Elliptic PDE with natural/critical growth in the gradient

Boyan Sirakov

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Given an elliptic operator

$$Lu = a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)u, \qquad F(D^2u, Du, u, x)$$

$$Lu = \mathrm{div}(A(x)\nabla u) + b_i(x)\partial_i u + c(x)u, \qquad \mathrm{div}(\mathcal{A}(x, u, \nabla u))$$

consider

$$-Lu = g(x, u, \nabla u)$$

Here the nonlinearity

g grows in ∇u like $|\nabla u|^2$.

Why natural or critical ?

- Natural,
  since invariant w.r. to diffeomorphic changes of function

$$u \to \psi(u), \qquad v(x) = \psi(u(x)).$$

- Since L in general form,
  also invariant w.r. to diffeomorphic changes of variable

$$x \to \Psi(x), \qquad v(x) = u(\Psi^{-1}(x))$$

- Critical,
  since gradient term has same scaling as hessian term

$$u(x) \to u_t(x) = u(tx), \qquad t > 0.$$

$$\begin{cases} -L_0 u = c(x)u + \langle M(x)\nabla u, \nabla u \rangle + h(x) & \text{in} \quad \Omega \\ u = g(x) & \text{on} \quad \partial\Omega \end{cases}$$

$$L_0 u = a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u$$

M is a bounded matrix,

and the other coefficients have the necessary regularity to ensure satisfactory theory for the linear problem ($M = 0$).

The size and sign of $c(x)$ matter, both for the solvability and for the uniqueness of solutions.

Some history

KAZDAN—KRAMER 1975, various partial results

BOCCARDO—MURAT—PUEL 1980-1985
succeeded to obtain a full solvability result for the general class of divergence form operators, in the case when

$$c(x) \leq -c_0 < 0$$

FERONE—MURAT 2000
solvability for the case

$$c \equiv 0$$

and observation that only possible if $|Mh|$ small.

BARLES-MURAT, B.-PORRETTA... 1995-2000 — uniqueness.

Related results by dall'Aglio, Giachetti and Puel; Maderna, Pagani and Salsa; Grenon, Murat and Porretta; Abdellaoui, dall'Aglio and Peral; Abdel Hamid and Bidaut-Véron; Boccardo, Gallouet, Murat...

B.S. ARMA 2010 —

1. Extension of the above to general class of (even fully nonlinear) NON-DIVERGENCE form equations, through maximum-principle type arguments.

2. And how about the case $c > 0$?

– solvability still holds for small $c > 0$, but in general not the uniqueness !

$$\begin{cases} -L_0 u = c(x)u + \langle M(x)\nabla u, \nabla u\rangle + h(x) & \text{in} \quad \Omega \\ u = g(x) & \text{on} \quad \partial\Omega \end{cases}$$

Theorem (ARMA 2010)

1. *This problem has a unique solution if* $c(x) \leq -c_0 < 0$
2. *There exists* $\delta_0 > 0$ *depending on* λ, Λ, $\|b\|_n$, *diam*(Ω), *such that if*

$$\|M_0|h| + c^+ + M_0 c^+ (\max_{\partial\Omega} g)\|_n \leq \delta_0$$

then there exists a solution. It is unique if $c^+ = 0$.

3. *If* $c \equiv c_0 > 0$, $L_0 = \Delta$, $M = \mu_0 I$, $g = h = 0$, *then there are at least two solutions.*

$$\begin{cases} -L_0 u = c(x)u + <M(x)\nabla u, \nabla u> + h(x) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

How about a more general non-uniqueness result ?

One might hope that whenever $\| M_0 |h| + c^+ \|$ is small and we have a hypothesis which prevents M or c to be zero, say $M \gneqq 0$ or $M \lneqq 0$, and $c \gneqq 0$, then we have at least two solutions.

Results for the model equation with $L_0 = \Delta, \quad M = \mu(x)I$,

$$-\Delta u = c(x)u + \mu(x)|\nabla u|^2 + h(x)$$

B.S. 2010: $c(x) = c_0 > 0$, $\mu(x) = \mu_0 > 0$, $h = 0$ — exponential change and a Gidas-Spruck blow-up.

