On Chern-Simons-Schrödinger equations including a vortex point

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We are concerned with the Chern-Simons-Schrödinger equations:

$$\begin{cases} iD_0\phi + (D_1D_1 + D_2D_2)\phi + |\phi|^{p-1}\phi = 0, \\ \partial_0 A_1 - \partial_1 A_0 = \mathrm{Im}(\bar{\phi} D_2 \phi), \\ \partial_0 A_2 - \partial_2 A_0 = -\mathrm{Im}(\bar{\phi} D_1 \phi), \\ \partial_1 A_2 - \partial_2 A_1 = \frac{1}{2}|\phi|^2. \end{cases} \tag{1}$$

Here $t \in \mathbb{R}$, $x = (x_1, x_2) \in \mathbb{R}^2$ and the unknowns are (ϕ, A_0, A_1, A_2), where $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ is the scalar field, $A_\mu : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ are the components of the gauge potential, namely $(A_0, A_1, A_2) = (A^0, -\mathbf{A})$, and $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative ($\mu = 0,\ 1,\ 2$).

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The initial value problem, as well as global existence and blow-up, has been addressed in [Bergé, de Bouard & Saut, 1995; Huh, 2009-2013] for the case $p = 3$.

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They seek the standing wave solutions of the form

$$\phi(t,x) = u(|x|)e^{i\omega t}, \qquad A_0(x) = k(|x|),$$
$$A_1(t,x) = -\frac{x_2}{|x|^2}h(|x|), \qquad A_2(t,x) = \frac{x_1}{|x|^2}h(|x|),$$

where $\omega > 0$ is a given frequency and u, k, h are real valued functions on $[0, \infty)$ and $h(0) = 0$.

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$$\phi(t,x) = u(|x|)e^{i(N\theta + \omega t)}, \qquad A_0(x) = k(|x|),$$
$$A_1(t,x) = -\frac{x_2}{|x|^2}h(|x|), \qquad A_2(t,x) = \frac{x_1}{|x|^2}h(|x|),$$

where $\tan\theta = x_2/x_1$.

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where ξ appears as an integration constant and

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Moreover

$$A_1(x) = -\frac{x_2}{|x|^2} \int_0^{|x|} \frac{s}{2} u^2(s)\, ds, \quad A_2(x) = \frac{x_1}{|x|^2} \int_0^{|x|} \frac{s}{2} u^2(s)\, ds.$$

Therefore one need only to solve, in $\mathbb{R}^2$, the equation:

$$-\Delta u + \left(\omega + \xi + \frac{(h_u(|x|) - N)^2}{|x|^2} + \int_{|x|}^{+\infty} \frac{h_u(s) - N}{s} u^2(s)\, ds \right) u = |u|^{p-1} u.$$

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Observe that the constant $\omega+\xi$ is a gauge invariant of the stationary solutions of the problem.

The equation

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Our aim is to solve, in $\mathbb{R}^2$, the nonlocal equation:

$$-\Delta u + \left(\omega + \frac{(h_u(|x|) - N)^2}{|x|^2} + \int_{|x|}^{+\infty} \frac{h_u(s) - N}{s} u^2(s)\, ds\right) u = |u|^{p-1} u. \quad (\mathcal{CSS})$$

where

$$h_u(r) = \int_0^r \frac{s}{2} u^2(s)\, ds.$$

The case $N = 0$

In [Byeon, Huh, Seok, JFA 2012] it is shown that $(\mathcal{CSS})$ is indeed the Euler-Lagrange equation of the energy functional:

$$I_\omega : H^1_r(\mathbb{R}^2) \to \mathbb{R},$$

defined as

$$I_\omega(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + \omega u^2\right)\, dx$$
$$+ \frac{1}{8}\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\left(\int_0^{|x|} s u^2(s)\, ds\right)^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^2} |u|^{p+1}\, dx.$$

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The nonlocal term is well defined in $H^1_r(\mathbb{R}^2)$.

