Vortex-type solutions to a magnetic nonlinear Choquard equation

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A magnetic nonlinear Choquard equation

We shall consider the following problem

$$\begin{cases} (-\mathrm{i}\nabla + A(x))^2 u + (V_\infty + V(x))u = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u, \\ u \in L^2(\mathbb{R}^N, \mathbb{C}), \\ \nabla u + \mathrm{i}\,A(x)u \in L^2(\mathbb{R}^N, \mathbb{C}^N), \end{cases}$$

- $(-\mathrm{i}\nabla + A(x))^2 = -\Delta - \mathrm{i}\,\mathrm{div}A - 2\,\mathrm{i}\,A\cdot\nabla + |A|^2$
- $A : \mathbb{R}^N \to \mathbb{R}^N$ is a $\mathcal{C}^1$-vector potential.
- i *is the imaginary unit.*
- $V : \mathbb{R}^N \to \mathbb{R} \in \mathcal{C}^0$, $V_\infty \in (0,\infty)$, $\inf_{x\in\mathbb{R}^N}\{V_\infty + V(x)\} > 0$, $\lim_{|x|\to\infty} V(x) = 0$.
- $*$ *is the convolution.*
- $\alpha \in (0,N)$, $\quad N \geq 3$, $\quad p \in \left(\frac{2N-\alpha}{N}, \frac{2N-\alpha}{N-2}\right)$.

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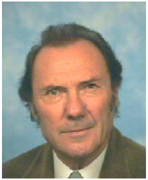
The Choquard equation

The physical model

$$-\Delta u + u = \left(\frac{1}{|x|} * |u|^2\right) u, \quad u \in H^1(\mathbb{R}^3).$$



- It was proposed by Philippe Choquard in 1976 to describe an electron trapped in its own hole, in a certain approximation to Hartree–Fock theory of one component plasma. (Lieb, Lieb–Simon, 1977)

- This equation arises in many interesting situations related to the quantum theory of large systems of non-relativistic bosonic atoms and molecules.

The Schrödinger-Newton equation

The physical model

$$-\Delta u + u = \left(\frac{1}{|x|} * |u|^2\right) u, \quad u \in H^1(\mathbb{R}^3).$$

- It was also proposed by Roger Penrose in 1996 as a model of self-gravitating matter, in a programme in which quantum state reduction is understood as a gravitational phenomenon.

The Choquard equation

On the most closely related known results

- **Lieb (1976)** proved the existence and uniqueness (modulo translations) of the ground state to

$$-\Delta u + u = \left(\frac{1}{|x|} * |u|^2\right) u, \quad u \in H^1(\mathbb{R}^3).$$

- **Lions (1980)** showed the existence of infinitely many radially symmetric solutions.

The general Choquard equation

Existence and properties of ground states

$$-\Delta u + u = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N),$$

where $N \geq 3$, $\alpha \in (0, N)$, $p \in \left(\frac{2N-\alpha}{N}, \frac{2N-\alpha}{N-2}\right)$.

- **Lions'** concentration compactness method yields the existence of a ground state.

Theorem (Moroz and van Schaftingen 2012)

Let ω be a ground state of

$$-\Delta u + u = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N).$$

Then $\omega \in L^1(\mathbb{R}^N) \cap \mathcal{C}^\infty(\mathbb{R}^N)$, ω does not change sign and it is radially symmetric and monotone decreasing in the radial direction with respect to some fixed point.

The general Choquard equation

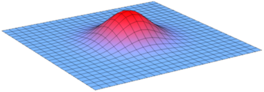
Properties of the ground states

Moreover, ω has the following asymptotic behaviour:

(i) *If $p > 2$ then* $\lim_{|x|\to\infty} |\omega(x)||x|^{\frac{N-1}{2}} e^{|x|} \in (0,\infty)$.

(ii) *If $p = 2$ then, for every $\varepsilon \in (0,1)$*

$$\lim_{|x|\to\infty} |\omega(x)||x|^{\frac{N-1}{2}} e^{\varepsilon|x|} \in (0,\infty).$$



Problem (Open)

Uniqueness of ground states is an open question for $N > 3$.