L. JEANJEAN–B.S. 2014: $\mu(x) = \mu_0 > 0$ — exponential change and a mountain-pass for a slow-growth nonlinearity.

D. ARCOYA, C. DE COSTER, L. JEANJEAN, K. TANAKA;
L. JEANJEAN, C. DE COSTER: $0 < \mu_1 \le \mu(x) \le \mu_2$
— degree theory, Rabinowitz type bifurcation point of view, dependence of solutions on a parameter ($c \to \lambda c$, $\lambda \in \mathbb{R}$).

The decisive uniform a priori bound depends heavily on the variational structure of the Laplacian.

Can we relax the strict lower bound on $\mu(x)$?

$$\begin{cases} -\Delta u &= c(x)u + \mu(x)|\nabla u|^2 + h(x) & \text{in} & \Omega \\ u &= 0 & \text{on} & \partial\Omega \end{cases}$$

The pictures suggest it might be hard to give a full answer.

SOUPLET:
It is necessary that the intersection of supp(μ) and supp(c) contains a ball.

If in addition

- $n = 2$; or
- $n = 3$ and power growth for μ and c close to $\partial\Omega$; or
- $n = 3, 4$, and $\mu(x) \geq \mu_1 > 0$ on supp(c).

then an uniform a priori bound holds.

Uses the variational structure of the Laplacian and elliptic theory in weighted L^p_δ-spaces.

We report on another method for proving a priori bounds, which uses only fundamental results valid for all elliptic operators, namely

- quantitative strong maximum principle;
- weak Harnack inequality;
- local maximum principle.

For the problem at hand, this method permits to us to

- Prove a priori bounds for any uniformly elliptic operator L_0;
- and do this, assuming that $\mu(x) \geq \mu_1 > 0$ on $\mathrm{supp}(c)$, in any dimension.

$$\begin{cases} -L_0 u = c(x)u + \langle M(x)\nabla u, \nabla u\rangle + h(x) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

Theorem

Let $M(x) \geq \mu_1 I$ on $supp(c)$, $\mu_1 > 0$.
If the problem with $c = 0$ has a solution,
then it has at least two solutions for $c \gneqq 0$ small, and
the same bifurcation diagrams are valid.

Some observations

If $c < \lambda_1$ then $M \geq 0$, $h \geq 0$ and $u \geq 0$ is generic.

$$-\Delta u = c(x)u \; + \mu(x)|\nabla u|^2 + \; h(x)$$
$$-\Delta \psi = c(x)\psi + h(x)$$

Then $v = u - \psi$ is positive (MP) and satisfies

$$-\Delta v + <\nabla \psi, \nabla v> = c(x)v \; + \mu(x)|\nabla v|^2 + \mu(x)|\nabla \psi|^2$$

modified elliptic operator.

Some observations

If lack of a priori bound, then lack of a priori bound on supp(c).

In $\mathcal{O}$ open in $\{c \equiv 0\}$, then in $\mathcal{O}$
both $u - \sup_{\partial\mathcal{O}} u$ and $u_0 - \inf_\Omega u_0$ are solutions of

$$-\Delta u = \mu(x)|\nabla u|^2 + \ h(x)$$

and they compare on the boundary, so by MP

$$\sup_{\mathcal{O}} u \leq \sup_{\partial\mathcal{O}} u + 2\|u_0\|_\infty$$

Some observations

So if lack of a priori bound, then for some $x_0 \in \overline{\Omega}$ and some (half-)ball B around x_0 we have an unbounded in B sequence of solutions of

$$-L_0 u = c(x)u + \mu(x)|\nabla u|^2 + h(x)$$

and in B

$$0 < \mu_1 \le \mu(x) \le \mu_2, \qquad c \gneqq 0.$$

standard exponential change $v_i = \mu_i^{-1}(e^{\mu_i u} - 1)$
In B, for large u

$$\begin{array}{rcl} -L_1 v_1 & \ge & f_1(x, v_1) \sim c_0\, c(x)\, v_1 \log(v_1) \\ -L_2 v_2 & \le & f_2(x, v_2) \sim C_0\, c(x)\, v_2 \log(v_2) \\ v_2 & \sim & v_1^A \end{array}$$

In B, for large u

$$\begin{aligned} -L_1 v_1 &\geq c_0\, c(x)\, v_1 \log(v_1) + h_1 \\ -L_2 v_2 &\leq C_0\, c(x)\, v_2 \log(v_1) + h_2 \\ v_2 &\sim v_1^A \end{aligned}$$

Theorem

There exists a constant C_0 depending on λ, Λ, $\|b\|_p$, $\|A_i c\|_\infty$, and a lower bound on $\int_B c$, such that $\|u\|_\infty \leq C_0$.