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The term

$$\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} \left(\int_0^{|x|} s u^2(s)\, ds - 2N \right)^2 dx$$

is not well defined in $H^1_r(\mathbb{R}^2)$, indeed, in particular, it contains

$$4N^2 \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} dx.$$

The case $N \neq 0$: the space $\mathcal{H}$

The functional I_ω is well defined in $\mathcal{H}$ is defined as

$$\mathcal{H} = \{u \in H^1_r(\mathbb{R}^2) : \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} dx < +\infty\},$$

endowed by the norm

$$\|u\|^2_{\mathcal{H}} = \int_{\mathbb{R}^2} |\nabla u(x)|^2 + \left(1 + \frac{1}{|x|^2}\right) u^2(x)\, dx.$$

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In [Byeon, Huh & Seok, 2014], it is shown that

$$\mathcal{H} \subset \{u \in C(\mathbb{R}^2) : u(0) = 0\} \cap L^\infty(\mathbb{R}^2).$$

Byeon-Huh-Seok results: case $p > 3$

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  If, instead, $N \neq 0$, the approach is based on the monotonicity trick, in [Byeon, Huh & Seok, 2014].
- Infinitely many (possibly sign-changing) solutions have been found in [Huh, JMP 2012] for $p > 5$ and $N = 0$: this case is more easy since (PS)-condition holds.

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Any standing wave solutions (ϕ, A_0, A_1, A_2) of the previous type have the following form:

$$(\phi, A_0, A_1, A_2) = \left(\frac{\sqrt{8}l(N+1)|lx|^N e^{i(N\theta+\omega t)}}{1+|lx|^{2(N+1)}}, \left(\frac{2l(N+1)|lx|^N}{1+|lx|^{2(N+1)}}\right)^2 - \omega, \right.$$
$$\left. \frac{-2l^2(N+1)x_2|lx|^{2N}}{1+|lx|^{2(N+1)}}, \frac{2l^2(N+1)x_1|lx|^{2N}}{1+|lx|^{2(N+1)}} \right),$$

where $l > 0$ is an arbitrary real constant.

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The case $N \neq 0$ is not treated.

On the boundedness from below of I_ω

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- *if $\omega > \omega_0$, then I_ω is bounded from below and coercive.*

ω_0 has an explicit expression:

$$\omega_0 = \frac{3-p}{3+p}\, 3^{\frac{p-1}{2(3-p)}}\, 2^{\frac{2}{3-p}} \left(\frac{m^2(3+p)}{p-1}\right)^{-\frac{p-1}{2(3-p)}},$$

with

$$m = \int_{-\infty}^{+\infty} \left(\frac{2}{p+1}\cosh^2\left(\frac{p-1}{2}r\right)\right)^{\frac{2}{1-p}} dr.$$

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- If unbounded, the sequence $\{u_n\}_n$ behaves as a soliton, if u_n is interpreted as a function of a single real variable.
- I_ω admits a natural approximation through a limit functional.
- The critical points of that limit functional, and their energy, can be found explicitly, so we can find ω_0.

The limit functional

Let u be a fixed even function which decays exponentially to zero at infinity, and define $u_\rho(r) = u(r - \rho)$. Let us now estimate $I_\omega(u_\rho)$ as $\rho \to +\infty$.

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$$\begin{aligned}(2\pi)^{-1} I_\omega(u_\rho) = & \frac{1}{2}\int_0^{+\infty} (|u_\rho'|^2 + \omega u_\rho^2) r \, dr \\ & + \frac{1}{8}\int_0^{+\infty} \frac{u_\rho^2(r)}{r} \left(\int_0^r s u_\rho^2(s)\, ds - 2N \right)^2 dr \\ & - \frac{1}{p+1}\int_0^{+\infty} |u_\rho|^{p+1} r\, dr.\end{aligned}$$

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$$(2\pi)^{-1} I_\omega(u_\rho) \sim \rho \left[\frac{1}{2} \int_{-\infty}^{+\infty} (|u'|^2 + \omega u^2)\, dr + \frac{1}{8} \int_{-\infty}^{+\infty} u^2(r) \left(\int_{-\infty}^{r} u^2(s)\, ds \right)^2 dr - \frac{1}{p+1} \int_{-\infty}^{+\infty} |u|^{p+1}\, dr \right].$$

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Actually, we can show that

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The critical points of J_ω, and their energy, can be found explicitly, so we can find ω_0.

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Let $p \in (1,3)$. We have:

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Theorem (Y. Jiang, A.P. & D. Ruiz)

- *For almost every $\omega \in (0, \omega_0]$, $(\mathcal{CSS})$ admits a positive solution.*

Moreover, there exist $\bar{\omega} > \tilde{\omega} > \omega_0$ such that:

- *if $\omega > \bar{\omega}$, then $(\mathcal{CSS})$ has no solutions different from zero;*
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The geometrical assumptions of the Mountain Pass Theorem are satisfied and we can apply the monotonicity trick [Struwe, Jeanjean], finding a solution for almost every $\omega \in (0, \omega_0]$.

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- Since I_ω is coercive, (PS) sequences are bounded.
- We find a second solution (a mountain-pass solution) which is at a positive energy level.

Thank you

for your attention!!!