Liouville theorems for superlinear parabolic problems

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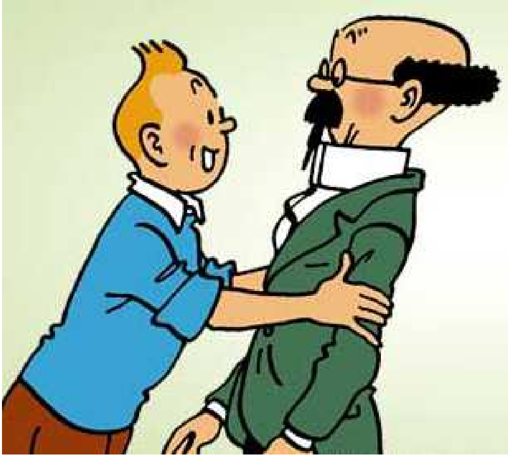

Tintin & Prof. Calculus (Tournesol)

Organizers:

D. Bonheure (chair), J. B. Casteras, J. Főldes, B. Noris, M. Nys, A. Saldaña, C. Troestler

Thank you!

Liouville-type theorems for entire solutions $(t \in (-\infty, \infty))$
of scaling invariant superlinear parabolic problems
guarantee optimal universal estimates for solutions of more general problems — including estimates of their singularities and decay.

We first consider several problems **with gradient structure** and show that **each positive bounded entire solution has to be time-independent**.

Then we consider a class of two-component systems **without gradient structure** and show that **the components of any positive bounded entire solution have to be proportional**.

Liouville theorems for the model problem

$$u_t - \Delta u = u^p, \qquad x \in \mathbb{R}^n,\ t \in \mathbb{R} \tag{1}$$

We will always assume $p > 1$ and $u \geq 0$.

Existence of stationary solutions: Gidas, Spruck 1981; Chen, Li 1991

(1) possesses positive stationary solutions iff $n > 2$ and $p \geq \frac{n+2}{n-2}$.

Sufficient conditions for nonexistence of positive solutions of (1)

(i) $p \leq \frac{n+2}{n}$ . . . [Fujita 1966; Hayakawa 1973; Kobayashi, Sirao, Tanaka 1977]

(ii) $p < \frac{n(n+2)}{(n-1)^2}$ . . . [Bidaut-Véron 1998]

(iii) $p < \frac{n+2}{n-2}$ *if* $u = u(|x|, t)$ . . . [Poláčik, Q., Souplet 2006-2007]

(iv) $p < \frac{n}{n-2}$ . . . [Q. 2015+]

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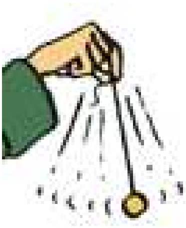
FROM ENTIRE SOLUTIONS TO STEADY STATES



$\rightarrow$

Idea of the proof: Assume that $u \geq 0$ is a solution of (1).
Doubling and scaling arguments $\Rightarrow$ we can assume $u \leq 1$.

Set

$$\mathcal{E}(\varphi) := \int_{\mathbb{R}^n} \Big(\frac{1}{2}|\nabla\varphi|^2 - \frac{1}{p+1}\varphi^{p+1}\Big)\,dx.$$

Formally:

$$\frac{d}{dt}\mathcal{E}(u(\cdot,t)) = -\int_{\mathbb{R}^n} u_t(x,t)^2\,dx$$

$\Rightarrow$ u should be either an equilibrium or a heteroclinic orbit between (two sets of) equilibria; elliptic Liouville $\Rightarrow$ $u \equiv 0$.

But: $\mathcal{E}(u(\cdot,t))$ need not be defined,

[Fila, Yanagida 2011]: $p > \frac{n+2}{n-2}$ $\Rightarrow$ $\exists$ homoclinic orbits:

