \in \mathbb{R}^m$
- $\delta_i = \sqrt{\epsilon}\left(d_0 + \epsilon^{\frac{m-6}{2m}} d_i\right) \in \mathbb{R}^+$
- ϕ is a remainder term
- Set $\mathbf{d} := (d_1, \ldots, d_k)$ and $\mathbf{q} := (q_1, \ldots, q_k)$

The problem is finding $(\mathbf{d}, \mathbf{q}) \in \left(\mathbb{R}^+\right)^k \times \left(\mathbb{R}^m\right)^k$



- Arguing as before, it is sufficient to find a critical point of the reduced energy

$$\widetilde{I}_\epsilon(\mathbf{d},\mathbf{q}) := kS + \epsilon^2 \frac{1}{|W_g(\xi_0)|_g^2} + \epsilon^{\frac{3(m-6)}{m}} \mathfrak{F}(\mathbf{d},\mathbf{q}) + h.o.t.$$

where

$$\mathfrak{F}(\mathbf{d},\mathbf{q}) := \mathcal{D}^2 |W_g(\xi_0)|_g^2(\mathbf{q},\mathbf{q}) - \sum_{\substack{i,j=1\\ i\neq j}}^{k} \frac{1}{|q_i - q_j|^{m-2}} - \sum_{i=1}^{k} d_i^2$$

- ξ_0 is a non-degenerate maximum point of $\xi \to |W_g(\xi)|_g^2$ $\Rightarrow$
the quadratic form $\mathcal{D}^2|W_g(\xi_0)|_g^2(\mathbf{q},\mathbf{q})$ is negatively definite $\Rightarrow$
the function $\mathfrak{F}$ has a maximum point $(\mathbf{d}_0,\mathbf{q}_0)$ which is C^1-stable $\Rightarrow$
if ϵ is small enough, there exists a critical point $(\mathbf{d}_\epsilon,\mathbf{q}_\epsilon)$ of the reduced energy $\widetilde{I}_\epsilon$. ■

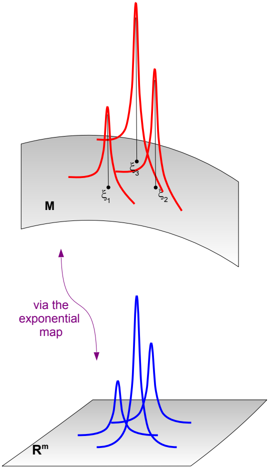
The towering: approximated solution

For any integer $k \geq 1$, we look for a solution as

$$u_\epsilon(x) = \sum_{i=1}^{k} W_i(x) + \sum_{i=1}^{k} \phi_i(x), \; x \in M, \text{ where}$$

$$W_i(x) = \begin{cases} \delta_i^{-\frac{m-2}{2}} U\left(\dfrac{d_{g_{\xi_i}}(x,\xi_i)}{\delta}\right) & \text{if } x \text{ is close to } \xi_i \\ 0 & \text{otherwise} \end{cases}$$

- $\xi_1 = \xi$ and $\xi_i = \xi + \epsilon^{s_i} q_i$, $q_i \in \mathbb{R}^m$, $i \geq 2$,
- $s_i := -\frac{1}{2} + \left(\frac{m-2}{m-6}\right)^{i-1}$, $\quad$ $s_1 = \frac{1}{2}$ and $s_i \nearrow$
- $\delta_i = d_i \epsilon^{s_i}$, $d_i \in \mathbb{R}^+$ $\quad$ $\frac{\delta_{i+1}}{\delta_i} \to 0$
- ϕ_i is a remainder term which only depends on δ_j and ξ_j for any $j \leq i-1$
- Set $\mathbf{d} := (d_1, \ldots, d_k)$ and $\mathbf{q} := (q_2, \ldots, q_k)$



The problem is finding $(\mathbf{d}, \mathbf{q}, \xi) \in (\mathbb{R}^+)^k \times (\mathbb{R}^m)^{k-1} \times M$

