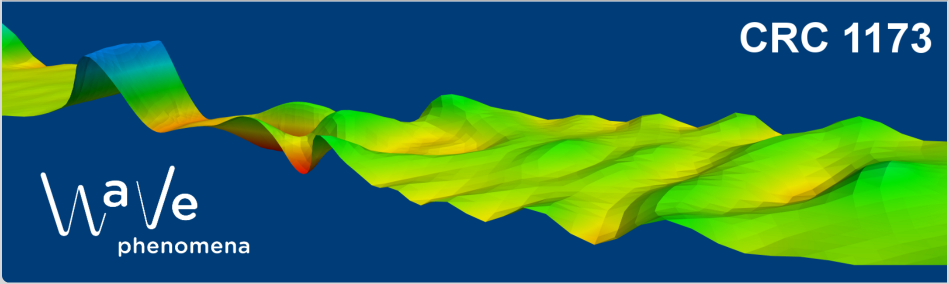
Standing waves for nonlinear curl-curl wave equations

Th. Bartsch, T. Dohnal, M. Plum, W. Reichel

Workshop in Nonlinear PDEs, Brussels • September 7–11, 2015

Institute for Analysis



Commercial announcement:

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CRC 1173 • Wave phenomena

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- People
- Projects
- iRTG
- Guests and events
- Equal opportunity
- Jobs
- Internal areas

PhD and Postdoc positions in Mathematics available

The Collaborative Research Center (CRC) 1173 »Wave phenomena – analysis and numerics« offers several PhD and Postdoc positions. Starting on July 1st, 2015 the CRC is funded by the German Research Foundation (DFG). It is jointly run by the Departments of Mathematics of Karlsruhe Institute of Technology (KIT), University of Stuttgart, and University of Tübingen in collaboration with KIT research groups in optics and photonics, biomedical engineering, and applied geophysics.

Research topics

The goal of this CRC is to analytically understand, numerically simulate, and eventually manipulate wave propagation under realistic scenarios by intertwining analysis and numerics. Follow the link to the list of project descriptions.

Job description

PhD students and Postdocs will work on one of the projects. The PhD students will be advised by at least two scientists and additionally will have a mentor. As members of the integrated Research Training Group (iRTG) they benefit from a variety of qualification programs including for example advanced and tailor-made courses on the topics of the CRC. PhD students and Postdocs will attend conferences, workshops and summer schools.

Required qualification

Candidates hold a Master degree (or equivalent) in Mathematics and have a strong background in analysis and/or numerical analysis of partial differential equations. They are open minded, active and have a good command of English language.

The problem

Find spatially localized, time-periodic $E : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ such that

$$\text{(quasi)} \qquad \nabla \times \nabla \times E + \partial_t^2\Big(V(x)E + \Gamma(x)|E|^{p-1}E\Big) = 0$$

$$\text{(semi)} \qquad \nabla \times \nabla \times E + V(x)\partial_t^2 E + q(x)E + \Gamma(x)|E|^{p-1}E = 0$$

with $p > 1$ & suitable conditions on $V, q, \Gamma : \mathbb{R}^3 \to \mathbb{R}$

Outline:

1. Physical background
2. Time-harmonic solutions: previous results/our results
3. Some details
4. Real-valued periodic solutions: previous results/our resuts
5. Some details

Maxwell's equations without sources

$$\nabla \times E + \partial_t B = 0, \qquad \nabla \cdot D = 0,$$
$$\nabla \times H - \partial_t D = 0, \qquad \nabla \cdot B = 0.$$

Material laws:

$$B = \mu_0 H, \quad D = \epsilon_0 E + P(x, E) = \epsilon_0(1 + \chi_1(x) + \chi_3(x)|E|^2)E$$

$\mu_0 \epsilon_0 = 1/c^2$ ($c =$ speed of light in vacuum)

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Quasilinear wave-equation for E:

$$\nabla \times \nabla \times E + \mu_0 \partial_t^2 \Big(\epsilon_0 (1 + \chi_1(x)) E + \epsilon_0 \chi_3(x) |E|^2 E \Big) = 0$$

