Some recent results on pseudo-relativistic Hartree equations

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Pseudo-relativistic Hartree equations

$$\mathrm{i}\varepsilon\frac{\partial\psi}{\partial t}=\left(\sqrt{-\varepsilon^2\Delta+m^2}-m\right)\psi+V\psi-\left(\frac{1}{|x|}*|\psi|^2\right)\psi \qquad (1)$$

where

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- $\varepsilon > 0$ is the semiclassical parameter (a scaled Planck constant)
- V is a bounded external potential in $\mathbb{R}^3$.

Physical motivation

- Equation (1) emerges as the correct evolution equation for the mean-field dynamics of many-body quantum systems modelling pseudo-relativistic boson stars in astrophysics (Elgart and Schlein, 2007).

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- Equation (1) emerges as the correct evolution equation for the mean-field dynamics of many-body quantum systems modelling pseudo-relativistic boson stars in astrophysics (Elgart and Schlein, 2007).
- The external potential V accounts for gravitational fields from other stars.
- Eq. (1) can also be derived from the coupling of a pseudo-relativistic Schrödinger equation and a Poisson equation:

$$\begin{cases} \mathrm{i}\varepsilon\frac{\partial\psi}{\partial t} = \left(\sqrt{-\varepsilon^2\Delta + m^2} - m\right) + V\psi - U\psi \\ -\Delta U = |\psi|^2 \end{cases}$$

Solitary waves

An *ansatz* like

$$\psi(t,x) = \mathrm{e}^{\mathrm{i}t\lambda/\varepsilon} u(x)$$

leads to the non-local stationary equation

$$\sqrt{-\varepsilon^2\Delta + m^2}\,u + Vu = \left(\frac{1}{|x|} * |u|^2\right) u$$

where we write V instead of $V + \lambda - m$.

More generally, we will study the equation

$$\sqrt{-\varepsilon^2\Delta + m^2}u + Vu = \left(I_\alpha * |u|^p\right)|u|^{p-2}u \quad \text{in } \mathbb{R}^N \tag{2}$$

where

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- $m > 0$, $2 \leq p < 2N/(N-1)$
- $V\colon \mathbb{R}^N \to \mathbb{R}$
- $I_\alpha(x) = c_{N,\alpha}|x|^{\alpha-N}$. We will assume for simplicity that $c_{N,\alpha} = 1$.

Essential bibliography for non-relativistic Hartree eq.

- J. Wei, M. Winter, *Strongly interacting bumps for the Schrödinger–Newton equation*, J. Math. Phys. **50** (2009), 012905.
- S. Cingolani, S. Secchi, M. Squassina, *Semi-classical limit for Schrödinger equations with magnetic field and Hartree-type nonlinearities*, Proceedings of the Royal Society of Edinburgh **140A** (2010), 973–1009.
- S. Cingolani, M. Clapp, S. Secchi, *Intertwining semiclassical solutions to a Schrödinger-Newton system*, Discrete Continuous Dynmical Systems Series S 6 (2013), 898–908.
- V. Moroz, J. Van Schaftingen, *Semi-classical states for the Choquard equations*, Calc. Var. Partial Differential Equations, to appear.

Essential bibliography for pseudo-relativistic H. eq. without external potentials

- E.H. Lieb, H.-T. Yau, *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics*, Comm. Math. Phys. **112** (1987), 147–174.
- E. Lenzmann, *Uniqueness of ground states for pseudo-relativistic Hartree equations*, Analysis and PDE **2** (2009), 1–27.
- V. Cozi Zelati, M. Nolasco, *Existence of ground states for nonlinear, pseudorelativistic Schrödinger equations*, Red. Lincei Mat. Appl. **22** (2011), 51–72.
- V. Cozi Zelati, M. Nolasco, *Ground states for pseudo-relativistic Hartree equations of critical type*, Rev. Mat. Iberoam. **22** (2013), 1421–1436.