The general Choquard equation

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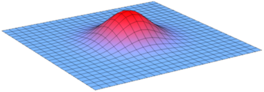

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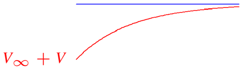
The magnetic nonlinear Choquard equation



$$(\wp_{A,V}) \quad \begin{cases} (-\mathrm{i}\nabla + A(x))^2 u + (V_\infty + V(x))u = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u, \\ u \in L^2(\mathbb{R}^N, \mathbb{C}), \end{cases}$$

Theorem (Cingolani-Clapp-Secchi 2012)

Assume that:

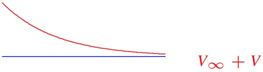
- $N \geq 3, \quad \alpha \in (0, N) \quad$ *and* $\quad p \in \left[2, \frac{2N-\alpha}{N-2}\right)$.
- $\lim_{|x|\to\infty} V(x) = 0$ *and there exist* $c_0 > 0$, $\rho > 0$ *and* $\kappa \in (0, 2\delta_\phi\sqrt{V_\infty})$ *such that*
$$|A(x)|^2 + V(x) \leq -c_0 e^{-\kappa|x|}, \quad \text{for all } x \in \mathbb{R}^N \text{ with } |x| \geq \rho.$$

Then problem $(\wp_{A,V})$ *has at least one nontrivial solution exhibiting a vortex type behavior.*

The magnetic nonlinear Choquard equation

Problem

- What can we say when the potential $V_\infty + V$ approaches to its limit at infinity exponentially from above?

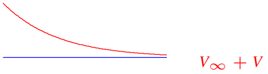


The local nonlinear Schrödinger equation

For the equation

$$-\Delta u + (V_\infty + V(x))u = |u|^{p-2}u, \qquad u \in H^1(\mathbb{R}^N),$$

Bahri and Lions solved the question of the existence, for potentials which approach to its limit from above, without any symmetry assumption.



A magnetic nonlinear Choquard equation

$$
\begin{cases}
(-\mathrm{i}\nabla + A(x))^2 u + (V_\infty + V(x))u = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u, \\
u \in L^2(\mathbb{R}^N, \mathbb{C}), \\
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\end{cases}
$$

- $N \geq 3, \quad \alpha \in (0, N), \quad p \in \left(\frac{2N-\alpha}{N}, \frac{2N-\alpha}{N-2}\right).$
- $A : \mathbb{R}^N \to \mathbb{R}^N$ is a $\mathcal{C}^1$-vector potential.
- $V \in \mathcal{C}^0(\mathbb{R}^N),\ V_\infty \in (0, \infty),\ \inf_{x \in \mathbb{R}^N}\{V_\infty + V(x)\} > 0,$ $\lim_{|x| \to \infty} V(x) = 0.$

The variational framework

- Assume without loss of generality that $V_\infty = 1$.
- Set $\nabla_A u := \nabla u + \mathrm{i}Au$.
- Consider the real Hilbert space

$$H^1_A(\mathbb{R}^N, \mathbb{C}) := \{u \in L^2(\mathbb{R}^N, \mathbb{C}) : \nabla_A u \in L^2(\mathbb{R}^N, \mathbb{C}^N)\}$$

with the scalar product

$$\langle u, v\rangle_{A,V} := \mathrm{Re} \int_{\mathbb{R}^N} \left(\nabla_A u \cdot \overline{\nabla_A v} + (1 + V(x))\, u\bar{v}\right).$$

- Assumption (V_0) guarantees that the induced norm

$$\|u\|_{A,V} := \left(\int_{\mathbb{R}^N} \left(|\nabla_A u|^2 + (1 + V(x))\, |u|^2\right)\right)^{1/2}$$

is equivalent to the usual one, defined by taking $V \equiv 0$.

The variational framework

Theorem (Diamagnetic inequality)

Let $A \in L^2_{loc}(\mathbb{R}^N)$ and $u \in H^1_A(\mathbb{R}^N, \mathbb{C})$. Then $|u| \in H^1(\mathbb{R}^N, \mathbb{R})$ and the diamagnetic inequality

$$|\nabla |u|(x)| \leq |(\nabla + \mathrm{i}A)u(x)| \quad a.e.\ x \in \mathbb{R}^N.$$

- $C_0^\infty(\mathbb{R}^N, \mathbb{C})$ is dense in $H^1_A(\mathbb{R}^N, \mathbb{C})$.