$\lim_{t\to\pm\infty} u(\cdot,t) = 0$, $u > 0$ is bounded,

radially symmetric and spatially decaying if $p < \frac{n-4}{n-10}$

For $k = 1, 2, \ldots$ set

$$w_k(y,s) := (k-t)^\beta u(y\sqrt{k-t}, t), \qquad s = -\log(k-t), \quad t < k,$$

where $\beta = \frac{1}{p-1}$. Then $w = w_k$ solve the problem

$$w_s = \frac{1}{\rho}\nabla \cdot (\rho \nabla w) - \beta w + w^p \qquad \text{in } \mathbb{R}^n \times \mathbb{R}, \quad \rho(y) = e^{-|y|^2/4}, \qquad (1^*)$$

$$E(\varphi) := \int_{\mathbb{R}^n} \Big(\tfrac{1}{2}|\nabla\varphi|^2 + \tfrac{\beta}{2}\varphi^2 - \tfrac{1}{p+1}\varphi^{p+1}\Big)\rho\, dx$$

is well defined for $\varphi = w(\cdot, s)$ and $\frac{d}{ds}E(w(\cdot,s)) = -\int_{\mathbb{R}^n} w_s(y,s)^2\, dy$.

$\int_{\mathbb{R}^n}[(1^*)\cdot\rho] \Rightarrow \quad \int_{\mathbb{R}^n} w(y,s)\rho(y)\, dy + \int_{s-1}^{s}\int_{\mathbb{R}^n} w^p(y,s)\rho(y)\, dy\, ds \le C$

Set $s_k := -\log k$. Then

$$\int_{s_k-1}^{s_k}\int_{\mathbb{R}^n}\left|\frac{\partial w_k}{\partial s}(y,s)\right|^2 \rho(y)\, dy\, ds = E(w_k(\cdot, s_k-1)) - E(w_k(\cdot, s_k)) \qquad \text{(E)}$$

$$\le \ldots\ldots\ldots \le C \sup_{s_k-2\le s\le s_k-1}\|w_k(\cdot,s)\|_\infty \le C(n,p)k^\beta$$

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Summary:

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$$\int_{s_k-1}^{s_k} \int_{\mathbb{R}^n} \left| \frac{\partial w_k}{\partial s}(y,s) \right|^2 \rho(y)\, dy\, ds \le C(n,p) k^\beta, \quad s_k := -\log k. \tag{E}$$

Set

$$v_k(z,\tau) := \lambda_k^{2/(p-1)} w_k(\lambda_k z, \lambda_k^2 \tau + s_k), \quad z \in \mathbb{R}^n,\ -k \le \tau \le 0,\ \lambda_k := \frac{1}{\sqrt{k}}.$$

Then

$$v_k(z,\tau) = e^{-\beta\tau/k} u\big(e^{-\tau/2k} z, k(1 - e^{-\tau/k})\big) \to u(z,\tau),$$

$$\int_{-k}^{0} \int_{|z|<\sqrt{k}} \left| \frac{\partial v_k}{\partial \tau} \right|^2 dz\, d\tau = \lambda_k^{-n+2+4/(p-1)} \int_{s_k-1}^{s_k} \int_{|y|<1} \left| \frac{\partial w_k}{\partial s} \right|^2 dy\, ds \to 0$$

due to $p < \frac{n}{n-2}$ and (E), hence $u_t \equiv 0$. Elliptic Liouville $\Rightarrow$ $u \equiv 0$.

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Vector valued generalization of (1) $\quad U = (u_1, u_2, \dots, u_m) \geq 0$

$$U_t - \Delta U = F(U) \quad \text{in } \mathbb{R}^n \times \mathbb{R}, \tag{2}$$

where

$$F = \nabla G, \quad G \in C^{2+\alpha}_{loc}(\mathbb{R}^m), \quad G(U) > G(0) \;\text{ for }\; U \neq 0,$$
$$F(\lambda U) = \lambda^p F(U) \quad \text{for } \; U \geq 0, \; \lambda > 0,$$
$$\xi \cdot F(U) > 0 \quad \text{for some } \; \xi \in (0,\infty)^m \; \text{ and all } \; U > 0.$$

Sufficient condition for nonexistence of positive solutions of (2)

$p < \frac{n}{n-2}$ (or $p < \frac{n+2}{n-2}$ *if* $U(\cdot, t)$ *is radially symmetric*)

Special case of (2): $U = (u, v) \geq 0$, $p = 2r + 1$, $\lambda < 1$

$$\left.\begin{aligned} u_t - \Delta u &= u^p - \lambda u^r v^{r+1} \\ v_t - \Delta v &= v^p - \lambda u^{r+1} v^r \end{aligned}\right\} \quad \text{in } \mathbb{R}^n \times \mathbb{R} \tag{2*}$$

Known sufficient conditions (in the non-radial case):

- Fujita-type results: $p \leq \frac{n+2}{n}$
- Bidaut-Véron's approach: $n = 1$, $\lambda < \frac{r}{3r+2}$ ... [Phan 2015]

  $p < \frac{n(n+2)}{(n-1)^2}$, $\lambda \leq 0$ ... [Phan, Souplet: preprint]

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Idea in the radial case: Let U be a positive radial solution of (2).