Simplified semilinear variant:

$$\nabla \times \nabla \times E + q(x) E + \mu_0 \epsilon_0 (1 + \chi_1(x)) \partial_t^2 E + \mu_0 \epsilon_0 \chi_3(x) |E|^2 E = 0$$

Time-harmonic solutions for (quasi) & (semi)

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Goal: time periodic, real-valued, spatially localized solutions

Ansatz: $E(x,t) = U(x)e^{i\omega t}$ leads to

$$\nabla\times\nabla\times U \underbrace{-\omega^2\epsilon_0\mu_0\Big(\underbrace{1+\chi_1(x)}_{=n^2(x)\geq 0}\Big)}_{V(x)\leq 0}U - \underbrace{\omega^2\epsilon_0\mu_0\chi_3(x)}_{=:\Gamma(x)}|U|^2U = 0 \text{ in } \mathbb{R}^3$$

$$\nabla\times\nabla\times U + \Big(\underbrace{q(x)-\omega^2\epsilon_0\mu_0(1+\chi_1(x))}_{V(x)}\Big)U + \underbrace{\chi_3(x)}_{=:-\Gamma(x)}|U|^2U = 0 \text{ in } \mathbb{R}^3$$

i.e. stationary, nonlinear Schrödinger-type problem

$$\nabla\times\nabla\times U + V(x)U = \Gamma(x)|U|^2U \quad \text{in} \quad \mathbb{R}^3$$

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i.e. stationary, nonlinear Schrödinger-type problem

$$\nabla \times \nabla \times U + V(x) U = \Gamma(x) |U|^2 U \quad \text{in} \quad \mathbb{R}^3$$

Results - Part I

(∗) $$\nabla \times \nabla \times U + V(x)U = \Gamma(x)|U|^{p-1}U \quad \text{in} \quad \mathbb{R}^3$$

0) $U(x_1, x_2, x_3) = (0, 0, u(x_1, x_2))^T$ leads to NLS (many results!)

$$-\Delta u + V(x)u = \Gamma(x)|u|^{p-1}u \text{ in } \mathbb{R}^2$$

1) Benci-Fortunato (2004) & Azzollini-Benci-d'Aprile-Fortunato (2006) & d'Aprile-Siciliano (2011):

$$\nabla \times \nabla \times U = W'(|U|^2)U \text{ in } \mathbb{R}^3$$

Existence of ground-states in subspaces of cylindrical symmetry

2) Bartsch-Mederski (2014):

$$\nabla \times \nabla \times U + \lambda U = \partial_U F(x, U) \text{ in } \Omega, \quad \nu \times U = 0 \text{ on } \partial\Omega.$$

(3) Mederski (2014): $f \approx |x|^{p+1}$ near 0, $f \approx |x|^{q+1}$ near ∞, $1 < p < 5 < q$.

$$\nabla \times \nabla \times U + V(x)U = f'(U) \text{ in } \mathbb{R}^3$$

(4) Bartsch-Dohnal-Plum-R. (2014) ... next

Results - Part II

$(*) \qquad \nabla \times \nabla \times U + V(x)U = \Gamma(x)|U|^{p-1}U \quad \text{in} \quad \mathbb{R}^3$

General assumption: $V = V(r, x_3), \Gamma = \Gamma(r, x_3), r = \sqrt{x_1^2 + x_2^2}$

Theorem (Defocusing case)

- $\Gamma(x) \le -C(1 + |x|^\alpha)$, $\alpha > \frac{3}{2}(p-1)$, $p > 1$,
- $V \in L^\infty(\mathbb{R}^3)$, $\sup V < 0$.

Then $(*)$ has a (restricted) ground-state.

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- $\inf \Gamma > 0$, $V, \Gamma \in L^\infty(\mathbb{R}^3)$ *are ω-periodic in x_3,*
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- $0 \notin \sigma(L)$ *with* $L = \nabla \times \nabla \times + V(x)$.

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Example of cylind. symm. V with $0 \notin \sigma(L)$. Unfortunately: $\sup V > 0$!