Essential bibliography for pseudo-relativistic H. eq. with external potentials

- M. Melgaard, F. Zongo, *Multiple solutions of the quasirelativistic Choquard equation*, J. Math. Phys. **53** (2012), no. 3, 033709, 12 pp.
- S. Cingolani, S. Secchi, *Ground states for the pseudo-relativistic Hartree equation with external potential*, Proc. of the Royal Soc. Edinburgh (2015)

Essential bibliography for semiclassical limits of pseudorelativistic H. eq.

G.L. Aki, P.A. Markowich, C. Sparber, *Classical limit for semirelativistic Hartree systems*, J. Math. Phys. **49** (2008), no. 10, 102110, 10 pp.

Scaling and main result

Replacing u by $y \mapsto \varepsilon^{\frac{\alpha}{2(1-p)}} u(\varepsilon y)$ transforms

$$\sqrt{-\varepsilon^2 \Delta + m^2} u + V u = \left(I_\alpha * |u|^p\right) |u|^{p-2} u \quad \text{in } \mathbb{R}^N$$

into

$$\sqrt{-\Delta + m^2} u + V_\varepsilon u = (I_\alpha * |u|^p)|u|^{p-2} u \quad \text{in } \mathbb{R}^N \tag{3}$$

with $V_\varepsilon(y) = V(\varepsilon y)$. Let us define $\mathscr{M} = \{y \in O \mid V(y) = V_0\}$.

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with $V_\varepsilon(y) = V(\varepsilon y)$. Let us define $\mathscr{M} = \{y \in O \mid V(y) = V_0\}$. We assume that

(V) $V \colon \mathbb{R}^N \to \mathbb{R}$ is a continuous and bounded function such that $V_{\min} = \inf_{\mathbb{R}^N} V > -m$ and there exists a bounded open set $O \subset \mathbb{R}^N$ with the property that

$$V_0 = \inf_O V < \min_{\partial O} V.$$

Theorem

Retain assumption (V) and assume that $2 \le p < 2N/(N-1)$ and $(N-1)p - N < \alpha < N$. Then, for every sufficiently small $\varepsilon > 0$, there exists a solution $u_\varepsilon \in H^{1/2}(\mathbb{R}^N)$ of equation (3) such that u_ε has a local maximum point y_ε satisfying

$$\lim_{\varepsilon \to 0} \mathrm{dist}(\varepsilon y_\varepsilon, \mathscr{M}) = 0,$$

and for which

$$u_\varepsilon(y) \le C_1 \exp\left(-C_2|y - y_\varepsilon|\right)$$

for suitable constants $C_1 > 0$ and $C_2 > 0$.

Theorem (continued)

Moreover, for any sequence $\{\varepsilon_n\}_n$ *with* $\varepsilon_n \to 0$, *there exists a subsequence, still denoted by the same symbol, such that there exist a point* $y_0 \in \mathscr{M}$ *with* $\varepsilon_n y_{\varepsilon_n} \to y_0$, *and a positive least-energy solution* $U \in H^{1/2}(\mathbb{R}^N)$ *of the equation*

$$\sqrt{-\Delta + m^2}\, U + V_0 U = (I_\alpha * U^p)\, U^{p-1}$$

for which we have

$$u_{\varepsilon_n}(y) = U(y - y_{\varepsilon_n}) + \mathscr{R}_n(y)$$

where $\lim_{n\to+\infty} \|\mathscr{R}_n\|_{H^{1/2}} = 0$.

Variational setting

We adapt the Dirichlet-to-Neumann operator used by Caffarelli and Silvestre to turn our equation into a *local* equation. Given $u \in \mathcal{S}(\mathbb{R}^N)$, we solve uniquely

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ v(0,y) = u(y) & \text{for } y \in \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+. \end{cases}$$

Here we have set

$$\mathbb{R}^{N+1}_+ = \left\{ (x,y) \mid x > 0,\ y \in \mathbb{R}^N \right\}$$

Setting $Tu(y) = -\frac{\partial v}{\partial x}(0, y)$, the function $w(x, y) = -\frac{\partial v}{\partial x}(x, y)$ solves

$$\begin{cases} -\Delta w + m^2 w = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ W(0, y) = Tu(y) & \text{for } y \in \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+. \end{cases}$$