1. Scaling, doubling and Liouville for $n = 1$
   $\Rightarrow$ decay estimates (as $|x| \to \infty$)
   $\Rightarrow$ $U(\cdot, t)$ belongs to the energy space
2. Lyapunov functional $\Rightarrow$ U is a connecting orbit between equilibria
3. Elliptic Liouville $\Rightarrow$ no positive equilibria $\Rightarrow$ contradiction

Vector valued generalization of (1) $\quad U = (u_1, u_2, \dots, u_m) \geq 0$

$$U_t - \Delta U = F(U) \quad \text{in } \mathbb{R}^n \times \mathbb{R}, \tag{2}$$

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Nonlinear boundary conditions

$$\left.\begin{aligned} u_t - \Delta u &= 0 && \text{in } \mathbb{R}^n_+ \times \mathbb{R}, \\ u_\nu &= u^q && \text{on } \partial\mathbb{R}^n_+ \times \mathbb{R}, \end{aligned}\right\} \qquad (3)$$

$$\mathbb{R}^n_+ := \{(x \in \mathbb{R}^n : x_1 > 0\}, \quad x = (x_1, \underbrace{x_2, \dots, x_n}_{=:\tilde{x}})$$

$$\nu = (-1, 0, 0, \dots, 0), \qquad q > 1, \qquad u \geq 0$$

Suff. conditions for nonexistence of bounded positive solutions of (3)

$q < \frac{n-1}{n-2}$ (or $q < \frac{n}{n-2}$ *if* u is axially symmetric: $u = u(x_1, |\tilde{x}|, t)$)

- Known results:
  Fujita-type: $q \leq \frac{n+1}{n}$ ... [Galaktionov, Levine 1996], [Deng, Fila, Levine 1994]
  Results for solutions with bounded derivatives if $n = 1$ ... [Q., Souplet 2011]
- Condition $q < \frac{n}{n-2}$ is optimal for the nonexistence of stationary solutions ... [Hu 1994].

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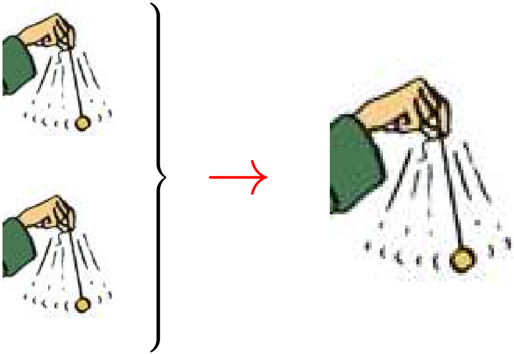
FROM SYSTEMS TO SCALAR EQUATIONS



$$\left.\begin{aligned} u_t - \Delta u &= u^r(-b_1 u^q + c_1 v^q) \\ v_t - \Delta v &= v^r(-b_2 v^q + c_2 u^q) \end{aligned}\right\} \quad \text{in } \mathbb{R}^n \times \mathbb{R}, \tag{4}$$

$$b_1, b_2, c_1, c_2 > 0, \quad c_1 c_2 > b_1 b_2 \tag{5}$$

Sufficient conditions for nonexistence of positive solutions of (4)

$q + r < \max\left(\frac{n(n+2)}{(n-1)^2}, \frac{n}{n-2}\right)$ (or $q + r < \frac{n+2}{n-2}$ if u, v are radially symmetric)

- $q = r = 1$ (Lotka-Volterra): sufficient (and necessary) condition $n \leq 5$
- $q = 2$, $r = 1$: sufficient (and necessary) condition $n \leq 3$

$$u_t - \Delta u = -b_1 u^3 + c_1 u v^2$$
$$v_t - \Delta v = -b_2 v^3 + c_2 u^2 v$$