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Variational set-up and symmetries

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$$J[U] = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla \times U|^2 + \frac{V(x)}{2}|U|^2 - \frac{\Gamma(x)}{p+1}|U|^{p+1}\,dx,$$

$$U \in X = H(\mathrm{curl};\mathbb{R}^3) \cap L^{p+1}_{|\Gamma|}(\mathbb{R}^3)$$

Here is the problem: $\|\nabla U\|^2_{L^2} = \|\nabla \times U\|^2_{L^2} + \|\nabla \cdot U\|^2_{L^2}$.

Constraint $\{U : \mathrm{div}\, U = 0\}$ does not solve it $\Rightarrow$ Lagrange-multiplier!

Symmetries! Look for invariance: $M^T U(M\cdot) = U(\cdot)$ for $M \in O(3)$

E.g. radial symmetry:

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NLS case: $u(x) = w(|x|)$

Differential equation:

$$-(w'' + \frac{n-1}{r}w') + V(r)w = \Gamma(r)|w|^{p-1}w$$

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Curl-Curl case: $U(x) = f(|x|)\frac{x}{|x|}$

"Differential" equation:

$$V(r)f(r) = \Gamma(r)|f(r)|^{p-1}f(r)$$

no interesting solutions

Cylindrical symmetry (r, x_3), $r^2 = x_1^2 + x_2^2$

$U(\cdot) = M^T U(M\cdot) \quad \forall M \in G_0$ where

$$G_0 = \left\{ \begin{pmatrix} cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} : \alpha \in [0, 2\pi) \right\}$$

Azzollini-Benci-d'Aprile-Fortunato: $U = Q + S + T$ with

$$Q = \frac{q(r, x_3)}{r} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \quad S = \frac{s(r, x_3)}{r} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}, \quad T = t(r, x_3) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Advantage of Q: $\nabla \cdot Q = 0$, $\|\nabla Q\|_{L^2} = \|\nabla \times Q\|_{L^2}$

Symmetric Sobolev-spaces $\mathcal{H}^1 = \left\{ Q : Q \in H^1(\mathbb{R}^3) \right\}$

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Existence results in $\mathcal{H}^1$

$$J[U] = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla \times U|^2 + \frac{V(x)}{2}|U|^2 - \frac{\Gamma(x)}{p+1}|U|^{p+1}\,dx, \quad U \in \mathcal{H}^1$$

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$$\mathcal{H}^1 = \mathcal{H}^+ \oplus \mathcal{H}^-$$

Least energy level:

$$c = \inf_N J[U], \quad N = \left\{U \in \mathcal{H}^1 \setminus \{0\} : J'[U]\phi = 0 \ \forall \phi \in \mathcal{H}^- \oplus [U]\right\}$$

- $J|_N$ bounded from below
- $\exists$ bounded, minimizing Palais-Smale sequence
- concentration compactness: $\exists$ minimizer

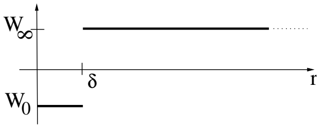
Example of a potential V with $0 \notin \sigma(L)$

Consider potential

$$V(r, x_3) = W(r) + P(x_3)$$

where P is periodic and

$$W(r) := \begin{cases} W_0, & 0 \leq r \leq \delta, \\ W_\infty, & \delta \leq r < \infty. \end{cases}$$



Spectrum:

$$\sigma(\nabla \times \nabla \times + V(r, x_3)) = \sigma_{\mathrm{rad}} + \sigma_{per}$$

$$\sigma_{\mathrm{per}} = [\nu_1, \nu_2] \cup [\nu_3, \nu_4] \cup \dots$$

Take any $\mu_0 < W_\infty$. $\Rightarrow \exists W_0, \delta$ s.t. $\sigma_{\mathrm{rad}} = \{\mu_0\} \cup [W_\infty, \infty)$.

