LARGE CONFORMAL METRICS WITH PRESCRIBED SIGN-CHANGING GAUSS CURVATURE

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The prescribed Gauss curvature problem

Let \((M, g)\) be a two-dimensional compact Riemannian manifold.

**Problem:** Given a real-valued, sufficiently smooth function \(\kappa(x)\) defined on \(M\), we want to know if \(\kappa\) can be realized as the Gaussian curvature \(K_{g_1}\) of \(M\) for a metric \(g_1\), which is in addition conformal to \(g\), namely \(g_1 = e^u g\) for some scalar function \(u\) on \(M\).

By the uniformization theorem, without loss of generality we may assume that \(M\) has constant Gaussian curvature for \(g\),

\[ K_g =: -\alpha. \]

The relation \(K_{g_1} = \kappa\) is equivalent to the following nonlinear partial differential equation

\[ \Delta_g u + \kappa e^u + \alpha = 0, \quad \text{on } M. \]
Integrating (1), assuming that $M$ has surface area equal to one, and using the Gauss-Bonet formula we obtain

\[ \int_M \kappa e^u d\mu_g = \int_M K_g d\mu_g = -\alpha = 2\pi \chi(M), \]

where $\chi(M) = 2(1 - g(M))$ is the Euler characteristic of the manifold $M$.

**Assumption:** $g(M) > 1$.

Thus: $\chi(M) < 0$, $\alpha > 0$.

**Necessary condition for existence:** $\kappa(x)$ has to be negative somewhere on $M$. Moreover, we must have $\int_M \kappa d\mu_g < 0$. Indeed testing (1) against $e^{-u}$ we get

\[ \int_M \kappa d\mu_g = -\int_M (|\nabla_g u|^2 + \alpha) e^{-u} d\mu_g < 0. \]
Solutions $u$ to (1) correspond to critical points in the Sobolev space $H^1(M, g)$ of the energy functional

$$E_\kappa(u) = \frac{1}{2} \int_M |\nabla_g u|^2 d\mu_g - \alpha \int_M u d\mu_g - \int_M \kappa e^u d\mu_g.$$ 

Since $\alpha > 0$, if $0 \not\equiv \kappa \leq 0$, then this functional is strictly convex and coercive in $H^1(M, g)$. It thus has a unique critical point $u_\kappa$ which is a global minimizer of $E_\kappa$.

Question: What happens when $\kappa$ changes sign? If $\sup_M \kappa > 0$, then $E_\kappa$ is no longer bounded below, hence a global minimizer cannot exist.

When $0 \not\equiv \kappa \leq 0$, there exists $C_0 > 0$ such that

$$\|h\|_1^2 \leq C_0 d^2 E_\kappa(u_\kappa)(h, h), \quad \text{for all } h \in H^1(M, g).$$

By the implicit function theorem, this gives existence also for certain sign-changing prescribed Gauss curvature functions, which can be characterized as relative minimizers of the associated energy.
One result in this direction.


Let $(M, g)$ such that $K_g < 0$, and let $\kappa \in C^\infty(M)$ with

$$K = \{ x \in M : 0 \leq \kappa(x) \} \not\subseteq M.$$ 

There exists a constant $C = C(K, M) > 0$ such that, if

$$\sup_M \kappa \leq C \sup_M (-\kappa),$$

then $\kappa$ is the Gauss curvature of a metric conformal to $g$. 

In what follows we focus in a special class of functions which change sign being nearly everywhere negative on $M$. Let $f$ be a function of class $C^3(M)$ such that

$$f \geq 0, \quad f \not\equiv 0, \quad \min_M f = 0.$$

For $\lambda > 0$, we let

$$\kappa_\lambda(x) = -f(x) + \lambda^2,$$

so that (1) now reads

(5) $$\Delta_g u - fe^u + \lambda^2 e^u + \alpha = 0, \quad \text{on } M.$$
Ding and Liu (Trans. Amer. Math. Soc., 1995) proved that there exists $\lambda_0 > 0$ such that the **global minimizer** of $E_{\kappa_0}$ persist as a **local minimizer** $u_\lambda$ of $E_{\kappa_\lambda}$, for any $0 < \lambda < \lambda_0$. From (3) we see that

$$\lambda_0 < \left( \int_M f \right)^{1/2}.$$ 

Moreover, they established the existence of a **second non-minimizing** solution $u_\lambda$ in this range. Besides, $u_\lambda \to u_0$, as $\lambda \to 0$, while $u_\lambda$ becomes **unbounded**.

**Figure**: Bifurcation diagram for solutions of Problem (5)
A recent result.

The proof of Ding and Liu does not provide any information on the **asymptotic blowing-up behavior** or about the **number** of such “large” solutions. Borer, Galimberti and Struwe (Comm. Math. Helv., 2014) provided a new construction of the “mountain pass solution” for small $\lambda$, which allowed them to identify further properties of $u_\lambda$ under the following generic assumption: **points of global minima of $f$ are non-degenerate. This means that if $f(p) = 0$ then $D^2 f(p)$ is positive definite.**

Their result can be described as follows: along any sequence $\lambda = \lambda_k \to 0$, there exist points $p^k_1, \ldots, p^k_n$, $1 \leq n \leq 4$, converging to $p_1, \ldots, p_n$ points of global minima of $f$ such that one of the following holds:
(i) There exist \( \varepsilon_1, \ldots, \varepsilon_n \), such that \( \varepsilon_i / \lambda \to 0 \), \( i = 1, \ldots, k \), and in local conformal coordinates around \( p_i \) there holds

\[
(6) \quad u_\lambda (\varepsilon_i \lambda x) - u_\lambda (0) + \log 8 \to w(x) := \log \frac{8}{(1 + |x|^2)^2},
\]

smoothly locally in \( \mathbb{R}^2 \). We note that

\[
\Delta w + e^w = 0, \quad \text{in} \ \mathbb{R}^2.
\]