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Hence we easily get $T \circ T = -\Delta + m^2$. Our nonlocal problem (3) is equivalent to

$$\begin{cases} -\Delta v_\varepsilon + m^2 v_\varepsilon = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ -\frac{\partial v_\varepsilon}{\partial x}(0, y) = -V_\varepsilon(y) v_\varepsilon(0, y) + & \\ \qquad + \left(I_\alpha * |v_\varepsilon(0, \cdot)|^p\right) |v_\varepsilon(0, y)|^{p-2} v_\varepsilon(0, y) & \text{for } y \in \mathbb{R}^N. \end{cases} \tag{4}$$

The Euler functional

A convenient setting to solve this Neumann system is $H = H^1(\mathbb{R}_+^{N+1})$ with the associated continuous trace operator $\gamma\colon H^1(\mathbb{R}_+^{N+1}) \to H^{1/2}(\mathbb{R}^N)$.

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A convenient setting to solve this Neumann system is $H = H^1(\mathbb{R}^{N+1}_+)$ with the associated continuous trace operator $\gamma \colon H^1(\mathbb{R}^{N+1}_+) \to H^{1/2}(\mathbb{R}^N)$.
We introduce the action function $\mathcal{E}_\varepsilon \colon H \to \mathbb{R}$ defined by

$$\mathcal{E}_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 \, dx\, dy + \frac{m^2}{2}\int_{\mathbb{R}^{N+1}_+} |v|^2 \, dx\, dy + \frac{1}{2}\int_{\mathbb{R}^N} V_\varepsilon |\gamma(v)|^2 \, dy - \frac{1}{2p}\int_{\mathbb{R}^N} \left(I_\alpha * |\gamma(v)|^p\right) |\gamma(v)|^p \, dy.$$

Remark

It is easy to check that $\mathcal{E}_\varepsilon \in C^1(H)$ and that $D\mathcal{E}_\varepsilon(v) = 0$ if and only if v (weakly) solves (4).

The limit problem

At least formally, the limit problem as $\varepsilon \to 0$ is

$$\sqrt{-\Delta + m^2}u + au = (I_\alpha * |u|^p)|u|^{p-2}u \tag{5}$$

whose Euler functional in the augmented space H is

$$\begin{aligned} L_a(v) = &\frac{1}{2}\int_{\mathbb{R}^{N+1}_+} \left(|\nabla v|^2 + m^2|v|^2\right) dx\, dy \\ &+ \frac{a}{2}\int_{\mathbb{R}^N} |\gamma(v)|^2\, dy - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |\gamma(v)|^p)|\gamma(v)|^p\, dy. \end{aligned}$$

Ground states

For $a > -m$ we consider

$$m_a = \inf \{L_a(v) \mid DL_a(v) = 0,\ v \in H \setminus \{0\}\}$$

and the set S_a of elements $v > 0$ such that $L_a(v) = m_a$.

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It is possible to prove that minimizers of

$$\tilde{m}_a = \inf_{\substack{u \in H^{1/2}(\mathbb{R}^N) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} \left| \sqrt{(m^2 - \Delta)^{1/2} - m} u \right|^2 + (a+m)|u|^2}{\left(\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \right)^{1/p}}$$

give rise to minimizers of m_a via a simple rescaling. By symmetrization on $\tilde{m}_a$, minimizers are positive and radially symmetric around some point.

Proposition

For any $a > -m$ *the set* S_a *is non-empty. Moreover, it is compact in* H *and for some* $C > 0$ *and any* $\sigma \in (-V_{\min}, m) \cap [0, +\infty)$ *the estimate*

$$v(x, y) \leq C\mathrm{e}^{-(m-\sigma)\sqrt{x^2+|y|^2}}\mathrm{e}^{-\sigma x}$$

for every $v \in S_a$*.*

A penalization scheme

For

$$\delta = \frac{1}{10}\,\mathrm{dist}(\mathscr{M}, \mathbb{R}^N \setminus O) \quad \text{and} \quad \beta \in (0, \delta)$$

we fix a cut-off $\varphi \in C_0^\infty(\mathbb{R}_+^{N+1})$ such that $0 \le \varphi \le 1$ everywhere, $\varphi(x, y) = 1$ if $x + |y| \le \beta$, and $\varphi(x, y) = 0$ if $x + |y| \ge 2\beta$.