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- $\mathcal{C}_0^\infty(\mathbb{R}^N, \mathbb{C})$ is dense in $H^1_A(\mathbb{R}^N, \mathbb{C})$.

The variational problem

The functional

The solutions to

$$(-\mathrm{i}\nabla + A(x))^2 u + (V_\infty + V(x))u = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u$$

are the critical points of the energy functional

$$J_{A,V} : H^1_A(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$$

$$J_{A,V}(u) := \frac{1}{2}\|u\|^2_{A,V} - \frac{1}{2p}\mathbb{D}(u),$$

where

$$\mathbb{D}(u) := \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^p = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^\alpha}dx\,dy$$

The variational problem

The Nehari manifold

$J_{A,V}$ is well defined and of class $\mathcal{C}^2$ if $p \in \left[2, \frac{2N-\alpha}{N-2}\right)$.

The nontrivial critical points of $J_{A,V}$ lie on the Nehari manifold

$$\begin{aligned}\mathcal{N}_{A,V} &:= \left\{u \in H^1_A(\mathbb{R}^N, \mathbb{C}) : J'_{A,V}(u)u = 0\right\} \\ &= \left\{u \in H^1_A(\mathbb{R}^N, \mathbb{C}) : u \neq 0,\ \|u\|^2_{A,V} = \mathbb{D}(u)\right\},\end{aligned}$$

which is of class $\mathcal{C}^2$ and radially diffeomorphic to the unit sphere in $H^1_A(\mathbb{R}^N, \mathbb{C})$.

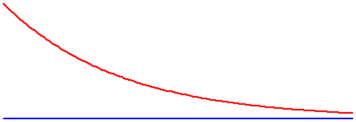
The variational problem

Nonexistence of minimizer

We set

$$c_{0,V} := \inf_{\mathcal{N}_{0,V}} J_{0,V}.$$

A minimizer does not necessarily exist



Proposition

If $V \geq 0$ and $V \not\equiv 0$, then $c_{0,V} = c_{0,\infty}$ and $c_{0,V}$ is not attained.

- $c_{0,\infty}$ is the energy of a ground state of the problem

$$\begin{cases} -\Delta u + u = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u, \\ u \in H^1(\mathbb{R}^N, \mathbb{C}). \end{cases}$$

The variational problem

Other solutions

Looking for solutions to $(\wp_V)$ *which are not ground states is hard, because still very little is known about the limit problem*

$$\begin{cases} -\Delta u + u = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u, \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

Symmetry Methods

Further assumptions

$$(\wp_{A,V}) \quad \begin{cases} (-\mathrm{i}\nabla + A(x))^2 u + (V_\infty + V(x))u = \left(\frac{1}{|x|^\alpha} * |u|^p\right) |u|^{p-2}u, \\ u \in L^2(\mathbb{R}^N, \mathbb{C}), \end{cases}$$

- We consider a closed subgroup G of the group $O(N)$ of linear isometries of $\mathbb{R}^N$ and denote by

$$Gx := \{gx : g \in G\}$$

the G-orbit of x.

- We assume that A and V satisfy

$$A(gx) = gA(x) \text{ and } V(gx) = V(x) \quad \text{for all } g \in G \text{ and } x \in \mathbb{R}^N.$$

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Symmetry Methods

Further assumptions

- We consider a continuous homomorphism $\phi : G \to \mathbb{S}^1$.
- We look for intertwining solutions u, i.e.

$$u(gx) = \phi(g)u(x) \quad \forall g \in G, \forall x \in \mathbb{R}^N.$$

- Note that, if u is intertwining, then $|u|$ is G-invariant, i.e.

$$|u(gx)| = |u(x)| \qquad \text{for all } g \in G \text{ and } x \in \mathbb{R}^N.$$

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The variational framework

for the symmetric problem

(The principle of symmetric criticality, Palais 1979)