(ii) In local conformal coordinates around \( p_i \), with a constant \( c_i \) there holds

\[
u_n (\lambda x) + 4 \log(\lambda) + c_i \to w_\infty (x),
\]

smoothly locally in \( \mathbb{R}^2 \), where \( w_\infty \) satisfies

\[
\Delta w_\infty + [1 - (Ax, x)] e^{w_\infty} = 0, \quad \text{in} \ \mathbb{R}^2,
\]

where \( A = \frac{1}{2} D^2 f(p_i) \).
In order to state our main result, we consider the singular problem

(7) \[ \Delta_g G - f e^G + 8\pi \sum_{i=1}^{n} \delta_{p_i} + \alpha = 0, \quad \text{on } M, \]

where \( \delta_{p_i} \) designates the Dirac mass at the point \( p_i \). We have the following result.

**Lemma 1**

*Problem 7 has a unique solution \( G \) which is smooth away from the singularities and in local conformal coordinates around \( p_i \) it has the form*

(8) \[ G(x) = -4 \log |x| - 2 \log \left( \frac{1}{\sqrt{2}} \log \frac{1}{|x|} \right) + \mathcal{H}(x), \]

*where \( \mathcal{H}(x) \in C(M) \).*
Main result

Theorem 2 (del Pino, Roman, Calc. Var. PDE 2015)

Let $p_1, \ldots, p_n$ be points such that $f(p_i) = 0$ and $D^2 f(p_i)$ is positive definite for each $i$. Then, there exists a family of solutions $u_\lambda$ to (5) with

$$\lambda^2 e^{u_\lambda} \rightarrow 8\pi \sum_{i=1}^{n} \delta_{p_i}, \quad \text{as } \lambda \rightarrow 0,$$

and $u_\lambda \rightarrow G$ uniformly in compacts subsets of $M \setminus \{p_1, \ldots, p_k\}$. We define

$$c_i = \frac{1}{2} e^{H(p_i)/2}, \quad \delta^i_\lambda = \frac{c_i}{|\log \lambda|}, \quad \varepsilon^i_\lambda = \lambda \delta^i_\lambda,$$

where $H$ is defined near $p_i$ by relation (8). In local conformal coordinates around $p_i$, there holds

$$u_\lambda(\varepsilon^i_\lambda x) + 4 \log \lambda + 2 \log \delta^i_\lambda \rightarrow \log \frac{8}{(1 + |x|^2)^2},$$

uniformly on compact sets of $\mathbb{R}^2$ as $\lambda \rightarrow 0$. 
In particular if $f$ has exactly $m$ non-degenerate global minimum points, then $2^m - 1$ distinct large solutions exist for all sufficiently small $\lambda$.

The key ingredient of the proof of Lemma 1 is the function

$$V(|x|) = -4 \log |x| - 2 \log \left( \frac{1}{\sqrt{2}} \log \frac{1}{|x|} \right),$$

which solves

$$\Delta V - |x|^2 e^V + 8\pi \delta_0 = 0.$$ 

This equation is important due to the fact that $p_1, \ldots, p_n$ are non-degenerate points of global minima of $f$.

The proof of our main result consists of the construction of a suitable first approximation of a solution as required, and then solving by linearization and a suitable Lyapunov-type reduction.
The “basic cells” for the construction of the first approximation are the radially symmetric solutions of the problem

\[
\begin{aligned}
\left\{ \begin{array}{l}
\Delta w + \lambda^2 e^w = 0 \quad \text{in } \mathbb{R}^2, \\
w(x) \to 0 \quad \text{as } |x| \to \infty.
\end{array} \right.
\end{aligned}
\]

which are given by the one-parameter family of functions

\[
w_\delta(|x|) = \log \frac{8\delta^2}{(\lambda^2 \delta^2 + |x|^2)^2},
\]

where \( \delta \) is any positive number. We define \( \varepsilon = \lambda \delta \).

To make the construction precise enough, we dealt with the equation

\[
(9) \quad \Delta F - \frac{\delta^2}{r^2} e^F = 0
\]

in the variable \( r = |x|/\varepsilon \) and we look for a radial solution \( F = F(r) \), away from \( r = 0 \). We solve (9) under the following conditions

\[
F(1/\delta) = 0, \quad F'(1/\delta) = 0.
\]
This problem has a unique regular solution, which blows-up at distance $O(1/\lambda)$ from the origin. We conclude that the solution $F(r)$ is defined for all $1/\delta \leq r \leq Ce^{1/\delta} = C/\lambda$, for some constant $C$. Besides, we extend by 0 the function $F$ for $r \in [0, 1/\delta)$.

In order to build a global approximation, we consider a smooth radial cut-off function $\eta$ such that $\eta(r) = 1$ if $r \leq C_1 \delta$ and $\eta(r) = 0$ if $r \geq C_2 \delta$, for constants $0 < C_1 < C_2$. We consider as initial approximation

$$U_\varepsilon = \eta u_\varepsilon + (1 - \eta)G,$$

where $G$ is the Green’s function,

$$u_\varepsilon(x) = \log \frac{8\delta^2}{(\varepsilon^2 + |x - k|^2)^2} + F,$$

and $k \in \mathbb{R}^2$ is a parameter related to translations.

**Choice of $\delta$:**

$$\log 8\delta^2 = -2 \log \left( \frac{1}{\sqrt{2}} \log \frac{1}{\lambda} \right) + \mathcal{H}(p).$$