A multiplicity result for a fractional Schrödinger equation in presence of a positive potential

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in collaboration with

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We are interested with the existence of positive solutions of the following

$$\varepsilon^{2s}(-\Delta)^s u + V(z)u = f(u) \quad \text{in } \mathbf{R}^N,\ N > 2s \qquad (\mathcal{P})$$

in the semiclassical limit, $\varepsilon \to 0^+$.

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- The fractional Laplacian ($s \in (0,1)$) is defined, for $u \in \mathcal{S}$, by

$$(-\Delta)^s u(x) = -\frac{1}{2} C_{N,s} \int \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy = (|\cdot|^{2s}\hat{u})^{\vee}(x);$$

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- V is a "Rabinowitz type" potential, i.e $V \in C(\mathbf{R}^N)$ and

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- $f \in C(\mathbf{R})$ is a "well behaved" nonlinearity, i.e.
  - $f(u) = 0$ for $u \leq 0$;
  - $\lim_{u\to 0} f(u)/u = 0$;
  - $\exists\, q \in (2, 2_s^* - 1)$ such that $\lim_{u\to\infty} f(u)/u^q = 0$, where $2_s^* := 2N/(N-2s)$;
  - $\exists\, \theta > 2$ such that $0 < \theta \int_0^u f(t)dt \leq uf(u)$ for all $u > 0$;
  - the function $u \to f(u)/u$ is strictly increasing in $(0,+\infty)$.

After a rescaling, the equation can be written as

$$(-\Delta)^s u + V(\varepsilon x)u = f(u) \quad \text{in } \mathbf{R}^N,\ N > 2s$$

and we are reduced to find the critical points of the C^1 functional

$$I_\varepsilon(u) = \frac{1}{2}\int |(-\Delta)^{s/2}u|^2 + \frac{1}{2}\int V(\varepsilon x)u^2 - \int F(u)$$

in $W_\varepsilon = \left\{u \in H^s(\mathbf{R}^N) : \int V(\varepsilon x)u^2 < +\infty\right\}$; as usual, $F(u) = \int_0^u f(t)dt$.

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Let

$$\mathcal{N}_\varepsilon = \left\{u \in H^s(\mathbf{R}^N) \setminus \{0\} : \int |(-\Delta)^{s/2}u|^2 + \int V(\varepsilon x)u^2 - \int f(u)u = 0\right\}$$

be the Nehari manifold associated to I_ε.

The problem "at infinity"

An important role is played by the problem at infinity:

$$(-\Delta)^s u + \mu u = f(u) \quad \text{in } \mathbf{R}^N, \ \mu > 0. \qquad (\mathcal{P}^\infty)$$

The solutions are critical points in $H^s(\mathbf{R}^N)$ of

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It is easy to see that E_μ has a MP geometry and setting

- c_μ^∞ the MP level,
- m_μ^∞ the ground state level (infimum on the Nehari manifold associated)

they coincide and are achieved on a positive (ground state) solution $\mathbf{U}_\mu$.

In the sequel, μ will be V_0 or V_∞ (if finite).

Analogously for I_ε (the functional associated to our initial equation) setting

- c_ε the MP level,
- m_ε the ground state level (infimum on $\mathcal{N}_\varepsilon$)

it holds

$$c_\varepsilon = m_\varepsilon \geq m_{V_0}^\infty > 0.$$

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It is possible to see that the functional I_ε in W_ε satisfies the $(PS)_c$ condition

1. for $c < m^\infty_{V_\infty}$, if $V_\infty < +\infty$ (and in particular "near" the level $m^\infty_{V_0}$),
2. for every $c \in \mathbf{R}$, if $V_\infty = +\infty$.

We are then reduced to find the critical points of I_ε on $\mathcal{N}_\varepsilon$, on which it is bounded below and satisfies the $(PS)_c$ condition.

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Theorem (Existence of ground state)

Under the above assumptions, there exists a ground state solution $\mathfrak{u}_\varepsilon \in W_\varepsilon$

1. *for every $\varepsilon \in (0, \bar{\varepsilon}]$, for some $\bar{\varepsilon} > 0$, if $V_\infty < +\infty$;*
2. *for every $\varepsilon > 0$, if $V_\infty = +\infty$.*

Let $\delta > 0$ be fixed (it will be conveniently chosen later) and

$$\eta(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq \delta/2 \\ 0 & \text{if } s \geq \delta. \end{cases} \quad \text{smooth and nonincresing.}$$

Let $\mathbf{U}_{V_0}$ be the g.s. solution of $(\mathcal{P}^\infty)$ with $\mu = V_0$,

$$M = \{x \in \mathbf{R}^N : V(x) = V_0\} \qquad \text{(the set of minima of } V\text{)}$$

and for any $y \in M$, define

$$\Psi_{\varepsilon,y}(x) := \eta(|\varepsilon x - y|)\mathbf{U}_{V_0}\left(\frac{\varepsilon x - y}{\varepsilon}\right).$$