The critical points of the restriction of $J_{A,V}$ to the fixed point space of the G-action, defined as

$$H^1_A(\mathbb{R}^N,\mathbb{C})^\phi := \{u \in H^1_A(\mathbb{R}^N,\mathbb{C}) : u \text{ is intertwining}\}.$$

are the intertwining solutions to $(\wp_{A,V})$

The variational framework

for the symmetric problem

- The nontrivial ones lie on the Nehari manifold

$$\mathcal{N}^{\phi}_{A,V} := \left\{ u \in H^1_A(\mathbb{R}^N, \mathbb{C})^{\phi} : u \neq 0,\ \|u\|^2_{A,V} = \mathbb{D}(u) \right\},$$

  which is of class $\mathcal{C}^2$ and radially diffeomorphic to the unit sphere in $H^1_A(\mathbb{R}^N, \mathbb{C})^{\phi}$.

- Set

$$c^{\phi}_{A,V} := \inf_{\mathcal{N}^{\phi}_{A,V}} J_{A,V} = \inf_{u \in H^1_A(\mathbb{R}^N,\mathbb{C})^{\phi} \smallsetminus \{0\}} \max_{t \geq 0} J_{A,V}(tu).$$

A compactness condition

Definition

The functional $J_{A,V} : H^1_A(\mathbb{R}^N, \mathbb{C})^\phi \to \mathbb{R}$ is said to satisfy the Palais-Smale condition $(PS)_c$ at the level c, if every sequence (v_n) such that

$$v_n \in H^1_A(\mathbb{R}^N, \mathbb{C})^\phi, \qquad J_{A,V}(v_n) \to c, \qquad \nabla J_{A,V}(v_n) \to 0,$$

contains a convergent subsequence.

Proposition (Cingolani-Clapp-Secchi 2012)

The functional $J_{A,V}$ satisfies condition $(PS)_c$ at each

$$c < \left(\min_{x \in \mathbb{R}^N \smallsetminus \{0\}} \#Gx \right) c_{0,\infty}.$$

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Vortex-type solutions

to a magnetic nonlinear Choquard equation

Theorem (Cingolani-Clapp-Secchi 2012)

If $\#Gx = \infty$ *for all* $x \neq 0$ *and if there exists* $x \in \mathbb{R}^N$ *such that*

$$G_x := \{g \in G : gx = x\} \subset \ker \phi,$$

then $(\wp_{A,V})$ *has infinitely many intertwining solutions.*

Vortex-type solutions

to a magnetic nonlinear Choquard equation

Problem

- What can we say about existence of intertwining solutions when there are finite G-orbits?

Vortex-type solutions

to a magnetic nonlinear Choquard equation

If
$$c^{\phi}_{A,V} := \inf_{\mathcal{N}^{\phi}_{A,V}} J_{A,V} < \left(\min_{x\in\mathbb{R}^N \setminus \{0\}} \#Gx \right) c_{0,\infty},$$

then $(\wp_{A,V})$ has an intertwining solution.

- **Our goal** is to give conditions on A and V so that

$$c^{\phi}_{A,V} < \left(\min_{x\in\mathbb{R}^N \setminus \{0\}} \#Gx \right) c_{0,\infty},$$

Vortex-type solutions

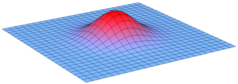
to a magnetic nonlinear Choquard equation

- Set

$$\Sigma^{\phi} := \{x \in \mathbb{R}^N : |x| = 1, \#Gx = \min_{x \in \mathbb{R}^N \setminus \{0\}} \#Gx,\ G_x \subset \ker \phi\}.$$

- Let ω be a ground state of the problem

$$\begin{cases} -\Delta u + u = \left(\frac{1}{|x|^{\alpha}} * |u|^p \right) |u|^{p-2} u, \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

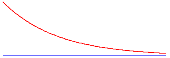


- For $z \in \Sigma^{\phi}$ let us consider

$$\sigma_{Rz}(x) := \sum_{gz \in Gz} \phi(g)\omega(x - Rgz), \quad R > 0.$$

Vortex-type solutions

to a magnetic nonlinear Choquard equation



$V_\infty + V$

Theorem (S. 2015)