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$$U_\varepsilon^{y_0}(x, y) = \varphi_\varepsilon\left(x, y - \frac{y_0}{\varepsilon}\right) U\left(x, y - \frac{y_0}{\varepsilon}\right)$$

Remark

$\mathscr{M}^\beta$ denotes the closed β-neighborhood of $\mathscr{M}$.

We also define, for all $\varepsilon > 0$,

$$\chi_\varepsilon(y) = \begin{cases} 0 & \text{if } y \in O_\varepsilon \\ \varepsilon^{-6/\mu} & \text{if } y \notin O_\varepsilon \end{cases}$$

and

$$Q_\varepsilon(v) = \left(\int_{\mathbb{R}^N} \chi_\varepsilon \gamma(v)^2 \, dy - 1\right)_+^{\frac{2p+1}{2}}$$

for $v \in H$.

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for $v \in H$.
Finally, let

$$\Gamma_\varepsilon(v) = \mathscr{E}_\varepsilon(v) + Q_\varepsilon(v), \quad v \in H.$$

Looking for a solution

For $\varepsilon > 0$ sufficiently small, we will look for a solution near the set of approximate solutions

$$X_\varepsilon = \left\{ U_\varepsilon^{y_0} \mid y_0 \in \mathscr{M}^\beta \text{ and } U \in S_{V_0} \right\}$$

Representation of (PS) sequences

Proposition

Let $d > 0$ be small enough, and let $\{\varepsilon_j\}_j$ be such that $\lim_{j\to+\infty}\varepsilon_j = 0$ and let $\{v_{\varepsilon_j}\} \subset X^d_{\varepsilon_j}$ be such that

$$\lim_{j\to+\infty}\Gamma_{\varepsilon_j}(v_{\varepsilon_j}) \le E_{V_0}, \quad \lim_{j\to+\infty}\Gamma'_{\varepsilon_j}(v_{\varepsilon_j}) = 0.$$

Then there exist — up to a subsequence — $\{\tilde{y}_j\}_j \subset \mathbb{R}^N$, a point $\bar{y} \in \mathscr{M}$ and $U \in S_{V_0}$ such that

$$\lim_{j\to+\infty}|\varepsilon_j\tilde{y}_j - \bar{y}| = 0$$

$$\lim_{j\to+\infty}\left\| v_{\varepsilon_j} - \varphi_{\varepsilon_j}(\cdot,\cdot - \tilde{y}_j)U(\cdot,\cdot - \tilde{y}_j)\right\| = 0.$$

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Proposition

For $\varepsilon > 0$ sufficiently small, the functional Γ_ε has a critical point $v_\varepsilon \in X_\varepsilon^d$, i.e. localized near X_ε.

To prove our main result, we now claim that v_ε is actually a critical point of $\mathcal{E}_\varepsilon$.

Recall that

$$\Gamma_\varepsilon(v) = \mathscr{E}_\varepsilon(v) + \left(\int_{\mathbb{R}^N} \chi_\varepsilon \gamma(v)^2 \, dy - 1\right)_+^{\frac{2p+1}{2}} .$$

To prove the claim, it is enough to show that $v_\varepsilon(x, y)$ is small when y is far from $\mathscr{M}$.

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To prove the claim, it is enough to show that $v_\varepsilon(x, y)$ is small when y is far from $\mathscr{M}$.
More precisely we can prove that

$$\lim_{\varepsilon \to 0} \sup_{(x,y) \in \mathbb{R}^{N+1}_+ \setminus \left([0,+\infty) \times (\mathscr{M}^{2\beta})_\varepsilon \right)} |v_\varepsilon(x, y)| = 0,$$

and our main theorem follows.

Thank you for your attention and...

Goodbye!