Sufficient for $0 \notin \sigma(L)$: $\quad -W_\infty < \nu_1 < \nu_2 < -\mu_0 < \nu_3$

Summary: time-harmonic curl-curl problem

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$$(\nabla \times \nabla \times + V(x))\, U = \Gamma(x)|U|^{p-1} U \text{ in } \mathbb{R}^3$$

- Use cylindrical symmetry: $U(r, x_3) = \frac{q(r,x_3)}{r}\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}$
- Defocusing case: restricted ground states exist if

$$\Gamma(x) \le -C(1 + |x|^\alpha), \quad \alpha > \frac{3}{2}(p-1), \quad p > 1, \quad \sup V < 0$$

- Focusing case: restricted ground states exist if

$$\inf \Gamma > 0, \quad V, \Gamma \;\; x_3\text{-periodic}, \quad 0 \notin \sigma(L), \quad 1 < p < 5$$

- Examples: $0 \notin \sigma(L)$ with $V(r, x_3) = W(r) + P(x_3)$

Real-valued time-periodic solutions of (semi)

(semi) $$\nabla\times\nabla\times E + V(x)\partial_t^2 E + q(x)E + \Gamma(x)|E|^{p-1}E = 0$$

(1) (scalar) Sine-Gordon breather: $u(x,t) = 4\arctan\left(\frac{m\sin(\omega t)}{\omega\cosh(mx)}\right)$, $m^2+\omega^2 = 1$

$$-u_{xx} + u_{tt} + \sin u = 0$$

(2) (scalar) Replace $\sin(u)$ by $g(u)$ with $g(0)=0, g'(0)=1 \Rightarrow$ in general breathers disappear

(3) (scalar) Blank, Chirilus-Bruckner, Lescaret, Schneider (2011): $\exists$ periodic $V, q, \Gamma \in L^\infty(R)$ such that

$$-u_{xx} + V(x)u_{tt} + q(x)u + \Gamma(x)u^3 = 0$$

has real-valued breather solutions.

(4) (vector) Plum, R. (2015): ... next

Existence result

(semi) $\nabla \times \nabla \times E + V(x)\partial_t^2 E + q(x)E + \Gamma(x)|E|^{p-1}E = 0$

Ansatz: $E(x,t) = \psi(r,t)\frac{x}{r}, \quad r = |x|.$

$$V(r)\ddot{\psi} + q(r)\psi + \Gamma(r)|\psi|^{p-1}\psi = 0$$

Theorem

Let $T > 0$.

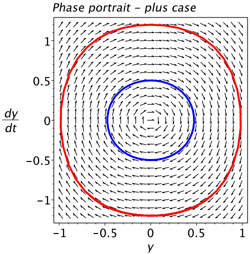
- $V, q, \Gamma > 0$ *radially symmetric* C^2*-functions,*
- $\sup \frac{q}{\Gamma} < \infty$,
- $T\sqrt{\frac{q(r)}{V(r)}} < 2\pi$ *on* $\mathbb{R}^3$,
- $$\left|2\pi - T\sqrt{\frac{q(r)}{V(r)}}\right|^{\frac{1}{p-1}} = \begin{cases} O(e^{-\alpha r}) \text{ as } r \to \infty, \\ o(1) \text{ in } C^2\text{-sense as } r \to 0. \end{cases}$$

Then $\exists$ T-periodic, real-valued, exponentially decaying solution.

The proof – solving an ODE

$$V(r)\ddot{\psi} + q(r)\psi + \Gamma(r)|\psi|^{p-1}\psi = 0$$

Rescale: $\psi(r,t) = \left(\frac{q(r)}{\Gamma(r)}\right)^{1/(p-1)} y\left(\sqrt{\frac{q(r)}{V(r)}}\, t\right)$



$$\ddot{y} + y + |y|^{p-1}y = 0$$

$$\dot{y}^2 + y^2 + \frac{2}{p+1}|y|^{p+1} = \text{cst.} = c$$

periodic orbits $y(t;c)$

- period $L(c) = 2\pi - O(c^{\frac{p-1}{2}})$
- $y(0;c) = N(c) := \max_{\mathbb{R}} y(t;c)$
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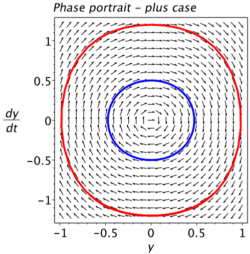

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How to choose $c = c(r)$?