Systems without gradient structure $\quad q \geq r > 0, \ \ q + r > 1$

$$\begin{aligned} u_t - \Delta u &= u^r(-b_1 u^q + c_1 v^q) =: f \\ v_t - \Delta v &= v^r(-b_2 v^q + c_2 u^q) =: g \end{aligned} \quad \text{in} \ \ \mathbb{R}^n \times \mathbb{R}, \tag{4}$$

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[Montaru, Souplet, Sirakov 2014] $\quad \exists K > 0 \quad \underbrace{(f - Kg)}_{w_t - \Delta w}\underbrace{(u - Kv)}_{=: w} \leq 0$

Aim: Show $u = Kv$ (i.e. $w = 0$), then $u_t - \Delta u = c u^{q+r}$ for some $c > 0$.

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Aim: Show $u = Kv$ (i.e. $w = 0$), then $u_t - \Delta u = cu^{q+r}$ for some $c > 0$.

Idea of the proof of $w = 0$ (for u, v bounded):

$$(w_t - \Delta w)\,\mathrm{sign}(w) \leq -h(|w|) \qquad \text{in } \mathbb{R}^n \times \mathbb{R},$$

where $h \in C([0, \infty))$, $h(s) > 0$ for $s > 0$.
Assume on the contrary $w \not\equiv 0$. W.l.o.g. $w(x^*, t^*) > 0$ for some x^*, t^*.
Then we arrive at a contradiction by considering the points of maxima of

$$w_\varepsilon(x, t) := w(x, t) - \varepsilon|x - x^*|^2 - \varepsilon\Big(\sqrt{(t - t^*)^2 + 1} - 1\Big),$$

cf. [Főldes 2011].

Systems without gradient structure $\quad q \geq r > 0, \;\; q + r > 1$

$$\begin{aligned} u_t - \Delta u &= u^r(-b_1 u^q + c_1 v^q) \\ v_t - \Delta v &= v^r(-b_2 v^q + c_2 u^q) \end{aligned} \qquad \text{in } \mathbb{R}^n \times \mathbb{R}, \tag{4}$$

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  Application: Existence of periodic solutions of

$$\left.\begin{aligned} \left.\begin{aligned} u_t - \Delta u &= u(a_1 - b_1 u + c_1 v) \\ v_t - \Delta v &= v(a_2 - b_2 v + c_2 u) \end{aligned}\right\} & \quad \text{in } \; \Omega \times [0, T], \\ u = v = 0 \qquad\qquad & \quad \text{on } \; \partial\Omega \times [0, T], \end{aligned}\right\} \tag{4_p}$$

where $\Omega \subset \mathbb{R}^n$ is smooth and bounded, $a_i, b_i, c_i \in C(\overline{\Omega} \times [0, T])$ are T-periodic in t and satisfy (5), $a_1, a_2 < \lambda_1$.

Theorem. If $n \leq 5$ then (4_p) possesses a positive T-periodic solution.

Systems without gradient structure $\quad q \geq r > 0, \;\; q + r > 1$

$$\begin{aligned} u_t - \Delta u &= u^r(-b_1 u^q + c_1 v^q) \\ v_t - \Delta v &= v^r(-b_2 v^q + c_2 u^q) \end{aligned} \qquad \text{in} \;\; \mathbb{R}^n \times \mathbb{R}, \qquad (4)$$

$$b_1, b_2, c_1, c_2 > 0, \quad c_1 c_2 > b_1 b_2 \qquad (5)$$

Sufficient conditions for nonexistence of positive solutions of (4)

$q + r < \max\left(\frac{n(n+2)}{(n-1)^2}, \frac{n}{n-2}\right)$ (or $q + r < \frac{n+2}{n-2}$ if u, v are radially symmetric)

- $q = r = 1$ (Lotka-Volterra): sufficient (and necessary) condition $n \leq 5$
- $q = 2,\; r = 1$: sufficient (and necessary) condition $n \leq 3$

$$\begin{aligned} u_t - \Delta u &= -b_1 u^3 + c_1 u v^2 \\ v_t - \Delta v &= -b_2 v^3 + c_2 u^2 v \end{aligned}$$

Thank you for your attention!