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Let $t_{\varepsilon,y} > 0$ the unique point such that $t_{\varepsilon,y}\Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon$, and let

$$\Phi_\varepsilon : y \in M \longmapsto t_{\varepsilon,y}\Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon.$$

It is easy to see that

- for any $y \in M$, $\Phi_\varepsilon(y)$ has compact support,
- Φ_ε is a continuous map,

- the function Φ_ε satisfies

$$\lim_{\varepsilon\to 0^+} I_\varepsilon(\Phi_\varepsilon(y)) = m_{V_0}^\infty, \quad \text{uniformly in } y \in M.$$

Hence $h(\varepsilon) := |I_\varepsilon(\Phi_\varepsilon(y)) - m_{V_0}^\infty| = o(1)$ for $\varepsilon \to 0^+$ uniformly in y, and then

$$\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)} := \left\{ u \in \mathcal{N}_\varepsilon : I_\varepsilon(u) \leq m_{V_0}^\infty + h(\varepsilon) \right\} \neq \emptyset$$

since for sufficiently small ε, $\Phi_\varepsilon(y) \in \mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}$.

Then

$$\Phi_\varepsilon : M \longrightarrow \mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}.$$

The barycenter map

Choose $\delta > 0$ so that

$$M_{2\delta} := \left\{ x \in \mathbf{R}^N : d(x, M) \leq 2\delta \right\} \cong_{hom} M.$$

Let $\rho = \rho(\delta) > 0$ be such that $M_{2\delta} \subset B_\rho$ and $\chi : \mathbf{R}^N \to \mathbf{R}^N$ be defined as

$$\chi(x) = \begin{cases} x & \text{if } |x| \leq \rho \\ \rho \frac{x}{|x|} & \text{if } |x| \geq \rho. \end{cases}$$

Let us define the barycenter map β_ε

$$\beta_\varepsilon(u) := \frac{\int \chi(\varepsilon x) u^2(x)}{\int u^2(x)} \in \mathbf{R}^N, \quad u \in W_\varepsilon, \ supp(u) \text{ is compact.}$$

The function β_ε satisfies

- $\lim_{\varepsilon\to 0^+} \beta_\varepsilon(\Phi_\varepsilon(y)) = y$, uniformly in $y \in M$,
- $\lim_{\varepsilon\to 0^+} \sup_{u\in\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}} \inf_{y\in M_\delta} \left|\beta_\varepsilon(u) - y\right| = 0$

which implies that there exists $\varepsilon^* > 0$ such that

$$\forall\, \varepsilon \in (0, \varepsilon^*] : \quad \sup_{u\in\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}} d(\beta_\varepsilon(u), M_\delta) < \delta/2.$$

Defining

$$M^+ := \left\{ x \in \mathbf{R}^N : d(x, M) \leq 3\delta/2 \right\} \cong_{hom} M$$

we see that, possibly reducing $\varepsilon^* > 0$,

$$\beta_\varepsilon : \mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)} \longrightarrow M^+.$$

Then it is easy to see that

$$M \xrightarrow{\Phi_\varepsilon} \mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)} \xrightarrow{\beta_\varepsilon} M^+$$ is homotopic to the inclusion map

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$$cat(\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}) \geq cat_{M^+}(M)$$ (the LS category)

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Recall that, if $V_\infty = \infty$, the PS condition is satisfied at any level.

In case $V_\infty < \infty$, the PS condition is satisfied in $(m_{V_0}^\infty, m_{V_0}^\infty + h(\varepsilon))$, and then, by the Ljusternick-Schnirelmann theory

Theorem (Multiplicity)

For suitable small ε, the problem has at least $cat(M)$ positive solutions.

Remark

Assume furthermore that M, the set of minima of V, is not contractible and bounded.

It is easy to see that $\mathcal{A} := \Phi_\varepsilon(M) \subset \mathcal{N}_\varepsilon$

- is not contractible in $\mathcal{N}_\varepsilon^{m_{V_0}^\infty + h(\varepsilon)}$
- is contractible in $\mathcal{N}_\varepsilon^c$, for a suitable $c > m_{V_0}^\infty + h(\varepsilon)$.

This implies the existence of another critical level for I_ε.

Finally, by implementing the Morse theory, it is possible to show that

Theorem

Replace conditions

- $\lim_{u\to 0} f(u)/u = 0$,
- $\exists\, q \in (2, 2_s^* - 1)$ *such that* $\lim_{u\to\infty} f(u)/u^q = 0$,

with

- $f \in C^1(\mathbf{R})$;
- $\lim_{u\to 0} f'(u) = 0$;
- $\exists\, q \in (2, 2_s^* - 1)$ *such that* $\lim_{u\to\infty} f'(u)/u^{q-1} = 0$,

(being understood the other assumptions on f and V).
Then the problem possesses at least

$$2\mathcal{P}_1(M) - 1$$

where $\mathcal{P}_1(M) = \sum_k \dim H_k(M)$.

Thanks for your attention!