1. (QSMP) $-Lu \geq 0$, $u > 0$ in $\Omega \Longrightarrow$ for each compact $K \subset \Omega$

$$\inf_K u \geq c \left(\int_K (-Lu)^{\varepsilon_0} \right)^{1/\varepsilon_0}.$$

2. (WHI) $-Lu \geq f$, $u > 0$ in $\Omega \Longrightarrow$ for each $K \subset K' \subset \Omega$

$$\left(\int_{K'} u^{\varepsilon} \right)^{1/\varepsilon} \leq C(\inf_K u + \|f\|_n).$$

3. (LMP) $-Lu \leq f$ in $\Omega \Longrightarrow$ for each $K \subset K' \subset \Omega$ and $p > 0$

$$\sup_K u \leq C \left(\left(\int_{K'} u^p \right)^{1/p} + \|f\|_n \right)$$

Need boundary versions of these !!

1. (BQSMP) $-Lu \geq 0$, $u > 0$ in B_2^+, $u = 0$ on $\{x_n = 0\} \Longrightarrow$

$$\inf_{B_1^+} \frac{u}{x_n} \geq c \left(\int_{B_1} (-Lu)^{\varepsilon_0} \right)^{1/\varepsilon_0}.$$

2. (BWHI) $-Lu \geq f$, $u > 0$ in B_2^+, $u = 0$ on $\{x_n = 0\} \Longrightarrow$

$$\left(\int_{B_1^+} \left(\frac{u}{x_n} \right)^{\varepsilon} \right)^{1/\varepsilon} \leq C(\inf_{B_{3/2}^+} \frac{u}{x_n} + \|f\|_n).$$

3. (BLMP) $-Lu \leq f$ in B_2^+, $u = 0$ on $\{x_n = 0\}$, $c \in L^{n+}$, $\Longrightarrow$

$$\sup_{B_1^+} \frac{u}{x_n} \leq C \left(\left(\int_{B_{3/2}^+} u^p \right)^{1/p} + \|f\|_n \right)$$

Krylov original approach of writing a degenerate equation for u/x_n.

$$
\begin{aligned}
-L_1 v_1 &\geq c_0\, c(x)\, v_1 \,\log(v_1) + h_1\\
-L_2 v_2 &\leq C_0\, c(x)\, v_2 \,\log(v_1) + h_2\\
v_2 &\sim v_1^A
\end{aligned}
$$

1^{st} ineq. + BQSMP $\Longrightarrow$

$$
\inf_{B_1^+} \frac{v_1}{x_n} \geq c_0 \inf_{B_1^+} \frac{v_1}{x_n} \log\left(\inf_{B_1^+} \frac{v_1}{x_n}\right) \left(\int_{B_1^+} c^\varepsilon x_n^{1+\varepsilon}\right)
$$

So $\inf\limits_{B_1^+} \dfrac{v_1}{x_n} \leq C$. Then 1^{st} ineq. + BWHI $\Longrightarrow$

$$
\left(\int_{B_1^+} (v_1)^\varepsilon\right)^{1/\varepsilon} \leq C \left(\int_{B_1^+} \left(\frac{v_1}{x_n}\right)^\varepsilon\right)^{1/\varepsilon} \leq C.
$$

2^{nd} ineq. + LMP + $(\log(v_1))^{n+1} \leq C v_1^\varepsilon \Longrightarrow$

$$
\sup_{B_1^+} v_2 \leq \left(\int_{B_1^+} (v_2)^{\varepsilon/A}\right)^{A/\varepsilon} \leq C \left(\int_{B_1^+} (v_1)^\varepsilon\right)^{A/\varepsilon} \leq C.
$$