If $p = 2$ and the following hold:

$(\tilde{Z}_0)$ *There exist $z \in \Sigma^\phi$ and $a_0 > 1$ such that*

$$|gz - hz| \geq a_0\, \mu^\phi(Gz) \qquad \textit{for all } g, h \in G \textit{ with } Re\big(\phi(g)\overline{\phi(h)}\big) < 0,$$

(AV) *There exist $c_0 > 0$ and $\kappa > \mu^\phi(Gz)\sqrt{V_\infty}$ such that*

$$\big||A(x)|^2 + V(x)\big| \leq c_0 e^{-\kappa|x|} \qquad \textit{for all } x \in \mathbb{R}^N,$$

then $(\wp_{A,V})$ has at least one vortex–type solution.

$$\mu^\phi(Gz) := \min\{|gz - hz| : g, h \in G,\ gz \neq hz, \mathrm{Re}\big(\phi(g)\overline{\phi(h)}\big) > 0\}$$

Vortex type solutions

An example

- Fix $k \in \mathbb{N}$, $k \geq 2$ and let $G_k = \langle \xi \rangle$ with $\xi := e^{\mathrm{i}\frac{2\pi}{k}}$.
- If N is even, G_k acts on $\mathbb{R}^N \equiv \mathbb{C}^{N/2}$ by complex multiplication on each complex coordinate.
- For each $m \in \mathbb{N}$, $m \geq 1$, consider

$$\phi_m : G_k \to \mathbb{S}^1, \quad \phi_m(\xi) = \xi^m.$$

- If $k > 4m$, then $(\tilde{Z}_0)$ is satisfied for any $z \in \mathbb{S}^{N-1}$.
- Hence if, additionally, A and V are compatible with this action and satisfy (AV), our theorem yields at least one solution to problem $(\wp_{A,V})$.

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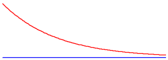

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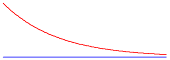
Vortex type solutions

Remark

- For $2 < p \leq 4$, the approach used in the nonmagnetic case to obtain the asymptotic estimates does not work.
- For $p > 4$, the inequality $p < \frac{2N-\alpha}{N-2}$ holds only for $N = 3$ and $\alpha \in (0,2)$; however condition $(\tilde{Z}_0)$ cannot be realized in dimension $N = 3$.

G-invariant solutions
to a magnetic nonlinear Choquard equation



$V_\infty + V$

- if $\phi \equiv 1$, we define

$$\mu_G := \inf_{z \in \Sigma^\phi} \mu^\phi(Gz)$$

Theorem (S. 2015)

If $p \geq 2$, $min_{x \in \mathbb{R}^N \setminus \{0\}} \# Gx \geq 3$ *and*

(AV_1) *There exist* $c_0 > 0$ *and* $\kappa > \mu_G \sqrt{V_\infty}$ *such that*

$$\left| |A(x)|^2 + V(x) \right| \leq c_0 e^{-\kappa |x|} \qquad \textit{for all } x \in \mathbb{R}^N,$$

then $(\wp_{A,V})$ *has at least one solution* u *which is G-invariant.*

G-invariant solutions

to a magnetic nonlinear Choquard equation

- When N is even, there are many groups satisfying the symmetry assumption in last theorem.
- When N is odd not many groups satisfy $\ell(G) \geq 3$. For example, if $N = 3$, the only subgroups of $O(3)$ which satisfy this condition are the rotation groups of the icosahedron, octahedron and tetrahedron, I, O and T, and the groups $I \times \mathbb{Z}_2^c$, $O \times \mathbb{Z}_2^c$, $T \times \mathbb{Z}_2^c$ and O^-.

Chossat, P., Lauterbach, R. and Melbourne, I.: *Steady-state bifurcation with O(3)-symmetry*. Arch. Ration. Mech. Anal. (1990).

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Further questions

Problem 1. Does problem $(\wp_{A,V})$ have a solution if

$$V(x) \leq c_0 e^{-\kappa|x|} \qquad \text{for all } x \in \mathbb{R}^N,$$

without any symmetry assumption?

- It will not be a ground state.

Thank you very much for your attention!