The towering: the reduced energy

- Arguing as before, it is sufficient to find a critical point of the reduced energy

$$\nabla_{d_1,\xi}\widetilde{I}_\epsilon(\mathbf{d},\mathbf{q},\xi) = \epsilon^2 \nabla_{d_1,\xi} \underbrace{\left(-|W_g(\xi)|_g^2 d_1^4 + d_1^2\right)}_{T_1(d_1,\xi)} + o\left(\epsilon^2\right) \text{ (only depending on } d_1 \text{ and } \xi\text{)},$$

$$\nabla_{d_2,q_2}\widetilde{I}_\epsilon(\mathbf{d},\mathbf{q},\xi) = \epsilon^{2s_2+1} \nabla_{d_2,q_2} \underbrace{\left[-U(q_2)\left(\frac{d_2}{d_1}\right)^{\frac{m-2}{2}} + d_2^2\right]}_{T_2(d_1,d_2,q_2)} + o\left(\epsilon^{2s_2+1}\right) \text{ (only depending on } d_1, d_2, \xi \text{ and } q_2\text{)},$$

$$\vdots$$

$$\nabla_{d_k,q_k}\widetilde{I}_\epsilon(\mathbf{d},\mathbf{q},\xi) = \epsilon^{2s_k+1} \nabla_{d_k,q_k} \underbrace{\left[-U(q_k)\left(\frac{d_k}{d_{k-1}}\right)^{\frac{m-2}{2}} + d_k^2\right]}_{T_k(d_1,\dots,d_k,q_2,\dots,q_k)} + o\left(\epsilon^{2s_k+1}\right),$$

- ξ_0 is a non-degenerate critical point of $\xi \to |W_g(\xi)|_g^2 \quad \Rightarrow$
  the function $(d_1, \xi) \to T_1(d_1, \xi)$ has a non-degenerate critical point $\quad\Rightarrow\quad$ there exists $d_{1\,\epsilon}, \xi_\epsilon$ such that
  $\nabla_{d_1,\xi}\widetilde{I}_\epsilon(d_{1\,\epsilon}, d_2, \dots, d_k, q_2, \dots, q_k, \xi_\epsilon) = 0$, with $\xi_\epsilon \to \xi_0$
  the function $(d_2, q_2) \to T_2(d_{1\,\epsilon}, d_2, q_2)$ has a non-degenerate maximum point $\quad\Rightarrow\quad$ there exists $d_{2\,\epsilon}, q_{2\,\epsilon}$ such that
  $\nabla_{d_2,q_2}\widetilde{I}_\epsilon(d_{1\,\epsilon}, d_{2\,\epsilon}, \dots, d_k, q_{2\,\epsilon}, \dots, q_k, \xi_\epsilon) = 0 \Rightarrow \dots \quad \Rightarrow$
  the function $(d_k, q_k) \to T_k(d_{1\,\epsilon}, \dots, d_{k-1\,\epsilon}, d_k, q_{2\,\epsilon}, \dots, q_{k-1\,\epsilon}, q_k)$ has a non-degenerate maximum point $\quad\Rightarrow$
  there exists $d_{k\,\epsilon}, q_{k\,\epsilon}$ such that $\nabla_{d_k,q_k}\widetilde{I}_\epsilon(d_{1\,\epsilon}, d_{2\,\epsilon}, \dots, d_{k\,\epsilon}, q_{2\,\epsilon}, \dots, q_{k\,\epsilon}, \xi_\epsilon) = 0 \quad \blacksquare$

A comparison with Brezis-Nirenberg problem

$$\begin{cases} -\Delta u + \epsilon u = u^{\frac{m+2}{m-2}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

($\Omega \subset \mathbb{R}^m$, $m \geq 3$, is open and bounded)

- $\epsilon > 0$ no solutions if Ω is starshaped (Pohozaev's identity)
- $\epsilon < 0$ and $m = 3$ no solutions if ϵ is small Brezis - Nirenberg (1983)
- $\epsilon < 0$ and $m \geq 4$ blowing-up solutions exist Musso - Pistoia (2002)
    - all the blow-up points are simple! Cerqueti (1999)!

$$\begin{cases} -\mathcal{L}_g u + \epsilon u = u^{\frac{m+2}{m-2}} & \text{in } M, \\ u > 0 & \text{in } M \end{cases}$$

- $\epsilon < 0$ no blowing up solutions if $m = 3, 4, 5$
  Druet 2004
- **Open problem**
    - if $\epsilon < 0$, is the problem compact?
- $\epsilon > 0$ and $m = 3$ no blowing-up solutions exist Li-Zhu 1999
- $\epsilon > 0$ and $m \geq 4$ blowing-up solutions exist
    - if $m \geq 7$ and (M, g) is not l.c.f. blow-up points are neither isolated (clustering) nor simple (towering)!
    - **Open problems**
        - what happens when $m = 6$?
        - if $m = 4, 5$ or (M, g) is l.c.f. are blow-up points simple?

THANK YOU FOR YOUR ATTENTION