$$\sqrt{\frac{q(r)}{V(r)}}T = L(c),\ c := L^{-1}\left(\sqrt{\frac{q(r)}{V(r)}}T\right)$$

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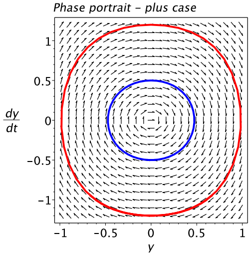

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Existence result (focusing & defocusing)

(semi) $$\nabla\times\nabla\times E + V(x)\partial_t^2 E + q(x)E \pm \Gamma(x)|E|^{p-1}E = 0$$

Ansatz: $E(x,t) = \psi(r,t)\frac{x}{r}, \quad r = |x|.$

$$V(r)\ddot{\psi} + q(r)\psi + \Gamma(r)|\psi|^{p-1}\psi = 0$$

Theorem

Let $T > 0$.

- $V, q, \Gamma > 0$ *radially symmetric C^2-functions,*
- $\sup \frac{q}{\Gamma} < \infty$,
- $T\sqrt{\frac{q(r)}{V(r)}} \lessgtr 2\pi$ *on* $\mathbb{R}^3$,
- $$\left|2\pi - T\sqrt{\frac{q(r)}{V(r)}}\right|^{\frac{1}{p-1}} = \begin{cases} O(e^{-\alpha r}) \text{ as } r\to\infty, \\ o(1) \text{ in } C^2\text{-sense as } r \to 0. \end{cases}$$

Then $\exists$ T-periodic, exponentially decaying solution.

The proof – changes in the defocusing case

$$V(r)\ddot{\psi} + q(r)\psi - \Gamma(r)|\psi|^{p-1}\psi = 0$$

Rescale: $\psi(r,t) = \left(\frac{q(r)}{\Gamma(r)}\right)^{1/(p-1)} y\left(\sqrt{\frac{q(r)}{V(r)}}t\right)$

$$\ddot{y} + y - |y|^{p-1}y = 0$$

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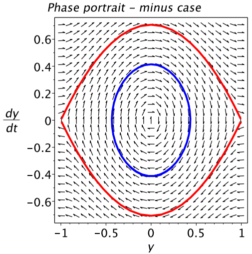

periodic orbits $y(t;c)$

- period $L(c) = 2\pi + O(c^{\frac{p-1}{2}})$
- $y(0;c) = N(c) := \max_{\mathbb{R}} y(t;c)$
- $\dot{y}(0;c) = 0$

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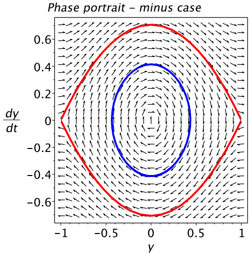

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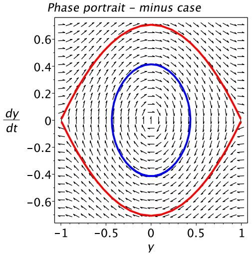

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Summary: real-valued curl-curl breathers

Karlsruhe Institute of Technology

(semi) $$\nabla \times \nabla \times E + V(x)\partial_t^2 E + q(x)E \pm \Gamma(x)|E|^{p-1}E = 0$$

- Use radial symmetry: $E(r,t) = \psi(r,t)\frac{x}{r}$
- T-periodic real breathers exist under natural conditions
  - $V, q, \Gamma > 0$,
  - $\sup \frac{q}{\Gamma} < \infty$,
  - $T\sqrt{\frac{q(r)}{V(r)}} \lessgtr 2\pi$ on $\mathbb{R}^3$,
  - $$\left|2\pi - T\sqrt{\frac{q(r)}{V(r)}}\right|^{\frac{1}{p-1}} = \begin{cases} O(e^{-\alpha r}) \text{ as } r \to \infty, \\ o(1) \text{ in } C^2\text{-sense as } r \to 0. \end{cases}$$
- Under the same assumptions:
  time-harmonic complex exponentially decaying solutions exist:
  $$E(x,t) = e^{i\frac{2\pi}{T}t}\psi(r)\frac{x}{r}$$