Large Conformal Metrics With Prescribed Sign-Changing Gauss Curvature

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The prescribed Gauss curvature problem

Let (M, g) be a two-dimensional compact Riemannian manifold.

Problem: Given a real-valued, sufficiently smooth function $\kappa(x)$ defined on M, we want to know if κ can be realized as the Gaussian curvature K_{g_1} of M for a metric g_1, which is in addition conformal to g, namely $g_1 = e^u g$ for some scalar function u on M.

By the uniformization theorem, without loss of generality we may assume that M has constant Gaussian curvature for g,

$$K_g =: -\alpha.$$

The relation $K_{g_1} = \kappa$ is equivalent to the following nonlinear partial differential equation

(1) $$\boldsymbol{\Delta_g u + \kappa e^u + \alpha = 0}, \quad \text{on } \boldsymbol{M}.$$

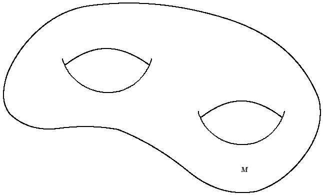
Integrating (1), assuming that M has surface area equal to one, and using the **Gauss-Bonet** formula we obtain

$$\int_M \kappa e^u d\mu_g = \int_M K_g d\mu_g = -\alpha = 2\pi\chi(M), \tag{2}$$

where $\chi(M) = 2(1 - g(M))$ is the Euler characteristic of the manifold M.

Assumption: $g(M) > 1$.

Thus: $\chi(M) < 0$, $\alpha > 0$.



Necessary condition for existence: $\kappa(x)$ has to be negative somewhere on M. Moreover, we must have $\int_M \kappa d\mu_g < 0$. Indeed testing (1) against e^{-u} we get

$$\int_M \kappa d\mu_g = -\int_M (|\nabla_g u|^2 + \alpha)e^{-u} d\mu_g < 0. \tag{3}$$

Solutions u to (1) correspond to **critical points** in the Sobolev space $H^1(M,g)$ of the energy functional

$$E_\kappa(u) = \frac{1}{2}\int_M |\nabla_g u|^2 d\mu_g - \alpha \int_M u\, d\mu_g - \int_M \kappa e^u d\mu_g. \tag{4}$$

Since $\alpha > 0$, if $0 \not\equiv \kappa \le 0$, then this functional is strictly convex and coercive in $H^1(M,g)$. It thus has a **unique critical point** u_κ which is a **global minimizer** of E_κ.

Question: What happens when κ changes sign? If $\sup_M \kappa > 0$, then E_κ is no longer bounded below, hence a global minimizer cannot exist.

When $0 \not\equiv \kappa \le 0$, there exists $C_0 > 0$ such that

$$\|h\|_1^2 \le C_0 d^2 E_\kappa(u_\kappa)(h,h), \quad \text{for all } h \in H^1(M,g).$$

By the **implicit function theorem**, this gives existence also for certain **sign-changing** prescribed Gauss curvature functions, which can be characterized as **relative minimizers** of the associated energy.

One result in this direction.

Theorem 1 (Bismuth, Bull. Sci. Math., 2000)

Let (M, g) *such that* $K_g < 0$, *and let* $\kappa \in C^\infty(M)$ *with*

$$\mathcal{K} = \{x \in M : 0 \leq \kappa(x)\} \not\subseteq M.$$

There exists a constant $C = C(\mathcal{K}, M) > 0$ *such that, if*

$$\sup_M \kappa \leq C \sup_M (-\kappa),$$

then κ *is the Gauss curvature of a metric conformal to* g.

In what follows we focus in a special class of functions which **change sign being nearly everywhere negative on** M. Let f be a function of class $C^3(M)$ such that

$$f \geq 0, \quad f \not\equiv 0, \quad \min_M f = 0.$$

For $\lambda > 0$, we let

$$\kappa_\lambda(x) = -f(x) + \lambda^2,$$

so that (1) now reads

(5) $$\boldsymbol{\Delta_g u - f e^u + \lambda^2 e^u + \alpha = 0, \quad \text{on } M.}$$

Ding and Liu (Trans. Amer. Math. Soc., 1995) proved that there exists $\lambda_0 > 0$ such that the **global minimizer** of E_{κ_0} persist as a **local minimizer** $\underline{u}_\lambda$ of E_{κ_λ}, for any $0 < \lambda < \lambda_0$. From (3) we see that

$$\lambda_0 < \left(\int_M f\right)^{1/2}.$$

Moreover, they established the existence of a **second non-minimizing** solution u_λ in this range. Besides, $\underline{u}_\lambda \to u_0$, as $\lambda \to 0$, while u_λ becomes **unbounded**.

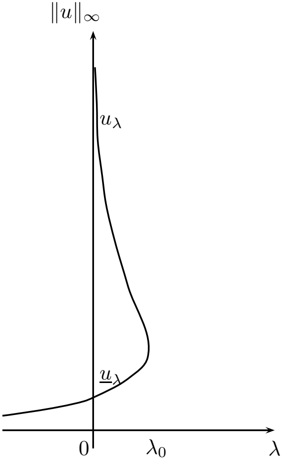


Figure : Bifurcation diagram for solutions of Problem (5)

A recent result.

The proof of Ding and Liu does not provide any information on the **asymptotic blowing-up behavior** or about the **number** of such "large" solutions. Borer, Galimberti and Struwe (Comm. Math. Helv., 2014) provided a new construction of the "mountain pass solution" for small λ, which allowed them to identify further properties of u_λ under the following generic assumption: **points of global minima of f are non-degenerate. This means that if $f(p) = 0$ then $D^2 f(p)$ is positive definite**.

Their result can be described as follows: along any sequence $\lambda = \lambda_k \to 0$, there exist points $p_1^k, \ldots, p_n^k$, $1 \leq n \leq 4$, converging to $p_1, \ldots, p_n$ points of global minima of f such that one of the following holds:

(i) There exist $\varepsilon_\lambda^1, \dots, \varepsilon_\lambda^n$, such that $\varepsilon_\lambda^i/\lambda \to 0$, $i = 1, \dots, k$, and in local conformal coordinates around p_i there holds

$$u_\lambda(\varepsilon_\lambda^i x) - u_\lambda(0) + \log 8 \to w(x) := \log \frac{8}{(1+|x|^2)^2}, \tag{6}$$

smoothly locally in $\mathbb{R}^2$. We note that

$$\Delta w + e^w = 0, \quad \text{in } \mathbb{R}^2.$$

(ii) In local conformal coordinates around p_i, with a constant c_i there holds

$$u_n(\lambda x) + 4\log(\lambda) + c_i \to w_\infty(x),$$

smoothly locally in $\mathbb{R}^2$, where w_∞ satisfies

$$\Delta w_\infty + [1 - (Ax, x)]e^{w_\infty} = 0, \quad \text{in } \mathbb{R}^2,$$

where $A = \frac{1}{2} D^2 f(p_i)$.

In order to state our main result, we consider the singular problem

$$\Delta_g G - f e^G + 8\pi \sum_{i=1}^{n} \delta_{p_i} + \alpha = 0, \quad \text{on } M, \tag{7}$$

where δ_{p_i} designates the Dirac mass at the point p_i. We have the following result.

Lemma 1

Problem 7 has a unique solution G which is smooth away from the singularities and in local conformal coordinates around p_i it has the form

$$G(x) = -4\log|x| - 2\log\left(\frac{1}{\sqrt{2}}\log\frac{1}{|x|}\right) + \mathcal{H}(x), \tag{8}$$

where $\mathcal{H}(x) \in C(M)$.

Main result

Theorem 2 (del Pino, Roman, Calc. Var. PDE 2015)

Let $p_1, \ldots, p_n$ be points such that $f(p_i) = 0$ and $D^2 f(p_i)$ is positive definite for each i. Then, there exists a family of solutions u_λ to (5) with

$$\lambda^2 e^{u_\lambda} \rightharpoonup 8\pi \sum_{i=1}^{n} \delta_{p_i}, \quad \text{as } \lambda \to 0,$$

and $u_\lambda \to G$ uniformly in compacts subsets of $M \setminus \{p_1, \ldots, p_k\}$. We define

$$c_i = \frac{1}{2} e^{\mathcal{H}(p_i)/2}, \quad \delta_\lambda^i = \frac{c_i}{|\log \lambda|}, \quad \varepsilon_\lambda^i = \lambda \delta_\lambda^i,$$

where $\mathcal{H}$ is defined near p_i by relation (8). In local conformal coordinates around p_i, there holds

$$u_\lambda(\varepsilon_\lambda^i x) + 4 \log \lambda + 2 \log \delta_\lambda^i \to \log \frac{8}{(1 + |x|^2)^2},$$

uniformly on compact sets of $\mathbb{R}^2$ as $\lambda \to 0$.

In particular if f has exactly m non-degenerate global minimum points, then $2^m - 1$ distinct large solutions exist for all sufficiently small λ.

The key ingredient of the proof of Lemma 1 is the function

$$V(|x|) = -4\log|x| - 2\log\left(\frac{1}{\sqrt{2}}\log\frac{1}{|x|}\right),$$

which solves

$$\Delta V - |x|^2 e^V + 8\pi\delta_0 = 0.$$

This equation is important due to the fact that $p_1, \ldots, p_n$ are non-degenerate points of global minima of f.

The proof of our main result consists of the construction of a suitable first approximation of a solution as required, and then solving by linearization and a suitable Lyapunov-type reduction.

The "basic cells" for the construction of the first approximation are the radially symmetric solutions of the problem

$$\begin{cases} \Delta w + \lambda^2 e^w = 0 & \text{in } \mathbb{R}^2, \\ w(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

which are given by the one-parameter family of functions

$$w_\delta(|x|) = \log \frac{8\delta^2}{(\lambda^2\delta^2 + |x|^2)^2},$$

where δ is any positive number. We define $\varepsilon = \lambda\delta$.
To make the construction precise enough, we dealt with the equation

$$\Delta F - \frac{\delta^2}{r^2} e^F = 0, \tag{9}$$

in the variable $r = |x|/\varepsilon$ and we look for a radial solution $F = F(r)$, away from $r = 0$. We solve (9) under the following conditions

$$F(1/\delta) = 0, \quad F'(1/\delta) = 0.$$

This problem has a unique regular solution, which blows-up at distance $O(1/\lambda)$ from the origin. We conclude that the solution $F(r)$ is defined for all $1/\delta \leq r \leq Ce^{1/\delta} = C/\lambda$, for some constant C. Besides, we extend by 0 the function F for $r \in [0, 1/\delta)$.

In order to build a global approximation, we consider a smooth radial cut-off function η such that $\eta(r) = 1$ if $r \leq C_1\delta$ and $\eta(r) = 0$ if $r \geq C_2\delta$, for constants $0 < C_1 < C_2$. We consider as initial approximation

$$U_\varepsilon = \eta u_\varepsilon + (1-\eta)G,$$

where G is the Green's function,

$$u_\varepsilon(x) = \log \frac{8\delta^2}{(\varepsilon^2 + |x-k|^2)^2} + F,$$

and $k \in \mathbb{R}^2$ is a parameter related to translations.

Choice of δ: $$\log 8\delta^2 = -2\log\left(\frac{1}{\sqrt{2}}\log\frac{1}{\lambda}\right) + \mathcal{H}(p).$$